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ON THE STABILITY OF SETS FOR DELAYED KOLMOGOROV-TYPE SYSTEMS

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Abstract. In this paper we consider Kolmogorov-type delay systems. Criteria on the uniform global asymptotic stability of sets are established for the above systems using Lyapunov functions and the Razumikhin technique.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|\,.\,|$ and distance $d(.,.)$, and let $\mathbb{R}_+ = [0,\infty)$. According to [2] and [16] for a given $h > 0$ we denote by $\mathcal{C}$ the space of continuous functions mapping $[-h,0]$ into $\mathbb{R}^n$, and for $\varphi \in \mathcal{C}$, $||\varphi|| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$.

In this paper we consider the n-dimensional Kolmogorov-type nonautonomous functional differential equations

$$\dot{x}_i(t) = x_i(t) f_i(t, x_i(t), x_t),\ i = 1, 2, ..., n, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$, $f = (f_1, f_2, ..., f_n)$, and $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t+\theta)$ for $-h \leq \theta \leq 0$.

The most efficient tool for the study of the stability of a given nonlinear system is provided by Lyapunov's second method. Its application to systems with delay has been developed in two main directions: the Lyapunov-Razumikhin technique ([7], [10], [14], [15]) and the Lyapunov-Krasovskii functional method ([2], [7], [9], [18]).

The study of the asymptotic stability of solutions of type (1.1) models has been extensively investigated and developed. Faria [4] gives sufficient conditions for the global asymptotic stability of the scalar Kolmogorov-type delay differential equation, without assuming that zero is a solution. By applying the basic theory of the Lyapunov functional method, Liu [9] established criteria for the global stability of the positive equilibriums of a class of systems of nonautonomous delay differential equations. As some applications, the global stability results for a nonautonomous Nicholson blowflies equation with patch structure and for a nonautonomous delay logistic equation with patch structure have been obtained. Using the properties of the characteristic equations of the corresponding linearized systems, some absolute stability, conditional stability, and bifurcation results for three classes of more general Kolmogorov-type predator-prey models with discrete delay are given by Ruan [11]. For more stability results, see also [17] and the references therein.

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It has been shown that many of the problems encountered in the study of stability and persistence of functional differential equations can be solved by using the Razumikhin technique ([6], [7], [10], [14], [15]). However, few authors applied the Lyapunov-Razumikhin function method to study stability and boundedness of the model (1.1) (see [3] and [5]). The first motivation for this paper is that the results obtained in such a manner can be applied more easily in comparison with the method of Lyapunov-Krasovskii functionals. By this technique (see [6] and [7]), one only needs to choose a Lyapunov function (instead of a Lyapunov functional) and to verify the nonpositivity of the derivative function for some initial data (instead of all initial data) under certain restrictions in order to have stability.

The second motivation comes from the question: How far can initial conditions be allowed to vary without disrupting the stability properties established in the immediate vicinity of equilibrium states? On this problem, Hale and Lunel [7] studied the stable set (or manifold) for delay differential equations. The notion of stability of sets, which includes as a special case stability of a solution, stability of invariant sets, stability of moving manifolds, etc., is one of the most important notions in stability theory. The stability of sets with respect to systems of ordinary differential equations without impulses has been considered by Yoshizawa in [18]. Similar ideas for different types of systems are presented in [12], [13], [15].

In the present paper the problem of global stability of sets of a sufficiently general type contained in some domain with respect to system (1.1) is considered by means of the Lyapunov-Razumikhin method. Some earlier results are generalized and improved.

The remaining part of this paper is organized as follows. In Section 2 we give some basic definitions and preliminary results. In Section 3, by applying the Lyapunov-Razumikhin technique, we establish several criteria of the global stability of sets with respect to system (1.1). In Section 4 we make some applications to Lotka-Volterra delay systems and to an economic model of price fluctuations.

2. Preliminaries

Let $\mathbb{R}^n_+ = \{y = (y_1, ..., y_n) \in \mathbb{R}^n : \ y_i \geq 0, \ i = 1, ..., n\}$ and $\mathcal{C}_+ = C([-h, 0], \mathbb{R}^n_+)$. Let $t_0 \geq 0$ and $\phi \in \mathcal{C}_+$, $\phi = (\phi_1, ..., \phi_n)$. Denote by $x(t) = x(t; t_0, \phi)$ the solution of system (1.1) with initial condition

$$x_{t_0} = \phi, \tag{2.1}$$

and by $J^+(t_0, \phi)$ the maximal interval of the type $[t_0, \beta)$ in which the solution $x(t; t_0, \phi)$ is defined.

