PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 3, March 2014, Pages 881–886
S 0002-9939(2013)11303-1
Article electronically published on November 27, 2013

MOCK MORREY SPACES

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(Communicated by Michael T. Lacey)

This paper is dedicated to the memory of
Guido Stampacchia
1922–1978

Abstract. For any function f belonging to $Q^{p,\lambda}$, a certain proper subspace of the classical Morrey space $L^{p,\lambda}$, a sharp capacity weak-type estimate is obtained for its Riesz potential $I_\alpha f$. This extends a 1969 result due to Franco Conti.

1. Introduction

The function spaces presented here were inspired by G. Stampacchia in the late 1960's (though the name Mock Morrey Spaces is mine). But for this, we first must consider the originals formulated in a 1938 paper [M]; C.B. Morrey, there, first wrote and used this condition, which defines a Morrey space. Their history goes through the theory of Campanato spaces of the early 1960's [P]. More recently, new developments have been added by J. Xiao and the author; see [AX1, AX2, AX3, AX4]. In particular, this note might be read in conjunction with this new Morrey theory.

We shall say that a real valued function $f(x)$ belongs to the Morrey space $L^{p,\lambda}$ on Euclidean $\mathbb{R}^N$, $N \geq 1$, if

$$(1) \qquad \left(\sup_{r>0,x} r^{\lambda-N} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p} \equiv \|f\|_{L^{p,\lambda}} < \infty,$$

where $B(x,r)$ is a ball in $\mathbb{R}^N$, centered at x and of radius r, $0 < \lambda < N$, $1 < p < \infty$. When $p = 1$ in the above, we admit signed Borel measures μ in the usual way; $|\mu| =$ total variation measure.

In fact, one might consider this article to be the fulfillment of an obligation that the author took on as a postdoc at the CNR in Rome, Italy, in 1970, when Stampacchia challenged him to consider this then-new space presented by F. Conti in 1969 [C]. Stampacchia knew at the time that the author was very much interested in the concept of "the capacity of a set" $E \subset \mathbb{R}^N$. So, some 40+ years later, we will try to bring that challenge into the perspective of the present technology, especially that of the recent developments in nonlinear potential theory, represented now by [AH], [MS2].

A Mock Morrey space, loosely speaking, will be any space, initially defined for arbitrary sets E (usually restricted to the class of all compact subsets of $\mathbb{R}^N$), which

Received by the editors April 8, 2011 and, in revised form, June 8, 2011.
2010 *Mathematics Subject Classification.* Primary 35B15, 42B37.

becomes exactly a Morrey space when that set E is everywhere replaced by a ball or a cube in $\mathbb{R}^N$, with sides parallel to the coordinate axes. Of main interest here are those $f(y)$ that satisfy

$$\left(\sup_E \frac{\int_E |f(y)|^p\,dy}{S(E)^{(N-\lambda)/\sigma}}\right)^{1/p} \equiv \|f\|_{Q^{p,\lambda}(S)} < \infty, \tag{2}$$

where S is a non-negative set function on subsets of $\mathbb{R}^N$ such that $S(B(x,r)) \sim r^\sigma$. Here $\sim$ means that the ratio is bounded above and below by positive constants for all sets under consideration — here the collection of all balls $B(x,r) \subset \mathbb{R}^N$. Standard set functions used below may include Lebesgue N-measure $\mathcal{L}_N$, Hausdorff capacity (content) Λ^d, $0 < d < N$, Riesz capacity $\dot{C}_{\alpha,p}$, $\alpha > 0$, and/or Bessel capacity $C_{\alpha,p}$; see [AH] for definitions and a full discussion of these latter set functions. When $S = \dot{C}_{\alpha,p}$ and $\lambda = \alpha p < N$, these Q spaces coincide with a space of Sobolev multipliers $\gamma \in M[\dot{L}^{\alpha,p} \to L^p]$; see [MS2], or [MS1].

Notice that one easily has the inclusions

$$Q^{p,\lambda}(\mathcal{L}_N) \supset Q^{p,\lambda}(\dot{C}_{\alpha,p}) \supset Q^{p,\lambda}(\dot{C}_{\beta,p}),$$

$0 < \alpha < \beta$, upon discovering that $\dot{C}_{\alpha,p}(B(x,r)) \sim r^{N-\alpha p}$, $0 < \alpha p < N$. Also,

$$Q^{p,\lambda}(\mathcal{L}_N) \subset L^{p,\lambda}.$$

Further, note that if $f \in Q^{p,\lambda}(\mathcal{L}_N)$, then f belongs to the Lorentz space $L^{q,\infty}$, i.e., weak-L^q for $q = Np/\lambda > p$. It is of some interest that this does not happen for arbitrary $f \in L^{p,\lambda}$; i.e., it was noted in [A3] that such f's cannot, in general, belong to any L^q_{loc} space, $q > p$. Thus, again we see $Q^{p,\lambda}(\mathcal{L}_N) \subsetneq L^{p,\lambda}$, as noted by Conti. Note also that

$$|y|^{-\lambda/p} \in Q^{p,\lambda}(\mathcal{L}_N). \tag{3}$$

Indeed,

$$\int_E |y|^{-\lambda}\,dy \le c\mathcal{L}_N(E)^{1-\lambda/N}$$

holds for some constant c independent of E, $0 < \lambda < N$.

