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ON THE GEOMETRY OF GROSS-PITAEVSKI VORTEX CURVES FOR GENERIC DATA

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ABSTRACT. We study an energy functional that arises as a Γ-limit of the Gross-Pitaevskii (GP) energy. This last functional is often used to model rotating Bose-Einstein condensates, and the functional we study represents the contribution to the GP energy of vortices, or whirlpools, in the condensate. For our energy, we give a rough description of its (local) minimizers using ODE techniques along with an isoperimetric inequality.

1. INTRODUCTION

Bose-Einstein condensates (BEC) are a particular kind of matter sometimes characterized by the property that the particles making up a sample act as a single *super-particle* rather than as a collection of individual ones. An energy functional that is often used to model such a condensate is the Gross-Pitaevskii energy, which has been considered in several papers. (See for instance [6], [3], [5] and [7]. See also [1] and [8] for good surveys on both mathematical and physical aspects of BEC.) An important feature of these condensates is that, when stirred, they develop whirlpools, often referred to as vortices. In dimension $n = 3$, these vortices can be thought of as curves in space, and their contribution to the GP energy can be formally expressed as

$$E_0(\gamma) = \int_a^b \{\rho(\gamma)\,|\gamma'| + B_0(\gamma)\cdot\gamma'\}\,.$$

Here $\Omega \subset \mathbb{R}^3$ represents the region occupied by the condensate, and $\gamma :]a, b[\to \Omega$ is a Lipschitz curve with no boundary in Ω that models a vortex. ρ is a real valued function representing a trapping potential that keeps the condensate in place, and $B_0 \in C^{1,1}(\Omega;\mathbb{R}^3)$ is a vector field determined both by the stirring applied to the condensate and the potential ρ.

From the references mentioned above, we know that the functional E_0 is in fact a Γ-limit of the GP energy. Γ-convergence is a convergence notion for functionals that yields various relations between minimizers of the functionals involved. For example, global minimizers of the converging functionals converge to global minimizers of the limiting one. Also, isolated local minimizers of the limit functional give raise to local minimizers of the converging functionals. In several works regarding the Γ-convergence of the GP energy to E_0, ρ and B_0 were rotationally symmetric or explicit. However, the results in [7] show that this Γ-limit result is valid for rather

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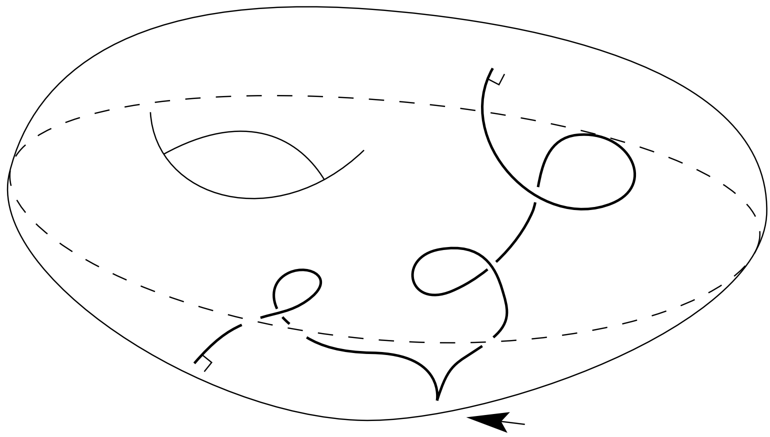

FIGURE 1. A critical point might have high curvature where its interior nears the boundary.

general ρ and B_0. In all these cases ρ and DB_0 vanish linearly at the boundary of Ω, and we would like to study local minimizers of E_0 in this context.

1.1. **Results.** The first term in the energy E_0 is just the length of the curve γ weighted by the function ρ, which we denote $L_\rho(\gamma)$. When ρ vanishes at the boundary of Ω, it is possible for curves to linger near the boundary, keeping bounded weighted length but attaining unbounded regular length. Our first result shows that this does not happen for critical points of E_0. Throughout this section we use the notation described in Section 2.1.