Suppose that f is smooth enough and is Lipschitz continuous with respect to its second and third arguments to guarantee existence, uniqueness, and continuous dependence of solutions of system (1.1) for $t \geq t_0$. Moreover, we assume that solutions of (1.1) with initial conditions (2.1) are nonnegative, and if $\phi_i > 0$ for some i, then $x_i(t) > 0$ for all $t \geq t_0$. Note that this assumption is natural from the biological point of view.

Let $M \subset [t_0 - h, \infty) \times \mathbb{R}^n_+$. We shall use the following notation:

$M(t) = \big\{x \in \mathbb{R}^n_+ : (t, x) \in M, \ t \in [t_0, \infty)\big\}$;

$M_0(t) = \big\{x \in \mathbb{R}^n_+ : \ (t, x) \in M, \ t \in [t_0 - h, t_0]\big\}$;

$d(x, M(t)) = \inf\limits_{y \in M(t)} |x - y|$ is the distance between $x \in \mathbb{R}^n_+$ and $M(t)$;

$M(t, \varepsilon) = \big\{x \in \mathbb{R}^n_+ : \ d(x, M(t)) < \varepsilon\big\}$ $(\varepsilon > 0)$ is an ε-neighborhood of $M(t)$;

$d_0(\phi, M_0(t)) = \sup_{t\in[t_0-h,t_0]} d(\phi(t-t_0), M_0(t)),\ \phi \in \mathcal{C}_+;$
$M_0(t,\varepsilon) = \{\phi \in \mathcal{C}_+ :\ d_0(\phi, M_0(t)) < \varepsilon\}$ is an ε-neighborhood of $M_0(t)$;
$\overline{S_\alpha} = \{x \in \mathbb{R}^n_+ :\ |x| \le \alpha\}$; $\overline{S_\alpha}(\mathcal{C}_+) = \{\phi \in \mathcal{C}_+ :\ ||\phi|| \le \alpha\}$.

We introduce the following assumptions:
A1. $M(t) \neq \emptyset$ for $t \in [t_0, \infty)$.
A2. $M_0(t) \neq \emptyset$ for $t \in [t_0 - h, t_0]$.
A3. For any compact subset F of $[t_0, \infty) \times \mathbb{R}^n_+$ there exists a constant $K > 0$ depending on F such that if (t, x), $(t', x) \in F$, then the following inequality is valid:

$$|d(x, M(t)) - d(x, M(t'))| \le K|t - t'|.$$

We shall use the following definitions:

Definition 2.1. The solutions of system (1.1) are said to be *uniformly M-bounded* if

$$(\forall\eta > 0)(\exists\beta = \beta(\eta) > 0)(\forall t_0 \in \mathbb{R}_+)(\forall\alpha > 0),$$
$$(\forall\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t,\eta)})(\forall t \ge t_0) :\ x(t; t_0, \phi) \in M(t, \beta).$$

Definition 2.2. The set M is said to be:
(a) *stable* with respect to system (1.1) if

$$(\forall t_0 \in \mathbb{R}_+)(\forall\alpha > 0)(\forall\varepsilon > 0)(\exists\delta = \delta(t_0, \alpha, \varepsilon) > 0)$$
$$(\forall\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap M_0(t, \delta))(\forall t \ge t_0) :$$
$$x(t; t_0, \phi) \in M(t, \varepsilon);$$

(b) *uniformly stable* with respect to system (1.1) if the number δ from point (a) depends only on ε;
(c) *uniformly globally attractive* with respect to system (1.1) if

$$(\forall\eta > 0)(\forall\varepsilon > 0)(\exists T = T(\eta, \varepsilon) > 0)$$
$$(\forall t_0 \in \mathbb{R}_+)(\forall\alpha > 0)(\forall\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t,\eta)})$$
$$(\forall t \ge t_0 + T) :\ x(t; t_0, \phi) \in M(t, \varepsilon);$$

(d) *uniformly globally asymptotically stable* with respect to system (1.1) if M is a uniformly stable and uniformly globally attractive set of system (1.1) and if the solutions of system (1.1) are uniformly M-bounded.