The main result of this article (as well as that of Conti, in the eyes of this author) is the establishment of the following capacity weak-type inequalities for the Riesz potential $I_\alpha f$ of any $f \in Q^{p,\lambda}(\dot{C}_{\alpha,p})$:

Theorem 1. *If $f \in Q^{p,\lambda}(\dot{C}_{\alpha,p})$, then there is a constant $A > 0$ such that*

$$\dot{C}_{\alpha,p}\left([|I_\alpha f| > t]\right) \le A t^{-q_0} \|f\|^{q_0}_{Q^{p,\lambda}(\dot{C}_{\alpha,p})} \tag{4}$$

for $q_0 = (N-\alpha p)p/(\lambda - \alpha p)$, $\alpha p < \lambda$, and

$$I_\alpha f(x) = \int_{\mathbb{R}^N} |x-y|^{\alpha - N} f(y)\,dy,$$

$0 < \alpha < N$.

Theorem 2. *If $\lambda = \alpha p$ in Theorem 1, then*

$$C_{\alpha,p}\left([|G_\alpha f| > t]\right) \le A e^{-at/\|f\|}, \tag{5}$$

for some constant $a > 0$ independent of f; $\|f\| =$ the $Q^{p,\lambda}(C_{\alpha,p})$ norm of f.

2. The classical operators on $Q^{p,\lambda}(\dot{C}_{\alpha,p})$

Our first result is to recall a key idea of Verbitsky (see [MV] and [MS1]): the Calderon-Zygmund operator K and the Hardy-Littlewood maximal operator M_0 are bounded operators on the spaces $Q^{p,\lambda}(\dot{C}_{\alpha,p})$. (This is also the case for the spaces $L^{p,\lambda}$; see [P] and [CF].) The key idea is that the Wolff potentials $W^{\mu}_{\alpha,p}(x)$, $\mu = \dot{C}_{\alpha,p}$ capacity measure for the compact set $E \subset \mathbb{R}^N$, when raised to the β power, $0 < \beta < (p-1)N/(N-\alpha p)$, are A_1-weights in the sense of Muckenhoupt; see [S]. For example,

$$\begin{aligned}\int_E (M_0 f)^p\,dx &\le \int (M_0 f)^p (W^{\mu}_{\alpha,p})^{\beta}\,dx \\ &\le A\int |f|^p (W^{\mu}_{\alpha,p})^{\beta}\,dx \\ &= A\int_0^{\mathcal{M}} \int_{E_t} |f|^p\,dx\,t^{\beta-1}\,dt \\ &\le A\int_0^{\mathcal{M}} C_{\alpha,p}(E_t)^{\delta} t^{\beta-1}\,dt \cdot \|f\|_Q^p \\ &\le A\int_0^{\mathcal{M}} t^{-(p-1)\delta} t^{\beta-1}\,dt \cdot \|f\|_Q^p\, C_{\alpha,p}(E)^{\delta},\end{aligned}$$

where μ is chosen so that $W \ge 1$ on E and

$$W^{\mu}_{\alpha,p}(x) = \int_0^{\infty} \left[r^{\alpha p - N}\mu(B(x,r))\right]^{p-1} \frac{dr}{r};$$

$\mathcal{M}$ is the L^{∞} bound of W, $E_t = [W > t]$, and $\delta = (N-\lambda)/(N-\alpha p) \le 1$. Thus

$$\int_E (M_0 f)^p\,dx \le \|f\|_Q^p\, C_{\alpha,p}(E)^{\delta},$$

since β needs only to exceed $(p-1)\delta$.

The same argument also shows that K is bounded on the $Q^{p,\lambda}(\dot{C}_{\alpha,p})$ spaces. Below, we will also need that the operators T_{α} of Dahlberg [D] are also bounded on the $Q^{p,\lambda}(\dot{C}_{\alpha,p})$ spaces. This is due to the L^p bounds of Dahlberg plus the usual extrapolation argument of [GCRdF].