Theorem 1.1. *Assume ρ, DB_0 vanishes to order k at the boundary of Ω (made precise in hypotheses (H1)-(H2)). Let $J \in \mathcal{I}(\Omega)$ be a critical point of E_0 in the sense of (2.14) with $L_\rho(J) < \infty$, and let γ be one of its regular irreducible Lipschitz curves. Then the curve γ is $C^{1,1}$ in its interior and has bounded curvature. Furthermore, γ has finite length, controlled by the weighted length*

$$L(\gamma) < A_k L_\rho(\gamma) \tag{1.1}$$

via an explicit constant A_k depending on the input data ρ, B_0, but independent of γ. The curve γ is either a closed-loop in the interior of Ω or it meets the boundary of Ω at each of its ends, doing so perpendicularly.

Remark. We do not show any uniformity of the curvature bound on γ. Indeed, we expect that there exist critical points J with high curvature where their interior nears the domain boundary (as indicated by the arrow in Figure 1).

With the regular length of critical points under control, it is possible to use the isoperimetric inequality for the domain Ω and (unweighted) length and area to rule out infinite-component local minimizers.

Theorem 1.2. *Assume that $\partial\Omega$ is path-connected. Let J be a* local minimum *of E_0 with respect to the flat metric, and assume that J has finite weighted length. Then J is a sum of* finitely many $C^{1,1}$ *curves and has finite total length according to* (1.1).

Under an additional assumption, we may use our tools to count the number of components.

Theorem 1.3. *Assume that $\partial\Omega$ is path-connected. Let J be a* local minimum *of energy E_0 with finite weighted length. If J is moreover minimal in the sense that subtracting any component of J increases energy, then J has the following properties:*

(i) *There is a uniform lower bound on lengths:*

$$||\rho||_{L^\infty} L(\gamma_i) \geq L_\rho(\gamma_i) \geq L_{min} := \left(||DB||_{L^\infty} C_{iso}(\Omega) A_k^2\right)^{-1} > 0.$$

Here $C_{iso}(\Omega)$ is an isoperimetric constant of the domain Ω and A_k is given by Theorem 1.1.

(ii) *There is a bound on the number of components*

$$n_{components} \leq M_\rho(J)/L_{min}$$

which is uniform in the sense that L_{min} does not depend on J.

Remark 1.4. It is interesting to compare E_0 with the energy obtained from it by replacing ρ by the function identically equal to 1 in Ω. This energy also corresponds to a Γ-limit, but of the Ginzburg-Landau energy from super-conductivity instead. In this last case, critical points of E_0 obey an Euler-Lagrange equation which is the equation of motion of a charged particle moving in the magnetic field $\nabla \times B_0$, often referred to as motion under the Lorentz force.

Remark 1.5. Here we do not address the existence of curves that locally minimize E_0 for ρ that vanishes on $\partial\Omega$. We know that such curves exist from concrete examples that can be found in [6] and [3], and our results apply to them.

2. Setup

The main data of our problem are the smooth domain $\Omega \subset \mathbb{R}^3$, the function ρ and the vector field B_0. We assume that ρ and DB_0 vanish to order k at the boundary of Ω, in the following sense:

(H1) $\rho = \omega^k$ for some integer $k \geq 1$ and a function $\omega \in C^{1,1}(\Omega;]0, \infty[)$ that satisfies $\omega = 0$ on $\partial\Omega$ and in Ω,

$$m < |\nabla\omega|^2 + \omega \quad < \quad M, \tag{2.1}$$

$$|D^2\omega| \quad < \quad M_2, \tag{2.2}$$

for positive constants m, M, M_2.

(H2) $B_0 \in C^{1,1}(\Omega; \mathbb{R}^3)$ and

$$|B_0| + |DB_0| \leq M_3\rho \text{ in } \Omega, \tag{2.3}$$

for some $M_3 > 0$.

Note in particular that the function $\omega \in C^{1,1}(\Omega;]0, \infty[)$ in (H1) has $\omega > 0$ in Ω, $\nabla\omega \neq 0$ on $\partial\Omega$, and satisfies

$$C_1 d_{\partial\Omega} \leq \omega \leq C_2 d_{\partial\Omega} \tag{2.4}$$

for $0 < C_1 \leq C_2$, where $d_{\partial\Omega}$ denotes the distance to the boundary function. The function ω could be taken to be a smoothed version of the distance to $\partial\Omega$.