For a specific choice of the set M, point (d) of Definition 2.2 is reduced to the following particular cases:
1) Lyapunov uniform global asymptotic stability of the zero solution of (1.1) if

$$M = [t_0 - h, \infty) \times \{x \in \mathbb{R}^n_+ : x_i \equiv 0,\ i = 1, ..., n\}.$$

2) Lyapunov uniform global asymptotic stability of a nonnull solution $x^*(t)$, $x^* = (x^*_1, ..., x^*_n)$ of (1.1), if

$$M = [t_0 - h, \infty) \times \{x \in \mathbb{R}^n_+ : x_i \equiv x^*_i,\ i = 1, ..., n\}.$$

3) Uniform global asymptotic stability of an invariant set M with respect to system (1.1) if $t_0 \in \mathbb{R}_+$, $\phi \in M_0(t)$ implies $x(t; t_0, \phi) \in M(t)$ for each $t \ge t_0$.
4) Uniform global asymptotic stability of conditionally invariant set B with respect to a set A, where $A \subset B \subset \mathbb{R}^n_+$, if $M(t) = B$ for $t \ge t_0$ and $M_0(t) = A$ for $t \in [t_0 - h, t_0]$.

5) Uniform global asymptotic conditional stability of system (1.1) with respect to an $(n-l)$-dimensional manifold $M(n-l)$ $(l < n)$ if $M_0(t) = M_0(n-l) = \{\phi(t-t_0) : \phi \in C[[t_0 - h, t_0], M(n-l)]\}$ and $M(t) = M(n-l)$ for $t \geq t_0$.

Definition 2.3. A function $V : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ *belongs to the class* C_0 if V is continuous in $\mathbb{R}_+ \times \mathbb{R}^n_+$ and locally Lipschitz continuous with respect to its second argument x.

If $V \in C_0$, then the *upper right-hand derivative of* V *with respect to system* (1.1) is defined by ([6])

$$D^+V(t, \phi(0)) = \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma}[V(t+\sigma, x(t+\sigma; t_0, \phi)) - V(t, \phi(0))],$$

where $(t, \phi) \in \mathbb{R}_+ \times \mathcal{C}$.

In the proof of the main results we shall use the following lemma.

Lemma 2.1 ([8])**.** *Assume that:*

1. The function $g : [t_0, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous in* $[t_0, \infty) \times \mathbb{R}_+$.

2. The maximal solution $u^+(t; t_0, u_0)$ *of the scalar problem*

$$\begin{cases} \dot{u}(t) = g(t, u(t)), & t > t_0, \\ u(t_0) = u_0 \geq 0 \end{cases}$$

is defined on the interval $[t_0, \infty)$.

3. The function $V \in C_0$ *is such that* $\sup\limits_{\theta \in [-h, 0]} V(t_0 + \theta, \phi(\theta)) \leq u_0$ *and*

$$D^+V(t, \phi(0)) \leq g(t, V(t, \phi(0))) \ \ \textit{if} \ \ V(t+\theta, \phi(\theta)) \leq V(t, \phi(0)),$$

$\theta \in [-h, 0]$ *for all* $t \geq t_0$, $\phi \in \mathcal{C}$.

Then

$$V(t, x(t; t_0, \phi)) \leq u^+(t; t_0, u_0), \quad t \in [t_0, \infty).$$

In the case when $g(t, u) = 0$ for $(t, u) \in [t_0, \infty) \times \mathbb{R}_+$, we deduce the following corollary from Lemma 2.1.

Corollary 2.1. *Assume that the function* $V \in C_0$ *is such that*

$$D^+V(t, \phi(0)) \leq 0 \ \ \textit{if} \ \ V(t+\theta, \phi(\theta)) \leq V(t, \phi(0)),$$

$\theta \in [-h, 0]$ *for all* $t \geq t_0$, $\phi \in \mathcal{C}$.

Then

$$V(t, x(t; t_0, \phi)) \leq V(t_0, \phi(0)), \quad t \in [t_0, \infty).$$

We shall use the following class of functions:

$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] :\ a(r)$ is strictly increasing and $a(0) = 0\}$.