3. The Conti-Stampacchia iteration scheme

To obtain Theorem 1, we iterate an inequality of the form

$$\phi(2^k) \le A 2^{-kp} \phi(2^{k-1})^{\delta}, \tag{6}$$

where $\phi(t)$ is a decreasing function of $t > 0$ and $\delta < 1$. For Theorem 2,

$$\phi(h_s) \le \frac{1}{e}\phi(h_{s-1}), \tag{7}$$

where $h_s = k_0 + s(Ae)^{1/p}$, $s > 0$; see [C] and [St].

In fact, in the first case, we argue that

$$\phi(2^k) \le \frac{A^{1+\delta+\delta^2+\cdots+\delta^{k-1}}\phi(1)^{\delta^k}}{2^{p\sum_1^{k-1}(k-j)\delta^{j-1}}},$$

and since

$$\sum_{j=1}^{k-1}(k-j)\delta^{j-1} = \frac{k}{1-\delta} - \frac{1-\delta^k}{(1-\delta)^2},$$

we get

$$\phi(t) \le \frac{\phi(1)A^{1/(1-\delta)}2^{p/(1-\delta)^2}}{t^{p/(1-\delta)}}. \tag{8}$$

Iterating (7) gives

$$\phi(t) \le Ae^{-t} \tag{9}$$

with $A = \phi(k_0)$.

4. The iteration inequality

Next we show how one can deduce inequality (6) for $\phi(t) = \dot{C}_{\alpha,p}\left([|I_\alpha f| > t]\right)$. Here, we use the techniques of [A2] and [D] for $\alpha p < \lambda$. First in the case $\alpha =$ positive integer $\in (0, N)$, we smoothly truncate $I_\alpha f$, with $H(t) \in C^\infty(0,\infty)$,

$$H(t) = \begin{cases} 0, & t \le 1/2, \\ 1, & t \ge 1. \end{cases}$$

Then it easily follows, as in [A2], that

$$\begin{aligned} \int \left| D^\alpha H\left(\frac{I_\alpha f}{2^k}\right)\right|^p dx &\le A\int_{E_{k-1}} (M_0 f)^p \, dx \cdot 2^{-kp} \\ &\le A\int (M_0 f)^p \left(W^\mu_{\alpha,p}\right)^\beta \cdot 2^{-kp}, \end{aligned}$$

$E_{k-1} = [I_\alpha f > 2^{k-1}]$, $f \ge 0$, where again we are using the Wolff potential to the power β as an A_1-weight. This then yields

$$\dot{C}_{\alpha,p}([I_\alpha f > 2^k]) \le A\|f\|^p_{Q^{p,\lambda}(C_{\alpha,p})} 2^{-kp}\dot{C}_{\alpha,p}([I_\alpha f > 2^{k-1}])^\delta$$

with $\delta = (N-\lambda)/(N-\alpha p)$, $\alpha p < \lambda$.

For the fractional case, we use the estimates of [D]. In particular,

$$H(I_\alpha f) = I_\alpha g$$

for some g, with

$$\|g\|_{Q^{p,\lambda}(\dot{C}_{\alpha,p})} \le A\|f\|_{Q^{p,\lambda}(\dot{C}_{\alpha,p})}$$

via the T_α operator generalizing the Strichartz fractional integral operator S_α; see [D]. Notice that support of $H(I_\alpha f/2^k) = I_\alpha g_k$ is contained in E_{k-1}.

Inequality (7) follows in a similar manner, again with $C_{\alpha,p}([G_\alpha f > h_s]) = \phi(h_s)$; cf. [St]. Here G_α is the Bessel potential operator; see [AH].

5. Some remarks

(a) Some readers might object to the severity of the above arguments. However, it seems to be a good idea here to present just the basic outline of the proofs as given rather than to repeat the calculations of others. And after 40+ years, the Conti results [C] haven't received much attention, so it is doubtful that these results are in the mainstream of analysis.

(b) It is interesting to compare the results (esp. Theorem 1) with the literature. We can easily get

$$M_p^{p,\lambda} \subset Q^{p,\lambda}(\dot{C}_{\alpha,p}) \subset L^{p,\lambda}, \tag{10}$$

where $M_q^{p,\lambda}$ was introduced in [A3]:

$$\sup_x \int_0^\infty \left(r^{\lambda-N} \int_{B(x,r)} |f(y)|^p\, dy \right)^{q/p} \frac{dr}{r} < \infty.$$

In fact, the following embedding (an extension of the result $I_\alpha \colon L^{p,\lambda} \to L^{\tilde{p},\lambda}$, $\tilde{p} = \lambda p/(\lambda - \alpha p)$, $\alpha p < \lambda$, first presented in [A1]) holds:

$$I_\alpha \colon M_q^{p,\lambda} \to M_r^{\tilde{p},\lambda},$$

with $r = \lambda q/(\lambda - \alpha p)$, as stated in [A3] — again a result of estimates for $I_\alpha f$ in terms of the fractional maximal function M_α; see [A1] and [A3].