2.1. **Line energy.** To model the vortex curves we use rectifiable 1-currents. These are essentially countable sums of Lipschitz curves, in a way which we make precise in this section. Here we define the objects that we need in a somewhat non-standard way.

The space of 1-currents in Ω is defined as the dual of the space of smooth vector fields $C_0^\infty(\Omega;\mathbb{R}^3)$ when this last space is endowed with the usual inductive limit topology. We denote such a space by $\mathcal{D}_1(\Omega)$. We say that $T \in \mathcal{D}_1(\Omega)$ is rectifiable if there are countably many (Lebesgue) measurable sets $I_i \subset \mathbb{R}$ and Lipschitz functions $\gamma_i : I_i \to \Omega$ such that

$$T(B) = \sum_{i=1}^{\infty} \int_{I_i} \langle B \circ \gamma_i, \gamma_i' \rangle \tag{2.5}$$

for every $B \in C_0^\infty(\Omega;\mathbb{R}^3)$. We use the notation $\langle \cdot, \cdot \rangle$ for the standard inner product in $\mathbb{R}^3$.

The boundary of T, denoted by ∂T, is the distribution acting on functions $\phi \in C_0^\infty(\Omega)$ through the formula

$$(\partial T)(\phi) = T(\nabla \phi). \tag{2.6}$$

A norm on $\mathcal{D}_1(\Omega)$ that we use frequently in this paper is the mass. For rectifiable 1-currents, which are essentially countable families of Lipschitz curves, this mass is just the total length of the curves. We require a weighted version of this that we denote by $M_\rho(T)$ and that can be expressed as

$$M_\rho(T) = \sup\{T(B) : B \in C_0^\infty(\Omega;\mathbb{R}^3), |B| \leq \rho \text{ in } \Omega\}. \tag{2.7}$$

We will always work with rectifiable 1-currents with zero boundary and finite weighted mass, and we will denote the collection of such currents by $\mathcal{I}(\Omega)$. In other words,

$$\mathcal{I}(\Omega) = \{T \in \mathcal{D}_1(\Omega) : T \text{ is rectifiable}, \ M_\rho(T) < \infty, \partial T = 0\}. \tag{2.8}$$

The following version of 4.2.25 from [4] can be found in [6].

Lemma 2.1. *Let $T \in \mathcal{I}(\Omega)$. One can choose the I_i and the curves γ_i in* (2.5) *in such a way that $I_i =]a_i, b_i[$ is an interval, possibly infinite,*

$$M_\rho(T) = \sum_{i=1}^{\infty} \int_{a_i}^{b_i} \rho(\gamma_i)\, |\gamma_i'| \,, \tag{2.9}$$

and each curve $\gamma_i : I_i \to \Omega$ is injective and of one of the following two types:

(i) Closed loop. The following limits exist, belong to the interior of Ω and agree:

$$\lim_{t\to a_i^+} \gamma_i(t) = \lim_{t\to b_i^-} \gamma_i(t).$$

(ii) Boundary-to-boundary curve:

$$\lim_{t\to a_i^+} \omega(\gamma_i(t)) = \lim_{t\to b_i^-} \omega(\gamma_i(t)) = 0. \tag{2.10}$$

Definition 2.2. We call each γ_i in the lemma above an *irreducible component* of T.

Next, we wish to endow $\mathcal{I}(\Omega)$ with a metric, usually referred to as the *flat metric*. To do this we need first to recall a few definitions.

Definition 2.3. A set $S \subset \mathbb{R}^3$ is called 2-rectifiable if there is a countable collection of bounded sets $A_k \subset \mathbb{R}^2$, along with Lipschitz functions $f_k : A_k \to \mathbb{R}^3$, $k \geq 1$, such that

$$H^{(2)}(S \setminus \bigcup_{k=1}^{\infty} f_k(A_k)) = 0.$$

Here, and throughout the paper, $H^{(2)}$ denotes the 2 dimensional Hausdorff measure.