Let $\lambda : [t_0, \infty) \to \mathbb{R}_+$ be a measurable function. Then, we say that $\lambda(t)$ is *integrally positive* if

$$\int_J \lambda(t)dt = \infty$$

whenever $J = \bigcup\limits_{k=1}^{\infty}[\alpha_k, \beta_k]$, $\alpha_k < \beta_k < \alpha_{k+1}$, and $\beta_k - \alpha_k \geq \Delta > 0$, $k = 1, 2, \ldots$.

3. Stability of sets

In the further considerations, we shall use piecewise continuous auxiliary functions $V : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$, which belong to the class C_0 and are such that the following condition holds:

A4. $V(t,x) = 0$ for $(t,x) \in M$, $t \geq t_0$k, and $V(t,x) > 0$ for $(t,x) \in \{[t_0,\infty) \times \mathbb{R}_+\} \setminus M$.

Theorem 3.1. *Assume that:*

1. Conditions A1-A3 hold.

2. There exists a function $V \in C_0$ such that A4 holds:

$$a(d(x,M(t))) \leq V(t,x) \leq b(d(x,M(t))),\ a,b \in K,\ (t,x) \in [t_0,\infty) \times \mathbb{R}^n_+, \tag{3.1}$$

where $a(u) \to \infty$ as $u \to \infty$, and

$$D^+V(t,\phi(0)) \leq -q(t)c(d(\phi(0),M(t))) \text{ if } V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)), \tag{3.2}$$

$\theta \in [-h,0]$ for all $t \geq t_0$, $\phi \in \mathcal{C}$, where, $q : [t_0,\infty) \to (0,\infty)$, $c \in K$.

3. $\displaystyle\int_0^\infty q(s)c[b^{-1}(\eta)]ds = \infty$ for each sufficiently small value of $\eta > 0$.

Then the set M is uniformly globally asymptotically stable with respect to system (1.1).

Proof. Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon) > 0$, $\delta < \varepsilon$ so that $b(\delta) < a(\epsilon)$.

Let $\alpha > 0$ be arbitrary, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap M_0(t,\delta)$ and $x(t) = x(t;t_0,\phi)$ be the solution of (1.1) through (t_0,ϕ).

Since all the conditions of Corollary 2.1 are met, then from (3.1) we get

$$\begin{aligned} a(d(x(t;t_0,\phi),M(t))) &\leq V(t,x(t)) \leq V(t_0,\phi(0)) \\ &\leq b(d(\phi(0),M_0(t_0))) \leq b(d_0(\phi,M_0(t))) < b(\delta) < a(\varepsilon),\ t \geq t_0; \end{aligned}$$

hence the set M is uniformly stable with respect to system (1.1).

Now let $\eta > 0$ and $\varepsilon > 0$ be given and let the number $T = T(\eta,\varepsilon) > 0$ be chosen so that

$$\int_{t_0}^{t_0+T} q(s)c[b^{-1}(a(\varepsilon))]ds > b(\eta). \tag{3.3}$$

(This is possible in view of condition 3 of Theorem 3.1.)

Let $\alpha > 0$ be arbitrary, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t,\eta)}$ and $x(t) = x(t;t_0,\phi)$. We want to show that there exists $t^* \in [t_0,t_0+T]$, such that

$$V(t^*,x(t^*)) < a(\epsilon). \tag{3.4}$$

Suppose the opposite: for any $t \in [t_0,t_0+T]$ the inequality $V(t,x(t)) \geq a(\epsilon)$ holds, whence we have

$$d(x(t),M(t)) \geq b^{-1}(V(t,x(t))) \geq b^{-1}(a(\varepsilon)) \tag{3.5}$$

for $t_0 \leq t \leq t_0 + T$. Then, by (3.2), it follows that

$$\int_{t_0}^{t_0+T} D^+V(s,x(s))ds \leq -\int_{t_0}^{t_0+T} q(s)c[b^{-1}(a(\varepsilon))]ds < -b(\eta). \tag{3.6}$$

On the other hand,

$$\int_{t_0}^{t_0+T} D^+V(s,x(s))ds \geq V(t_0+T,x(t_0+T)) - V(t_0,\phi(0)),$$

whence, in view of (3.6), it follows that $V(t_0+T, x(t_0+T)) < 0$, which contradicts (3.1). This proves the existence of $t^* \in [t_0, t_0+T]$ such that inequality (3.4) is valid. Since V does not increase along the solution $x(t) = x(t; t_0, \phi)$, then $V(t, x(t)) < a(\epsilon)$ for $t \geq t^*$ (hence for any $t \geq t_0 + T$ as well). This implies, $x(t) \in M(t, \epsilon)$ for $t \geq t_0 + T$. Hence, the set M is uniformly globally attractive with respect to system (1.1).

