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CONVERGENCE OF EINSTEIN YANG-MILLS SYSTEMS

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ABSTRACT. In this paper, we prove a convergence theorem for sequences of Einstein Yang-Mills systems on $U(1)$-bundles over closed n-manifolds with some bounds for volumes, diameters, L^2-norms of bundle curvatures, and $L^{\frac{n}{2}}$-norms of curvature tensors. This result is a generalization of earlier compactness theorems for Einstein manifolds.

1. Introduction

A Riemannian metric g on a smooth manifold M is called an Einstein metric with Einstein constant λ if g has constant Ricci curvature λ, i.e.,

$$\mathrm{Ric} = \lambda g$$

(cf. [5]). Einstein metrics have many nice properties and have been widely studied. Nonetheless, it is well-known that some manifolds wouldn't admit any Einstein metric (cf. [5], [12]). In [16] and [18], the notion of a Einstein Yang-Mills system is introduced as a generalization of Einstein metrics by coupling Einstein equations with Yang-Mills equations. Let $\mathcal{L}$ be a principal $U(1)$-bundle over a smooth manifold M. If a Riemannian metric g on M and a connection A of $\mathcal{L}$ satisfy the equations

$$\left\{ \begin{array}{c} \mathrm{Ric} - \frac{1}{2}\eta = \lambda g, \\ d^* F = 0\ , \end{array} \right. \tag{1}$$

where F is the curvature of A and $\eta_{ij} = g^{kl}F_{ik}F_{jl}$, then $(M, \mathcal{L}, g, F, \lambda)$ is called an Einstein Yang-Mills (EYM for short) system and λ is called the Einstein Yang-Mills constant. Besides Einstein metrics with flat $U(1)$-bundles, other solutions of (1) are obtained in Section 2 of [16]. The parabolic version of (1), the so-called Ricci Yang-Mills flow, is studied for solving (1) in [16] and [18]. A solution of (1) also solves Einstein-Maxwell equations, which are studied in the literature of both physics and mathematics (cf. [13] and the references in it). In this paper, we study the compactness of families of EYM systems.

The convergence of Einstein manifolds in the Gromov-Hausdorff topology has been studied by many authors (cf. [1], [2], [4], [7], [17], etc.). Gromov's precompactness theorem says that if (M_i, g_i) is a family of Riemannian n-manifolds with diameters bounded from above and Ricci curvature bounded from below, a subsequence of (M_i, g_i) converges to a compact length space (X, d_X) in the Gromov-Hausdorff sense (cf. [9]). In addition, if the Ricci curvatures $|\mathrm{Ric}(g_i)| < \mu$, the

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volumes $\mathrm{Vol}_{M_i} > v > 0$ for constants μ and v independent of i, and the $L^{\frac{n}{2}}$-norms of the curvature tensors of g_i have a uniform bound, then it is shown in [2] that (X, d_X) is an orbifold with finite singular points $\{p_k\}_{k=1}^N$. Furthermore, d_X is induced by a $C^{1,\alpha}$-metric g_∞ on the regular part $X \backslash \{p_k\}_{k=1}^N$, and by passing to a subsequence, g_i $C^{1,\alpha}$-converges to g_∞ in the Cheeger-Gromov sense. If g_i are Einstein metrics with bounded Einstein constants, then g_∞ is an Einstein metric and g_i converge to g_∞ in the C^∞-topology (cf. [1], [4], [17]). In the present paper, we prove an analogue convergence theorem for EYM systems.

Providing $(M, \mathcal{L}, g, F, \lambda)$ is an Einstein Yang-Mills system, for any $m \in \mathbb{N}^+$, $(M, \mathcal{L}^m, m^2 g, mF, \frac{\lambda}{m^2})$ also satisfies (1). By choosing an appropriate m, we can normalize the Einstein Yang-Mills system such that the EYM constant $|\frac{\lambda}{m^2}| \leq 1$. Thus, in this paper we only consider EYM systems with EYM constants belonging to $[-1, 1]$.