Remark 2.4. It is well known that when $S \subset \mathbb{R}^3$ is 2-rectifiable and $H^{(2)}$-measurable, at $H^{(2)}$-almost every $x \in S$, S possesses an approximate tangent plane. Because we are in $\mathbb{R}^3$, this is equivalent to saying that at almost every $x \in S$ there is a vector ν_x normal to S which is $H^{(2)}$-measurable (cf. 3.2.19 and 3.2.25 of [4]).

Definition 2.5. Given a 1-current $T \in \mathcal{I}(\Omega)$ and a 2-rectifiable, $H^{(2)}$-measurable set $S \subset \Omega$, we say that $T = \partial S$ (relative to Ω) if, for every $B \in C_0^\infty(\Omega; \mathbb{R}^3)$, we have the identity

$$T(B) = \int_S \langle \nu, \nabla \times B \rangle \, dH^{(2)}.$$

Here ν denotes the normal vector to S, and $\nabla \times B$ the curl of B.

With these definitions in hand, we can now define the flat metric: for $T_1, T_2 \in \mathcal{I}(\Omega)$, define

$$d_\flat(T_1, T_2) = \inf\{H^{(2)}(\Sigma) : \Sigma \subset \Omega,\ \text{2-rectifiable},\ H^{(2)}\text{-measurable such that } \partial\Sigma = T_1 - T_2\}. \tag{2.11}$$

We define $d_\flat(T_1, T_2) = \infty$ if the set on the right-hand side of (2.11) is empty. We study the local minima of

$$E_0(T) = M_\rho(T) + T(B_0) \tag{2.12}$$

in the sense of $d_\flat$, where B_0 is the vector field from (H2). Note that when a current $T \in \mathcal{I}(\Omega)$ is comprised by a single Lipschitz curve $\gamma :]a, b[\to \Omega$, this definition becomes

$$E_0(\gamma) = L_\rho(\gamma) + \int_a^b \langle B(\gamma), \gamma' \rangle \, dt, \tag{2.13}$$

where L_ρ denotes weighted length

$$L_\rho(\gamma) = \int_a^b \rho(\gamma) \, |\gamma'| \, .$$

This is the form of E_0 that we used in the discussion of the introduction. Note that the terms in the sum of the right-hand side of (2.9) can be interpreted as the weighted lengths of the irreducible components of T, mentioned in Definition 2.2.

Recall now that the first variation of E_0 at the curve γ in the direction of a smooth vector field $Y \in C_0^\infty(]a, b[, \mathbb{R}^3)$, which we denote by $Y_* E_0(\gamma)$, is defined by the formula

$$Y_* E_0(\gamma) = \left.\frac{d}{d\xi}\right|_{\xi=0} E_0(\gamma + \xi Y).$$

Note that for sufficiently small ξ, $\gamma + \xi Y$ maps the (compact) support of Y to a compact subset of Ω that lies at a positive distance from $\partial\Omega$. Thus for ξ small enough, $t \mapsto \gamma(t) + \xi Y(t)$ is a curve lying in Ω and is in the domain of E_0.

Definition 2.6. We say $T \in \mathcal{I}(\Omega)$ is a *critical point of* E_0 *with respect to interior variations* if each irreducible component $\gamma :]a, b[\to \Omega$ of T has

$$Y_* E_0(\gamma) = 0 \tag{2.14}$$

for all $Y \in C_0^\infty(]a, b[, \mathbb{R}^3)$.

Remark 2.7. We remark that if $T \in \mathcal{I}(\Omega)$ is a local minimum of E_0 with respect to $d_\flat$, then it must be a critical point in the above sense. This is because as $\xi \to 0$, and for every γ irreducible component of T, the curves $\gamma_\xi = \gamma + \xi Y$ may be connected to γ by ruled surfaces of arbitarily small area.

3. Euler-Lagrange equation and finiteness of length

In this section we show how weighted length controls length for critical curves of E_0, proving (1.1) of Theorem 1.1. The key idea is to vary large subarcs of such a curve in the direction of $\nabla\rho$.