Finally we shall prove that the solutions of system (1.1) are uniformly M-bounded.

Let $\eta > 0$ and let $\beta = \beta(\eta) > 0$ be such that $a(\beta) > b(\eta)$.

Choose arbitrary $\alpha > 0$, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \bigcap \overline{M_0(t, \eta)}$ and let $x(t) = x(t; t_0, \phi)$. Then for $t \geq t_0$ the following inequalities are valid:

$$a(d(x(t), M(t))) \leq V(t, x(t)) \leq V(t_0, \phi(0)) \leq b(d(\phi(0), M_0(t_0)))$$

$$\leq b(d_0(\phi, M_0(t))) \leq b(\eta) < a(\beta).$$

Hence $x(t) \in M(t, \beta)$ for $t \geq t_0$.

Theorem 3.2. *Assume that:*

1. *Condition 1 of Theorem* 3.1 *holds.*
2. *There exists a function $V \in C_0$ such that A4 and* (3.1) *hold, and*

$$D^+V(t, \phi(0)) \leq -\lambda(t)c(d(\phi(0), M(t))) \text{ if } V(t+\theta, \phi(\theta)) \leq V(t, \phi(0)), \tag{3.7}$$

$\theta \in [-h, 0]$ *for all $t \geq t_0$, $\phi \in \mathcal{C}$, where $c \in K$ and $\lambda(t)$ is an integrally positive function.*

Then the set M is uniformly globally asymptotically stable with respect to system (1.1).

Proof. The fact that the set M is uniformly stable with respect to system (1.1) and the uniform M-boundedness of the solutions of system (1.1) are proved as in the proof of Theorem 3.1.

Now, we shall prove that the set M is uniformly globally attractive with respect to system (1.1).

Again let $\varepsilon > 0$ and $\eta > 0$ be given. Choose the number $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$. We shall prove that there exists $T = T(\varepsilon, \eta) > 0$ such that for any solution $x(t) = x(t; t_0, \phi)$ of system (1.1) for which $t_0 \in \mathbb{R}_+$, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t, \eta)}$ ($\alpha > 0$ - arbitrary) and for any $t^* \in [t_0, t_0+T]$ the following inequality is valid:

$$d(x(t^*), M(t^*)) < \delta(\varepsilon). \tag{3.8}$$

Suppose that this is not true. Then, for any $T > 0$ there exists a solution $x(t) = x(t; t_0, \phi)$ of system (1.1) for which $t_0 \in \mathbb{R}_+$, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t, \eta)}$, $\alpha > 0$, such that

$$d(x(t), M(t)) \geq \delta(\varepsilon),\ t \in [t_0, t_0+T]. \tag{3.9}$$

From (3.7), it follows that

$$V(t, x(t)) - V(t_0, \phi(0)) \leq \int_{t_0}^{t} D^+V(s, x(s))ds$$

$$\leq -\int_{t_0}^{t} \lambda(s)c(d(x(s), M(s)))ds,\ t \geq t_0. \tag{3.10}$$

From the properties of the function $V(t, x(t))$ in the interval $[t_0, \infty)$, it follows that there exists the finite limit

$$\lim_{t\to\infty} V(t, x(t)) = v_0 \geq 0. \tag{3.11}$$

Then from (3.7), (3.9), (3.10) and (3.11), it follows that

$$\int_{t_0}^{\infty} \lambda(t)c(d(x(t), M(t)))dt \leq b(\eta) - v_0.$$

From the integral positivity of the function $\lambda(t)$, it follows that the number T can be chosen so that

$$\int_{t_0}^{t_0+T} \lambda(t)dt > \frac{b(\eta) - v_0 + 1}{c(\delta(\varepsilon))}.$$