Theorem 1.1. *Let $\{(M_i, \mathcal{L}_i, g_i, F_i, \lambda_i)\}$ be EYM systems with EYM constants $\lambda_i \in [-1, 1]$, where $\{M_i\}$ is a family of connected closed n-manifolds. Assume that there are constants $\Omega > 0$, $v > 0$, $D > 0$, $C_0 > 0$ and $c_0 > 0$ independent of i such that*

(i) $\mathrm{Vol}_{M_i} \geq v > 0$, *and* $\mathrm{diam}_{M_i} \leq D$,

(ii)
$$\int_{M_i} |F_i|^2 \, d\mu_i \leq \Omega,$$

(iii) $b_2(M_i) \leq c_0$, *for $n = 4$, or*
$$\int_{M_i} |\mathrm{Rm}(g_i)|^{\frac{n}{2}} \, d\mu_i \leq C_0 \text{ for } n > 4.$$

Then a subsequence of (M_i, g_i) converges, without changing the subscripts, in the Gromov-Hausdorff sense, to a connected Riemannian orbifold (M_∞, g_∞) with finite singular points $\{p_k\}_{k=1}^N$, each having a neighborhood homeomorphic to the cone $C(S^{n-1}/\Gamma_k)$, with Γ_k a finite subgroup of $O(n)$. The metric g_∞ is a C^0 Riemannian orbifold metric on M_∞, which is smooth off the singular points. Furthermore, there is a $U(1)$-bundle $\mathcal{L}_\infty$ on the regular part $M_\infty^o = M_\infty \backslash \{p_k\}_{k=1}^N$, a Yang-Mills connection A_∞ of $\mathcal{L}_\infty$ with curvature F_∞, and a constant $\lambda_\infty \in [-1, 1]$ such that $(M_\infty^o, \mathcal{L}_\infty, g_\infty, F_\infty, \lambda_\infty)$ is an EYM system. Also, for any compact subset $K \Subset M_\infty^o$, there are embeddings $\Phi_K^i : K \to M_i$ such that $\Phi_K^{i,-1}\mathcal{L}_i \cong \mathcal{L}_\infty|_K$ for $i \gg 1$,
$$\Phi_K^{i,*} g_i \to g_\infty, \quad \Phi_K^{i,*} F_i \to F_\infty, \quad \text{and} \quad \lambda_i \to \lambda_\infty,$$
when $i \to \infty$ in the C^∞-sense.

Remark 1.2. If we assume that $\lambda_i > \kappa > 0$ for a uniform positive constant κ, then the Ricci curvature of g_i is bounded below by κ, since η is a non-negative symmetric tensor. By Myers' Theorem, the diameters of g_i are uniformly bounded from above. Therefore, the condition of diameter bounds in Theorem 1.1 can be removed.

Remark 1.3. If M_i in Theorem 1.1 are odd-dimensional oriented manifolds, by Corollary 2.8 in [2], there are no singular points in M_∞, and the EYM systems $\{(M_i, \mathcal{L}_i, g_i, F_i, \lambda_i)\}$ smoothly converge to a smooth EYM system.

Remark 1.4. If we replace the assumption $\mathrm{Vol}_{M_i} \geq v > 0$ and $\int_{M_i} |\mathrm{Rm}|^{\frac{n}{2}}\, d\mu \leq C_0$ by the condition of injective radius bounded from below, we can obtain C^∞ subconvergence of the sequence $\{(M_i, \mathcal{L}_i, g_i, F_i, \lambda_i)\}$ to a smooth EYM system $(M_\infty, \mathcal{L}_\infty, g_\infty, F_\infty, \lambda_\infty)$ by Theorem 1.1 in [2] and similar arguments in the proof of Theorem 1.1.

In Section 2, some properties of EYM systems are studied, and in Section 3, Theorem 1.1 is proved.