Consider an irreducible component Γ of J_0, a Lipschitz map with $\Gamma : I \to \mathbb{R}^3$, where I is a possibly unbounded interval, and $L_\rho(\Gamma)$ finite. Pick $t_0 \in I$ and consider the *signed arc-length* of γ,

$$s(t) = \int_{t_0}^{t} |\Gamma'| \, (\bar{t}) \, d\bar{t}. \tag{3.1}$$

Let $\gamma : I' \to \mathbb{R}^3$ be the reparameterized curve, $\gamma(s(t)) = \Gamma(t)$. Set $s_a = \lim_{t \to a^+} s(t)$ and $s_b = \lim_{t \to b^-} s(t)$ so that $I' =]s_a, s_b[$. Note that I' may be unbounded.

As in Section 2, we consider a vector field $V \in C_0^\infty(I'; \mathbb{R}^3)$ and a family of curves

$$\gamma_\xi(s) = \gamma(s) + \xi V(s).$$

A standard computation shows that

$$V_* E_0(\gamma) = \int_{s_a}^{s_b} \left\{ \langle V, \nabla\rho \rangle + \rho \langle \frac{dV}{ds}, \frac{d\gamma}{ds} \rangle + \langle (DB_0)V, \frac{d\gamma}{ds} \rangle + \langle B_0, \frac{dV}{ds} \rangle \right\}.$$

Integrating the last term by parts we obtain

$$V_* E_0(\gamma) = \int_{s_a}^{s_b} \left\{ \rho \langle \frac{dV}{ds}, \frac{d\gamma}{ds} \rangle + \langle V, \nabla\rho + (DB_0^T - DB_0) \frac{d\gamma}{ds} \rangle \right\}. \tag{3.2}$$

Thus the condition $V_* E_0(\gamma) = 0$ becomes the weak ODE

$$\frac{d}{ds} \left\{ \rho(\gamma) \frac{d\gamma}{ds} \right\} = \nabla\rho + (DB_0^T - DB_0) \frac{d\gamma}{ds} \tag{3.3}$$

for γ. By standard regularity for ODEs (see for instance chapter VIII of [2]) we obtain that $\gamma \in C^{1,1}_{loc}(]s_a, s_b[)$. It is a simple matter to check that the last equation for γ in dimension 3 can be re-written as

$$\frac{d}{ds} \left\{ \rho(\gamma) \frac{d\gamma}{ds} \right\} = \nabla\rho + \frac{d\gamma}{ds} \times \nabla \times B_0. \tag{3.4}$$

We are now in a position to prove (1.1). To do this, we will establish that for every integer $j \in \{0, ..., k-1\}$, we have

$$\int_{s_a}^{s_b} \omega^{k-j-1} \leq A_j \int_{s_a}^{s_b} \omega^{k-j}, \tag{3.5}$$

for some constant A_j that depends only on ρ and B_0. Note that this reduces the power in ρ successively from k all the way down to 0.

We establish (3.5) by varying a piece of γ in direction $\nabla\omega$. In ODE terms this means taking the inner product of (3.3) with the vector field $z(s) = \nabla\omega(\gamma(s))$ and then integrating on $[s_0, s_1] \subset I'$:

$$-\int_{s_0}^{s_1} \rho(\gamma)\langle\frac{d\gamma}{ds}, \frac{dz}{ds}\rangle \quad = \quad \int_{s_0}^{s_1} \langle z, \nabla\rho + \frac{d\gamma}{ds} \times \nabla \times B_0\rangle - \rho(\gamma)\langle\frac{d\gamma}{ds}, z\rangle\Big|_{s_0}^{s_1}. \tag{3.6}$$

Recalling hypothesis (H1), we obtain from (3.6) that

$$\begin{aligned} k\int_{s_0}^{s_1} \omega^{k-1}\left|\nabla\omega\right|^2(\gamma) \quad &= \quad -\int_{s_0}^{s_1}\left\{\rho(\gamma)\langle\frac{d\gamma}{ds}, D^2\omega\frac{d\gamma}{ds}\rangle + \langle(\nabla\omega)(\gamma), \frac{d\gamma}{ds} \times \nabla \times B_0\rangle\right\} \\ &\quad + \quad \rho(\gamma)\langle\frac{d\gamma}{ds}, (\nabla\omega)(\gamma)\rangle\Big|_{s_0}^{s_1}. \end{aligned} \tag{3.7}$$