Notice now the following:

$$I_\alpha \colon \begin{cases} L^{p,\lambda} \to L^q_{\mathrm{loc}}(\dot{C}_{\alpha,p}), & q < \frac{p(N-\alpha p)}{\lambda-\alpha p)} = q_0, \\ Q^{p,\lambda}(\dot{C}_{\alpha,p}) \to L^{q_0,\infty}_{\mathrm{loc}}(\dot{C}_{\alpha,p}), & q = q_0, \\ M_p^{p,\lambda} \to L^{q_0}_{\mathrm{loc}}(\dot{C}_{\alpha,p}). \end{cases}$$

The first result above can be found in [AX3]. In particular, these results reflect the proper inclusions (10).

(c) It is well-known that

$$G_\alpha \colon L^{p,\lambda} \to BMO$$

for $\lambda = \alpha p$. This together with the well-known J-N Lemma [JN] agrees with estimate (5).

References

[A1] David R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), no. 4, 765–778. MR0458158 (56 #16361)

[A2] David R. Adams, *On the existence of capacitary strong type estimates in R^n*, Ark. Mat. **14** (1976), no. 1, 125–140. MR0417774 (54 #5822)

[A3] D.R. Adams. *Lectures on L^p-potential theory.* Dept. Math., University of Umeå, Sweden, 1981.

[AH] David R. Adams and Lars Inge Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314, Springer-Verlag, Berlin, 1996. MR1411441 (97j:46024)

[AX1] David R. Adams and Jie Xiao, *Nonlinear potential analysis on Morrey spaces and their capacities*, Indiana Univ. Math. J. **53** (2004), no. 6, 1629–1663, DOI 10.1512/iumj.2004.53.2470. MR2106339 (2005h:31015)

[AX2] David R. Adams and Jie Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), no. 2, 201–230, DOI 10.1007/s11512-010-0134-0. MR2961318

[AX3] David R. Adams and Jie Xiao, *Morrey potentials and harmonic maps*, Comm. Math. Phys. **308** (2011), no. 2, 439–456, DOI 10.1007/s00220-011-1319-5. MR2851148

[AX4] David R. Adams and Jie Xiao, *Regularity of Morrey commutators*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4801–4818, DOI 10.1090/S0002-9947-2012-05595-4. MR2922610

[CF] Filippo Chiarenza and Michele Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. (7) **7** (1987), no. 3-4, 273–279 (1988). MR985999 (90f:42017)

[C] Franco Conti, *Su alcuni spazi funzionali e loro applicazioni ad equazioni differenziali di tipo ellittico*, Boll. Un. Mat. Ital. (4) **2** (1969), 554–569 (Italian, with English summary). MR0264388 (41 #8983)

[D] Björn E. J. Dahlberg, *Regularity properties of Riesz potentials*, Indiana Univ. Math. J. **28** (1979), no. 2, 257–268, DOI 10.1512/iumj.1979.28.28018. MR523103 (80g:31004)

[GCRdF] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985. MR0807149 (87d:42023)

[JN] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR0131498 (24 #A1348)

[MS1] V. G. Maz′ya and T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Monographs and Studies in Mathematics, vol. 23, Pitman (Advanced Publishing Program), Boston, MA, 1985. MR785568 (87j:46074)

[MS2] Vladimir G. Maz′ya and Tatyana O. Shaposhnikova, *Theory of Sobolev multipliers. With applications to differential and integral operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 337, Springer-Verlag, Berlin, 2009. MR2457601 (2010d:46039)

[MV] Vladimir G. Maz′ya and Igor E. Verbitsky, *Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers*, Ark. Mat. **33** (1995), no. 1, 81–115, DOI 10.1007/BF02559606. MR1340271 (96i:26021)

[M] Charles B. Morrey Jr., *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), no. 1, 126–166, DOI 10.2307/1989904. MR1501936

[P] Jaak Peetre, *On the theory of $\mathcal{L}_{p,\,\lambda}$ spaces*, J. Functional Analysis **4** (1969), 71–87. MR0241965 (39 #3300)

[St] Guido Stampacchia, *Some limit cases of L^p-estimates for solutions of second order elliptic equations*, Comm. Pure Appl. Math. **16** (1963), 505–510. MR0152735 (27 #2710)

[S] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, with the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43, Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993. MR1232192 (95c:42002)

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