2. Preliminary properties of EYM systems

To start things off we present some basic preliminary properties of EYM systems which are used in the proof of Theorem 1.1. First, we have the following lemma for an EYM system.

Lemma 2.1. *If $(M^n, \mathcal{L}, g, F, \lambda)$ is an EYM system, then the scalar curvature R of g and $|F|^2$ are constants.*

Proof. By taking a trace of the first equation of (1), we have

$$R - \frac{1}{2}|F|^2 = n\lambda.$$

Hence

$$\begin{aligned} 0 &= \nabla_k \left(R - \frac{1}{2}|F|^2 \right) \\ &= 2g^{ij}\nabla_i R_{jk} - \frac{1}{2}\nabla_k |F|^2 \\ &= 2g^{ij}\nabla_i \left(\lambda g_{jk} + \frac{1}{2}\eta_{jk} \right) - \frac{1}{2}\nabla_k |F|^2 \\ &= g^{ij}\nabla_i \eta_{jk} - \frac{1}{2}\nabla_k |F|^2 \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} g^{ij}\nabla_i\eta_{jk} &= g^{ij}\nabla_i \left(g^{pq} F_{jp} F_{kq} \right) \\ &= g^{ij} g^{pq} F_{jp} \nabla_i F_{kq} + g^{ij} g^{pq} F_{kq} \nabla_i F_{jp} \\ &= g^{ij} g^{pq} F_{jp} \nabla_i F_{kq} - g^{pq} F_{kq} d^* F_p \\ &= -g^{ij} g^{pq} F_{jp} \left(\nabla_k F_{qi} + \nabla_q F_{ik} \right) \\ &= \frac{1}{4}\nabla_k |F|^2 \ , \end{aligned}$$

where we have used the Yang-Mills equation of the EYM system (1) and the Bianchi identity. Thus we have $\nabla_k |F|^2 = 0$, and then $\nabla_k R = 0$. We obtain the conclusion; i.e. R and $|F|^2$ are constants. □

As a consequence of Lemma 2.1, the $W^{1,2p}$ estimate of the bundle curvature F of an EYM system is easily obtained.

Proposition 2.2. *Let $(M, \mathcal{L}, g, F, \lambda)$ be an EYM system with $|\lambda| \leq 1$. If the volume of the underlying n-manifold M is bounded from above by V, then for any $1 < p < +\infty$,*

$$\left(\int_M |\nabla F|^{2p}\, d\mu \right)^{\frac{1}{p}} \leq 2\max\{1, 2V^{\frac{1}{p}}\} |F|^2 \left(1 + \left(\int_M |\mathrm{Rm}|^p\, d\mu \right)^{\frac{1}{p}} \right).$$

Proof. We know that $|F|^2$ is a constant due to Lemma 2.1. Hence by the Bochner formula of 2-forms,

$$(\Delta_d F)_{ij} = (\Delta F)_{ij} + 2g^{kp}g^{lq}R_{ijkl}F_{pq} - g^{kl}R_{ik}F_{lj} - g^{kl}R_{jk}F_{il},$$

we have

$$\begin{aligned}
0 &= \Delta |F|^2 = 2\langle \Delta F, F\rangle + 2|\nabla F|^2 \\
&= 2\langle \Delta_d F, F\rangle - 2g^{is}g^{jt}\left(2g^{kp}g^{lq}R_{ijkl}F_{pq} - g^{kl}R_{ik}F_{lj} - g^{kl}R_{jk}F_{il}\right)F_{st} + 2|\nabla F|^2 \\
&= -4g^{is}g^{jt}g^{kp}g^{lq}R_{ijkl}F_{pq}F_{st} + 4\langle \mathrm{Ric}, \eta\rangle + 2|\nabla F|^2 .
\end{aligned}$$