Now by Lemma 2.1, γ falls into one of two cases:

(i) It is a closed loop. In this case, its image lies a positive distance away from $\partial\Omega$. So $\rho(\gamma(t)) > c > 0$ and $L(\gamma) < c^{-1}L_\rho(\gamma)$. We may reparametrize γ by arclength to get a curve $\gamma : I' \to \mathbb{R}^3$ on the finite interval $I' = [s_a, s_b]$. Moreover, we must have γ' agreeing at the endpoint of I'. (To see this we could reparametrize γ under a cyclic shift and use the above ODE regularity again.)

(ii) It is "boundary to boundary", and we get

$$\lim_{s\to s_a} \omega(\gamma(s)) = \lim_{s\to s_b} \omega(\gamma(s)) = 0.$$

Then, taking the limits as $s_0 \to s_a^+$ and $s_1 \to s_b^-$, the boundary terms in (3.7) vanish and we obtain

$$k\int_{s_a}^{s_b} \omega^{k-1}\left|\nabla\omega\right|^2(\gamma) \leq \left\{\left|\left|D^2\omega\right|\right|_\infty + ||\nabla\omega||_\infty\,||(\nabla \times B_0)/\rho||_\infty\right\}\int_{s_a}^{s_b} \rho(\gamma).$$

We now recall hypothesis (H1) (equation (2.1)) and find

$$\int_{s_a}^{s_b}\left\{\omega^{k-1}\left|\nabla\omega\right|^2(\gamma) + \omega^k\right\} \geq m\int_{s_a}^{s_b} \omega^{k-1}.$$

These last two estimates imply that

$$\int_{s_a}^{s_b} \omega^{k-1} \leq \frac{1}{mk}\left\{k + \left|\left|D^2\omega\right|\right|_\infty + ||\nabla\omega||_\infty\,||(\nabla \times B_0)/\rho||_\infty\right\}\int_{s_a}^{s_b} \rho(\gamma) \tag{3.8}$$

$$= A_1 L_\rho(\gamma), \tag{3.9}$$

where A_1 is a constant that depends on k, ρ and B_0. We have established (3.5) for $j = 0$.

We now iterate this procedure as follows: choose $0 \leq j \leq k-1$ and set

$$z(s) = \omega^{-j}(\gamma)(\nabla\omega)(\gamma(s)) \tag{3.10}$$

in (3.6). After some re-arrangement, one obtains

$$\begin{aligned}\int_{s_0}^{s_1} \omega^{k-1-j} \left\{ k \left|\nabla\omega\right|^2(\gamma) - j\left(\nabla\omega \cdot \frac{d\gamma}{ds}\right)^2 \right\} &= \int_{s_0}^{s_1} \omega^{-j}(\gamma)\langle \frac{d\gamma}{ds}, \nabla\omega \times \nabla \times B_0\rangle \\ &- \int_{s_0}^{s_1} \omega^{k-j}\langle D^2\omega \frac{d\gamma}{ds}, \frac{d\gamma}{ds}\rangle \\ &- \omega^{k-j}(\gamma)\langle \frac{d\gamma}{ds}, D^2\omega \frac{d\gamma}{ds}\rangle\Big|_{s_0}^{s_1}. \qquad (3.11)\end{aligned}$$

Here we now recall that $j \in \{0, ..., k-1\}$, that $\left|\frac{d\gamma}{ds}\right| = 1$, that $\left|D^2\omega\right|$ is bounded, and that (2.3) implies $|\nabla \times B_0| \leq M_3\rho = M_3\omega^k$. We let $s_0 \to s_a^+$ and $s_1 \to s_b^-$ to obtain (3.5). A straightforward induction argument now shows that for $\rho = \omega^k$

$$L(\gamma) \leq A_k L_\rho(\gamma),$$

where again A_k depends on ρ, B_0 and k.