Then, we obtain

$$b(\eta) - v_0 \geq \int_{t_0}^{\infty} \lambda(t)c(d(x(t), M(t)))dt$$

$$\geq \int_{t_0}^{t_0+T} \lambda(t)c(d(x(t), M(t)))dt \geq c(\delta(\varepsilon)) \int_{t_0}^{t_0+T} \lambda(t)dt > b(\eta) - v_0 + 1.$$

The contradiction obtained shows that there exists a positive constant $T = T(\epsilon, \eta)$ such that for any solution $x(t) = x(t; t_0, \phi)$ of system (1.1) for which $t_0 \in \mathbb{R}_+$, $\phi \in \overline{S_\alpha}(\mathcal{C}_+) \cap \overline{M_0(t, \eta)}$, $\alpha > 0$, there exists $t^* \in [t_0, t_0 + T]$ such that inequality (3.8) holds.

As in the proof of Theorem 3.1, we can prove that $V(t, x(t)) < a(\epsilon)$ for $t \geq t^*$ (hence for any $t \geq t_0 + T$ as well). Therefore, $x(t) \in M(t, \epsilon)$ for $t \geq t_0 + T$, which proves that the set M is uniformly globally attractive with respect to system (1.1). □

4. Applications

4.1. Consider (1.1) for $n = 1$; i.e. consider the scalar equation

$$\dot{x}(t) = x(t)f(t, x(t), x_t), \tag{4.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \times C[[-h, 0], \mathbb{R}] \to \mathbb{R}$, and $x_t \in C[[-h, 0], \mathbb{R}]$ is defined by $x_t(\theta) = x(t + \theta)$ for $-h \leq \theta \leq 0$.

Let $x(t; 0, \phi)$ be the solution of (4.1) with *admissible* ([4]) initial condition

$$x_0 = \phi,$$

where $\phi \in C[[-h, 0], \mathbb{R}]$, $\phi(\theta) \geq 0$, $\theta \in [-h, 0)$ and $\phi(0) > 0$.

As for the system (1.1), we assume that the solutions $x(t; 0, \phi)$ of (4.1) are nonnegative, and if $\phi > 0$, then $x(t; 0, \phi) > 0$ for $t \geq 0$.

Let the sets $M = \{(t, 0) : t \in [-h, \infty)\}$, $d(x, M(t)) = |x|$, and $d_0(\phi, M_0(t)) = ||\phi||$.

Consider the function $V(t, x) = \frac{1}{2}x^2$. Then for $t \geq 0$, we have

$$D^+V(t, \phi(0)) = \phi(0)\dot{\phi}(0) = \phi^2(0)f(t, \phi(0), \phi).$$

If there exist continuous functions $A, B : \mathbb{R}_+ \to (0, \infty)$ such that

$$f(t, x(t), x_t) \leq -A(t)x(t) + B(t) \sup_{-h\leq\theta\leq 0} x(t + \theta),$$

$$A(t) - B(t) \geq q(t) > 0, \; t \geq 0,$$

then

$$D^+V(t,\phi(0)) \leq \phi^2(0)\Big(- A(t)\phi(0) + B(t) \sup_{-h\leq\theta\leq 0} \phi(\theta)\Big)$$
$$= \phi(0)\Big(- A(t)\phi^2(0) + B(t)\phi(0) \sup_{-h\leq\theta\leq 0} \phi(\theta)\Big).$$

Using the inequality $2|u||v| \leq u^2+v^2$, for $V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)),\ \theta\in[-h,0]$, we have

$$D^+V(t,\phi(0))$$
$$\leq \phi(0)\Big\{ - A(t)\phi^2(0) + B(t)\Big[\frac{1}{2}\Big(\phi^2(0) + (\sup_{-h\leq\theta\leq 0} \phi(\theta))^2\Big)\Big]\Big\}$$
$$\leq -q(t)\phi^3(0) = -q(t)d^3(\phi(0),M(t)),\ t\geq 0.$$

Hence by Theorem 3.1, if $\int_0^\infty q(s)ds = \infty$, the set M is uniformly globally asymptotically stable with respect to (4.1), which means that the zero solution of (4.1) is uniformly globally asymptotically stable.