It follows that

$$\begin{aligned}
|\nabla F|^2 &\le 2|F|^2|\mathrm{Rm}| - 2\langle \mathrm{Ric}, \eta\rangle \\
&= 2|F|^2|\mathrm{Rm}| - 2\left\langle \frac{1}{2}\eta + \lambda g, \eta\right\rangle \\
&\le 2|F|^2|\mathrm{Rm}| - 2\lambda|F|^2 \\
&\le 2|F|^2(1+|\mathrm{Rm}|),
\end{aligned}$$

by $|\lambda| \le 1$. So

$$\begin{aligned}
\left(\int_M |\nabla F|^{2p}d\mu\right)^{\frac{1}{p}} &\le \left(\int_M \left(2|F|^2(1+|\mathrm{Rm}|)\right)^p d\mu\right)^{\frac{1}{p}} \\
&\le 2|F|^2\left[\left(\int_{|\mathrm{Rm}|\le 1} 2^p d\mu\right)^{\frac{1}{p}} + \left(\int_{|\mathrm{Rm}|>1} |\mathrm{Rm}|^p d\mu\right)^{\frac{1}{p}}\right] \\
&\le 2|F|^2\left[\left(\int_M 2^p d\mu\right)^{\frac{1}{p}} + \left(\int_M |\mathrm{Rm}|^p d\mu\right)^{\frac{1}{p}}\right] \\
&\le 2\max\{1, 2V^{\frac{1}{p}}\}|F|^2\left(1 + \left(\int_M |\mathrm{Rm}|^p d\mu\right)^{\frac{1}{p}}\right).
\end{aligned}$$

Combining this with the fact that $|F|^2$ is a constant, we have

$$\|F\|^2_{W^{1,2p}} \le C\left(1 + \|\mathrm{Rm}\|_{L^p}\right). \qquad \square$$

Beginning with the classical Cheeger-Gromov Theorem, much work has been done on convergence of families of manifolds. Many such results have applications to solving geometric problems (cf. [3], [9] and [11], etc.). A common aspect of these convergence results is the use of harmonic coordinates, coordinate systems where the corresponding coordinate functions are harmonic functions. One reason to use harmonic coordinates is that Ricci tensors are elliptic operators of metric tensors, as we shall see in the proof of the following lemma.

Given tensors ξ and ζ, $\xi * \zeta$ denotes some linear combination of contractions of $\xi \otimes \zeta$ in this paper.

Lemma 2.3. *Let $(M, \mathcal{L}, g, F, \lambda)$ be an EYM system and $u\colon U \to R^n$ be a harmonic coordinate system of the underlying manifold M. Then in this coordinate, the EYM equations* (1) *are*

$$-\frac{1}{2}g^{kl}\frac{\partial^2 g_{ij}}{\partial u^k \partial u^l} - Q_{ij}(g, \partial g) - \frac{1}{2}g^{kl}F_{ik}F_{jl} - \lambda g_{ij} = 0, \tag{3}$$

(4) $$g^{kl}\frac{\partial^2 F_{ij}}{\partial u^k \partial u^l} + P_{ij}(g, \partial g, \partial F) + T_{ij}(g, \partial g, F) = 0,$$

where

$$Q(g, \partial g) = \left(g^{-1}\right)^{*2} * (\partial g)^{*2},$$
$$P(g, \partial g, \partial F) = \left(g^{-1}\right)^{*2} * \partial g * \partial F,$$

and

$$T(g, \partial g, \partial^2 g, F) = \left(g^{-1}\right)^{*3} * (\partial g)^{*2} * F + \left(g^{-2}\right)^{*2} * \partial^2 g * F.$$

Proof. The Ricci tensor under the harmonic coordinate system is given by (cf. [15])

$$R_{ij} = -\frac{1}{2}g^{kl}\frac{\partial^2 g_{ij}}{\partial u^k \partial u^l} - Q_{ij}(g, \partial g)$$

with

$$Q_{ij}(g, \partial g) = \left(g^{-1}\right)^{*2} * (\partial g)^{*2}.$$

The first equation of the EYM equations becomes

$$-\frac{1}{2}g^{kl}\frac{\partial^2 g_{ij}}{\partial u^k \partial u^l} - Q_{ij}(g, \partial g) - \frac{1}{2}g^{kl}F_{ik}F_{jl} - \lambda g_{ij} = 0.$$