4. Behavior near boundary

In this section we want to determine the behavior near $\partial\Omega$ of a boundary-to-boundary critical curve γ. In order to do this, we note that, by the previous section, γ has finite length, and in the notation of the previous section, both the limits

$$\lim_{s\to s_a^+} \gamma(s) \text{ and } \lim_{s\to s_b^-} \gamma(s)$$

exist and in fact belong to $\partial\Omega$. We show next that whenever this happens, γ meets $\partial\Omega$ perpendicularly at these points. Note that we can choose $s_a = 0$ and $s_b \in]0, \infty[$, which we do from now on. Choose $s \in]0, s_b[$, and integrate the equation (3.4) over $]0, s[$. Because $\omega = 0$ on $\partial\Omega$ and $|\gamma'| \equiv 1$, we obtain

$$\omega^k \gamma' = k \int_0^s \omega^{k-1} \nabla\omega \, d\xi + \int_0^s T_B \gamma' \, d\xi.$$

Here we use the notation $T_B = DB^T - DB$, and abbreviate $\omega(\gamma(s))$, $(\nabla\omega)(\gamma(s))$, etc., by $\omega(s)$, and so on. This implies that

$$\begin{aligned}\omega^k \gamma' &= k \int_0^s \omega^{k-1}(\xi) \, d\xi (\nabla\omega)(s) \\ &+ k \int_0^s \omega^{k-1}(\xi)((\nabla\omega)(\xi) - (\nabla\omega)(s)) \, d\xi + \int_0^s T_B \gamma' \, d\xi. \qquad (4.1)\end{aligned}$$

Next, note that

$$|(\nabla\omega)(\xi) - (\nabla\omega)(s)| \leq \sup_{x\in\Omega} \left|(D^2\omega)(x)\right| s$$

and

$$|T_B(\xi)| \leq C\omega^k(\xi) \leq C\, s\, \omega^{k-1}(\xi),$$

both for all $\xi \in]0, s[$. Denoting the orthogonal projection onto the vector space orthogonal to γ' by $\pi_{\gamma'^\perp}$, we obtain from (4.1) that

$$\begin{aligned}\pi_{\gamma'^\perp} \nabla\omega(s) &= \frac{-1}{\int_0^s \omega^{k-1} \, d\xi} \pi_{\gamma'^\perp} \int_0^s \omega^{k-1}(\xi)((\nabla\omega)(\xi) - (\nabla\omega)(s)) \, d\xi \\ &+ \frac{-1}{k \int_0^s \omega^{k-1} \, d\xi} \pi_{\gamma'^\perp} \int_0^s T_B \gamma'. \qquad (4.2)\end{aligned}$$

From this last identity we obtain

$$\left|\pi_{\gamma'^{\perp}}(\nabla\omega)(s)\right| \le C\, s, \tag{4.3}$$

so that as $s \to 0$, $\pi_{\gamma'^{\perp}}(\nabla\omega)(s) \to 0$. This shows that γ' and $\nabla\omega$ become parallel as $s \to 0$ (recall that $\nabla\omega \neq 0$ near $\partial\Omega$). Since $\omega = 0$ on $\partial\Omega$ and ω is smooth, $\nabla\omega$ on $\partial\Omega$ is parallel to the normal of this last set. This shows that whenever γ meets $\partial\Omega$, it does so at a 90° angle.

4.1. **Curvature.** We show next that the curvature of γ is bounded. To do this we note that, by (H1) and (2.4), we can always find $\mu > 0$ so that $|\nabla\omega| \ge \mu$ in the set

$$\Omega_\mu = \{x \in \Omega : \mathrm{dist}(x;\partial\Omega) \le \mu\}.$$

Then, for $s^* > 0$ such that $0 < \max\{C, 1\}s^* \le \mu/2$, where $C > 0$ is the constant from (4.3) and all $s \in]0, s^*[$, we have

$$\begin{aligned}\left|(I-\pi_{\gamma'^{\perp}})\nabla\omega\right|(s) &= \left(|\nabla\omega|^2 - \left|\pi_{\gamma'^{\perp}}(\nabla\omega)(s)\right|^2\right)^{1/2}\\ &\ge \left(\mu^2-\mu^2/4\right)^{1/2} \ge \frac{\sqrt{3}}{2}\mu.\end{aligned} \tag{4.4}$$