4.2. Consider the following n-species delayed Lotka-Volterra-type system:

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)x_j(t-h_{ij}(t))\right], \tag{4.2}$$

where $n\geq 2,\ t\geq 0,\ a_{ij},h_{ij}\in C[\mathbb{R}_+,\mathbb{R}_+],\ b_i\in C[\mathbb{R}_+,\mathbb{R}],\ 0\leq h_{ij}(t)\leq h,\ h = const$. Here, $x_i(t)$ represents the density of species i at the moment t; $b_i(t)$ is the reproduction rate function; and $a_{ij}(t)$ are functions which describe the effect of the j-th population upon the i-th population, which is positive if it enhances and negative if it inhibits the growth.

Let $x^*(t) = x^*(t;0,\varphi) = col(x_1^*(t;0,\varphi), x_2^*(t;0,\varphi),...,x_n^*(t;0,\varphi))$ be a strictly positive (component-wise) solution of (4.2) and let $x(t) = x(t;0,\phi) = col(x_1(t;0,\phi), x_2(t;0,\phi),...,x_n(t;0,\phi))$ be any other solution of (4.2) with initial conditions

$$x_0^* = \varphi,\ x_0 = \phi,$$

where $\varphi,\phi\in\mathcal{C}_+$.

Furthermore, we will restrict our attention to only those solutions which evolve in the phase space $\mathbb{R}^n_+$, and we will assume that there exist positive constants r and R such that

$$r\leq x_i(t)\leq R,\quad t\in\mathbb{R}_+,\ i=1,...,n. \tag{4.3}$$

We point out that efficient sufficient conditions which guarantee the permanence of the solutions of such systems are given in [16].

Consider the set $M=\{(t,x^*(t)) : t\in[-h,0]\}$, and let $d(x,M(t))=\sum_{i=1}^{n}|x_i-x_i^*(t)|$.

Theorem 4.1. *Assume that:*

1. *The assumption* (4.3) *holds.*
2. *There exist integrally positive functions* $\lambda_i:\mathbb{R}_+\to\mathbb{R}_+,\ i=1,...,n,$ *such that*

$$r\min_{t\in\mathbb{R}_+} a_{ii}(t) - R\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\max_{t\in\mathbb{R}_+} a_{ij}(t)\right) \geq \lambda_i(t),\quad t\in\mathbb{R}_+,$$

for all $1\leq i\leq n$.

Then the set M is uniformly globally asymptotically stable with respect to system (4.2).

Proof. Define a Lyapunov function

$$V(t, x, x^*) = \sum_{i=1}^{n} |\ln \frac{x_i}{x_i^*}|.$$

By the Mean Value Theorem and by (4.3) it follows that

$$\frac{1}{R}|x_i(t) - x_i^*(t)| \leq |\ln x_i(t) - \ln x_i^*(t)| \leq \frac{1}{r}|x_i(t) - x_i^*(t)|. \tag{4.4}$$

Consider the upper right derivative $D^+V(t, \phi(0), \varphi(0))$ with respect to system (4.2). For $t \geq 0$ we derive the estimate

$$D^+V(t, \phi(0), \varphi(0)) = \sum_{i=1}^{n} \left(\frac{\dot{\phi}_i(0)}{\phi_i(0)} - \frac{\dot{\varphi}_i(0)}{\varphi_i(0)} \right) sgn\,(\phi_i(0) - \varphi_i(0))$$

$$\leq \sum_{i=1}^{n} \left[-a_{ii}(t)|\phi_i(0) - \varphi_i(0)| + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t)|\phi_j(-h_{ij}(t)) - \varphi_j(-h_{ij}(t))| \right]$$

$$\leq \sum_{i=1}^{n} \left[-\min_{t \in \mathbb{R}_+} a_{ii}(t)|\phi_i(0) - \varphi_i(0)| + \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\max_{t \in \mathbb{R}_+} a_{ij}(t) \right) |\phi_j(-h_{ij}(t)) - \varphi_j(-h_{ij}(t))| \right].$$