Note that the Yang-Mills equation $d^* F = 0$ is equivalent to $\Delta_d F = 0$ on compact manifolds; i.e. F is harmonic. We use the Bochner formula again. Observe that $\Delta u^k = 0$, which is equivalent to $g^{ij}\Gamma^k_{ij} = 0$ for all $k = 1, \cdots, n$. It is easy to compute that in this coordinate chart,

$$\begin{aligned}(\Delta_d F)_{ij} \;\; = \;\; & g^{kl}\frac{\partial^2 F_{ij}}{\partial u^k \partial u^l} + g^{kp}g^{lq}R_{ijkl}F_{pq} \\ & -2g^{pq}\{\Gamma^l_{pi}\frac{\partial F_{lj}}{\partial u^q} + \Gamma^l_{pj}\frac{\partial F_{il}}{\partial u^q}\} + \frac{\partial g^{pq}}{\partial u^i}\Gamma^l_{pq}F_{lj} + \frac{\partial g^{pq}}{\partial u^j}\Gamma^l_{pq}F_{il} \\ & +2g^{pq}\{\Gamma^m_{iq}\Gamma^l_{mp}F_{lj} + \Gamma^m_{jq}\Gamma^l_{mp}F_{il} + \Gamma^m_{jq}\Gamma^l_{ip}F_{lm}\}.\end{aligned}$$

Then the Yang-Mills equation becomes

$$g^{kl}\frac{\partial^2 F_{ij}}{\partial u^k \partial u^l} + P_{ij}(g, \partial g, \partial F) + T_{ij}(g, \partial g, \partial^2 g, F) = 0,$$

where

$$P_{ij}(g, \partial g, \partial F) = \left(g^{-1}\right)^{*2} * \partial g * \partial F \quad \text{and}$$
$$T_{ij}(g, \partial g, \partial^2 g, F) = \left(g^{-1}\right)^{*3} * (\partial g)^{*2} * F + \left(g^{-2}\right)^{*2} * \partial^2 g * F.$$

□

A compact Riemannian n-manifold (M, g) is said to have an $(r, \sigma, C^{l,\alpha})$ for $0 < \alpha < 1$ (resp. $(r, \sigma, W^{l,p})$ for $1 < p < \infty$) adapted harmonic atlas if there is a covering $\{B_{x_k}(r)\}_{k=1}^{\sigma}$ of M by geodesic r-balls, for which the balls $B_{x_k}\left(\frac{r}{2}\right)$ also cover M and the balls $B_{x_k}\left(\frac{r}{4}\right)$ are disjoint such that each $B_{x_k}(10r)$ has a harmonic coordinate chart $u\colon B_{x_k}(10r) \to U \subset \mathbb{R}^n$, and the metric tensor in these coordinates are $C^{l,\alpha}$ (resp. $W^{l,p}$) bounded; i.e. if $g_{ij} = g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$ on $B_{x_k}(10r)$, then

$$C^{-1}\delta_{ij} \leq g_{ij} \leq C\delta_{ij}$$

and

$$\|g_{ij}\|_{C^{l,\alpha}} \leq C \quad (\text{resp.} \quad \|g_{ij}\|_{W^{l,p}} \leq C,)$$

for some constant $C > 1$, where the norms are taken with respect to the coordinates $\{u^j\}$ on $B_{x_k}(10r)$.

In [2], Anderson shows a local existence of an adapted atlas.