In particular, $(\nabla\omega)(s)\cdot\gamma'(s) \ge \sqrt{3}\mu/2$ for all $s \in]0, s^*[$, and then

$$\omega(s) = \int_0^s (\nabla\omega)(\xi)\cdot\gamma'(\xi)\,d\xi \ge \frac{\sqrt{3}\mu}{2}s. \tag{4.5}$$

We note next that (3.4) can be re-written as

$$\gamma'' = \frac{k}{\omega}\pi_{\gamma'^{\perp}}\nabla\omega + \omega^{-k}T_B\gamma'. \tag{4.6}$$

Recalling that $T_B = DB^T - DB$, we apply (2.3), (4.3) and (4.5) in this last identity to obtain

$$|\gamma''| \le C,$$

for all $s \in]0, s^*[$. The same argument applies near $s_1 > 0$. Finally, we may assume that away from the endpoints of $]0, s_1[$, γ remains away from $\partial\Omega$, for otherwise we just shrink the interval $]0, s_1[$ enough to satisfy this property. This shows that $|\gamma''|$ is uniformly bounded in $]0, s_1[$, which implies that γ has bounded curvature. This concludes the proof of the theorem.

5. Finiteness

In this section we prove Theorems 1.2 and 1.3. Throughout this section we assume that $\partial\Omega$ is path-connected.

Lemma 5.1. *If γ is a critical point of E_0 in the sense of Definition* 2.6*, then either*

$$L_\rho(\gamma) \ge L_{min} := \left(||DB||_{L^\infty}\, C_{iso}(\Omega)A_k^2\right)^{-1}$$

or $E_0(\gamma) > 0$.

Proof. Let $\gamma \in \mathcal{I}(\Omega)$ be a critical point of E_0. The assumption that $\partial\Omega$ is path-connected allows us to appeal to the Isoperimetric Inequality (for instance 4.5.14 of [4] will do) to obtain a 2-rectifiable set Σ with $\partial\Sigma = \gamma$ relative to Ω and with

$$H^{(2)}(\Sigma) \le C_{iso}(\Omega)L(\gamma)^2.$$

We may use Stokes' theorem in the second term of E_0 to estimate the energy as follows:

$$E_0(\gamma) \geq L_\rho(\gamma) - ||DB||_{L^\infty} C_{iso}(\Omega) L(\gamma)^2.$$

But then we apply Theorem 1.1 to find

$$E_0(\gamma) \geq L_\rho(\gamma) - ||DB||_{L^\infty} C_{iso}(\Omega) A_k^2 L_\rho(\gamma)^2. \tag{5.1}$$

So we see that if

$$L_\rho(\gamma) < L_{min} := \left(||DB||_{L^\infty} C_{iso}(\Omega) A_k^2\right)^{-1},$$

then (5.1) reveals that γ contributes positive energy. □

We now prove our last two theorems.

Proof of Theorem 1.2. Let $\epsilon > 0$. Because $L_\rho(J)$ is finite we know that if J has infinitely many irreducible components, we may find one γ^* with weighted length less than ϵ. Because J is a local minimum with respect to the flat metric, we know γ^* is a critical point in the sense of Definition 2.6 (see the remark after the definition). By the Isoperimetric Inequality, there is a 2-rectifiable set Σ with $\partial\Sigma = \gamma$ and

$$H^{(2)}(\Sigma) < C_{iso}(\Omega) L(\gamma)^2 < C_{iso}(\Omega) A_k \epsilon^2.$$

By Lemma 5.1, for small enough ϵ, we have $E_0(\gamma^*) > 0$, so $J - \gamma$ has less energy. Also, by definition, $d_\flat(J - \gamma, J) \leq H^{(2)}(\Sigma)$. So taking ϵ to zero we find rectifiable 1-currents which are arbitrarily close to J in the flat metric and have strictly less energy. This contradiction proves the theorem. □

Proof of Theorem 1.3. Claim (i) follows from Lemma 5.1. Claim (ii) follows from the fact that every component has non-negative weighted length, and if there were more than the claimed number, this would violate Lemma 5.1. □

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