From (4.4) using the Razumikhin condition $V(t+\theta, \phi(\theta), \varphi(\theta)) \leq V(t, \phi(0), \varphi(0))$, $\theta \in [-h, 0]$, we have

$$\frac{1}{R} \sum_{j=1}^{n} |\phi_j(\theta) - \varphi_j(\theta)| \leq V(t + \theta, \phi(\theta), \varphi(\theta))$$

$$\leq V(t, \phi(0), \varphi(0)) \leq \frac{1}{r} \sum_{j=1}^{n} |\phi_j(0) - \varphi_j(0)|, \quad \theta \in [-h, 0],$$

and hence

$$\sum_{j=1}^{n} |\phi_j(-h_{ij}(t)) - \varphi_j(-h_{ij}(t))| \leq \frac{R}{r} \sum_{j=1}^{n} |\phi_j(0) - \varphi_j(0)|.$$

Then

$$D^+V(t, \phi(0), \varphi(0)) \leq -\frac{\lambda(t)}{r} \sum_{j=1}^{n} |\phi_j(0) - \varphi_j(0)|, \ t \geq 0,$$

where $\lambda(t) = \inf_{t \in \mathbb{R}_+} \{\lambda_1(t), ..., \lambda_n(t)\}$.

Since all conditions of Theorem 3.2 are satisfied, the set M is uniformly globally asymptotically stable with respect to system (4.2). Hence, the solution x^* of (4.2) is uniformly globally asymptotically stable. □

Remark 4.1. The above theorem can be applied to any solution which is of interest: an equilibrium, a periodic solution, an almost periodic solution, etc. Therefore, the results obtained in this paper provide one more general set of criteria for determining the stability of the n-species Kolmogorov-type system.

4.3. In considering the dynamics of price, production, and consumption of a particular commodity, Belair and Mackey [1] assumed that relative variations in market price $p(t)$ are governed by the equation

$$\dot{p}(t) = p(t)F(D(p_D), S(p_S)),\ t \geq 0, \tag{4.5}$$

where $D(\ .\ )$ and $S(\ .\ )$, respectively, denote the *demand* and *supply* functions for the commodity in question. The arguments of functions D and S are given by p_D (*demand price*) and p_S (*supply price*), respectively, rather than simply the current market price p.

The Kolmogorov-type model (4.5) and its generalizations have been studied by many authors (see [15] and the references cited therein).

Let α, a, b, c, $d > 0$. Consider the following special case of model (4.5) with time-varying delay $0 \leq h(t) \leq h$:

$$\dot{p}(t) = \alpha\left[\frac{a+c}{p(t)} - b - d\frac{p(t-h(t))}{p(t)}\right]p(t). \tag{4.6}$$

It is easy to verify that $p^* = \frac{a+c}{b+d}$ is an equilibrium of (4.6). Consider the set

$$M = \{(t, p^*),\ t \in [-h, \infty)\},$$

and let the distance $d(p, M(t)) = |p - p^*|$.

We shall show that if there exists an integrally positive function $\beta(t)$ such that $d \leq b - \beta(t)$ for $t \geq 0$, then the set M is uniformly globally asymptotically stable with respect to (4.6).

Let $V(t, p) = \frac{1}{2}(p - p^*)^2$. Then, for $t \geq 0$, we have

$$D^+V(t, \varphi(0)) = \alpha(\varphi(0) - p^*)[a - b\varphi(0) + c - d\varphi(-h(t))]$$

$$= \alpha(\varphi(0) - p^*)[-b(\varphi(0) - p^*) - d(\varphi(-h(t)) - p^*)].$$

From the last relation, if $V(t + \theta, \varphi(\theta)) \leq V(t, \varphi(0)),\ \theta \in [-h, 0]$, we obtain the estimate

$$D^+V(t, \varphi(0)) \leq \alpha[-b + d](\varphi(0) - p^*)^2 \leq -\alpha\beta(t)(\varphi(0) - p^*)^2,\ t \geq 0.$$

Therefore, by Theorem 3.2 the set M is uniformly globally asymptotically stable with respect to (4.6); hence, the equilibrium p^* of (4.6) is uniformly globally asymptotically stable.

5. Conclusions

By using the Lyapunov-Razumikhin technique, sufficient conditions for the global asymptotic stability of sets of a sufficiently general type contained in some domain with respect to a general Kolmogorov-type system are obtained. These criteria are independent of the delay parameters and are very precise in some cases. Compared with the method of Lyapunov functionals as in most previous studies, our method is simpler and more effective for stability analysis. Our results are quite general in their applicability to different Kolmogorov-type systems.

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