Proposition 2.4 (Proposition 2.5, Lemma 2.2 and Remarks 2.3 in [2])**.** *Let* (M, g) *be a Riemannian manifold with*

$$|\mathrm{Ric}_M| \leq \Lambda, \quad \mathrm{diam}_M \leq D \quad \textit{and} \quad \mathrm{Vol}_{B(r)} \geq v_0 > 0$$

for a ball $B(r)$. *Then, for any* $C > 1$ *there exist positive constants* $\sigma = \sigma(\Lambda, v_0, n, D)$, $\epsilon = \epsilon(\Lambda, v_0, n, C)$ *and* $\delta = \delta(\Lambda, v_0, n, C)$ *such that for any* $1 < p < \infty$, *one can obtain a* $(\delta, \sigma, W^{2,p})$ *adapted atlas on the union* U *of those balls* $B(r)$ *satisfying*

$$\int_{B(4r)} |\mathrm{Rm}|^{\frac{n}{2}} \, d\mu \leq \epsilon.$$

More precisely, on any $B(10\delta) \subset U$, *there is a harmonic coordinate chart such that for any* $1 < p < \infty$,

$$C^{-1}\delta_{ij} \leq g_{ij} \leq C\delta_{ij}$$

and

$$\|g_{ij}\|_{W^{2,p}} \leq C.$$

3. Convergence of EYM metrics

This section is devoted to the proof of Theorem 1.1. We begin with a review of a local version of the Cheeger-Gromov compactness theorem (cf. Theorem 2.2 in [1], Lemma 2.1 in [2], and [11]).

Proposition 3.1. *Let* V_i *be a sequence of domains in closed* C^∞ *Riemannian manifolds* (M_i, g_i) *such that* V_i *admits an adapted harmonic atlas* $(\delta, \sigma, C^{l,\alpha})$ *for a constant* $C > 1$. *Then there is a subsequence which converges uniformly on compact subsets in the* $C^{l,\alpha'}$ *topology,* $\alpha' < \alpha$, *to a* $C^{l,\alpha}$ *Riemannian manifold* V_∞.

Now we need the following lemma before proving Theorem 1.1.

Lemma 3.2. *Let* $(M, \mathcal{L}, g, F, \lambda)$ *be an EYM system with* $\lambda \in [-1, 1]$. *Assume that*

$$\mathrm{Vol}_M \geq v > 0 \quad \textit{and} \quad \int_M |F|^2 \, d\mu \leq \Omega.$$

Then there is a constant $C_1 = C_1(v, \Omega, n)$ *such that*

$$|\mathrm{Ric}\,(g)| \leq C_1.$$

Furthermore, if M *is a* 4*-manifold and*

$$\mathrm{diam}_M \leq D, \quad b_2(M) \leq c_0,$$

then there is a constant $C_2 = C_2(v, \Omega, D, c_0)$ *such that*

$$\int_M |\mathrm{Rm}\,(g)|^2 \, d\mu \leq C_2.$$

Proof. Since $(M, \mathcal{L}, g, F, \lambda)$ is an EYM system, by Lemma 2.1, $|F|$ is a constant and

$$|F|^2 \, \mathrm{Vol}_M = \int_M |F|^2 \, d\mu \leq \Omega.$$

By the assumption $\mathrm{Vol}_M \geq v > 0$, we have

$$|F|^2 \leq \frac{\Omega}{v}.$$

Then it is clear that

$$\begin{aligned} |\mathrm{Ric}\,(g)|^2 &= \left|\frac{1}{2}\eta\,(g,F) + \lambda g\right|^2 \\ &\leq \frac{1}{4}|F|^4 + \lambda\,|F|^2 + n\lambda^2 \\ &\leq C_1(\Omega, v, n). \end{aligned}$$

From now on, we assume that the dimension of M is 4, $\mathrm{diam}_M \leq D$, and $b_2\,(M) \leq c_0$. The Bishop comparison theorem implies that the volume of M is bounded above by a constant $c\,(\Omega, v, D, n)$, i.e.

$$\mathrm{Vol}_M \leq c\,(\Omega, v, D, n)\,.$$

By the standard Gauss-Bonnet-Chern formula (cf. [5]), the Euler characteristic $\chi\,(M)$ of an oriented compact 4-manifold M can be expressed as follows:

$$\chi\,(M) = \frac{1}{8\pi^2}\int_M \left(|\mathrm{Rm}|^2 - 4\,|\mathrm{Ric}|^2 + R^2\right) d\mu.$$

Thus

$$\begin{aligned} \frac{1}{8\pi^2}\int_M |\mathrm{Rm}\,(g)|^2\,d\mu &= \chi\,(M) + \frac{1}{8\pi^2}\int_M \left(4\,|\mathrm{Ric}\,(g)|^2 - R^2\,(g)\right) d\mu \\ &\leq 2 + b_2\,(M) + \frac{\max_M |\mathrm{Ric}\,(g)|^2}{2\pi^2}\mathrm{Vol}_M \\ &\leq C_2(v, \Omega, D, c_0). \end{aligned}$$

□

Now we are ready to prove the main theorem of this paper.

Proof of Theorem 1.1. By Lemma 3.2 we know that under the assumption of Theorem 1.1 the Ricci curvatures and the $L^{\frac{n}{2}}$-norms of the curvature tensors of g_i have uniform bounds. By Theorem 2.6 in [2], a subsequence of (M_i, g_i) converges to a Riemannian orbifold (M_∞, g_∞) with finite isolated singular points in the Gromov-Hausdorff sense. Furthermore, g_i $C^{1,\alpha}$-converges to g_∞ on the regular part in the Cheeger-Gromov sense. Now we improve the convergence to the C^∞-sense and study the convergence of F_i.

For a given $r > 0$, let $\{B^i_{x_k}(r)\}$ be a family of metric balls of radius r such that $\{B^i_{x_k}(r)\}$ covers (M_i, g_i), and $B^i_{x_k}(\frac{r}{2})$ are disjoint. Denote

$$G_i(r) = \bigcup \left\{ B^i_{x_k}(r) \Big| \int_{B^i_{x_k}(4r)} |\mathrm{Rm}\,(g_i)|^2\,d\mu_i \leq \epsilon \right\},$$

where $\epsilon = \epsilon\,(\Omega, v, D, n, C_3) > 0$ is obtained in Proposition 2.4 for a constant $C_3 > 1$. So $G_i(r)$ are covered by a $\left(\delta, \sigma, W^{2,p}\right)$ (for any $1 < p < \infty$) adapted atlas with the harmonic radius uniformly bounded from below by virtue of Proposition 2.4. In these coordinates we have $W^{2,p}$ bounds for the metrics, i.e.

$$C_3^{-1}\delta_{jk} \leq g_{i,jk} \leq C_3\delta_{jk}$$

and

$$\|g_i\|_{W^{2,p}} \leq C_3.$$

Since the curvature tensor $\mathrm{Rm}\,(g) = \partial^2 g + \partial g * \partial g$, for any $1 < p < \infty$, we have

$$\|\mathrm{Rm}(g_i)\|_{L^p} \leq C\left(n, \|g\|_{W^{2,2p}}\right).$$

By Proposition 2.2 and Lemma 3.2, we get a uniform $W^{1,2p}$ estimate for the 2-forms F_i,

$$\begin{aligned}\|F_i\|^2_{W^{1,2p}} &\leq C\left(1+\|\mathrm{Rm}(g_i)\|_{L^p}\right)\\ &\leq C(p,n,\Omega,v,C_3).\end{aligned}$$

By the Sobolev embbeding theorem we know that for all $0<\alpha\leq 1-\frac{n}{2p}$,

$$\|g_i\|_{C^{1,\alpha}}\leq C_4,$$

$$\|F_i\|^2_{C^{\alpha}}\leq C_5,$$

where C_4 and $C_5>0$ are two constants.

Note that the EYM equations are elliptic equations (3) and (4) in harmonic coordinates by Lemma 2.3. Now we apply the Schauder estimate for elliptic partial differential equations (cf. [10]) to (3) and obtain

$$\|g_i\|_{C^{2,\alpha}}\leq C(n,\|g_i\|_{C^{1,\alpha}},\|F_i\|^2_{C^{\alpha}})\leq C_6,$$

for a constant $C_6>0$ independent of i. By applying L^p-estimates for elliptic differential equations (cf. [10]) to (4), we obtain

$$\|F_i\|_{W^{2,2p}}\leq C(n,\|g_i\|_{C^{2,\alpha}},\|F_i\|^2_{L^{2p}})\leq C_7,$$

where $C_7>0$ is a constant independent of i. Then

$$\|F_i\|_{C^{1,\alpha}}\leq C_8,$$

by the Sobolev embedding theorem. By standard elliptic theory (cf. [10]), for any $l\geq 0$ and $0<\alpha<1$, we obtain uniform $C^{l,\alpha}$ bounds for the metrics g_i and the bundle curvatures F_i:

$$\|g_i\|_{C^{l,\alpha}}\leq C(l,\alpha,n,\Omega,v,C_3),$$

$$\|F_i\|_{C^{l-1,\alpha}}\leq C(l,\alpha,n,\Omega,v,C_3).$$

Hence all the covariant derivatives of the curvature tensor have uniform bounds. By Proposition 3.1, there is a subsequence of $G_i(r)$ that converges in the C^∞ topology to an open manifold G_r with a smooth metric g_r and a harmonic 2-form F_r.

Because $\int_{M_i}|\mathrm{Rm}(g_i)|^{\frac{n}{2}}d\mu_i$ is uniformly bounded, there are finitely many disjoint balls $B(r)$ on which $\int_{B(4r)}|\mathrm{Rm}|^2 d\mu>\epsilon$, and the number of such disjoint balls is independent of r.

Now choose $r_j\to 0$, with $r_{j+1}\leq\frac{1}{2}r_j$. Let

$$G_i^j=\{x\in M|\, x\in G_i(r_m)\ \text{for some}\ m\leq j\};$$

then

$$G_i^1\subset G_i^2\subset G_i^3\subset\cdots\subset M_i.$$

By using the diagonal argument, a subsequence of $\left(G_i^j,g_i,F_i,\lambda_i\right)$ converges, in the $C^{l,\alpha}$ topology for all $l\geq 0$, to an open manifold M_∞^o with a smooth Riemannian metric $\widehat{g}_\infty$, a harmonic 2-form F_∞ and a constant $\lambda_\infty\in[-1,1]$ such that $(\widehat{g}_\infty,F_\infty,\lambda_\infty)$ satisfies the EYM equations. More precisely, for any compact subset $K\Subset M_\infty^o$, there are embeddings $\Phi_K^i:K\to M_i$ such that

$$\Phi_K^{i,*}g_i\overset{C^{l,\alpha}}{\to}\widehat{g}_\infty,\quad \Phi_K^{i,*}F_i\overset{C^{l,\alpha}}{\to}F_\infty,\quad\text{and}\quad\lambda_i\to\lambda_\infty,\tag{5}$$

when $i\to\infty$ for any $l\geq 0$ and $0<\alpha<1$.

For a compact subset $K \Subset M_\infty^o$, we fix a decomposition $H^2(K;\mathbb{Z}) \cong H_F^2(K;\mathbb{Z}) \oplus H_T^2(K;\mathbb{Z})$ where $H_T^2(K;\mathbb{Z})$ is the torsion part and $H_F^2(K;\mathbb{Z})$ is the free part, which is a lattice in $H^2(K;\mathbb{R})$. Then the first Chern class $c_1(\Phi_K^{i,-1}\mathcal{L}_i) = c_{1,i}^F + c_{1,i}^T$, where $c_{1,i}^F \in H_F^2(K;\mathbb{Z})$, $c_{1,i}^T \in H_T^2(K;\mathbb{Z})$, and $\frac{\sqrt{-1}}{2\pi}\Phi_K^{i,*}F_i$ represents $c_{1,i}^F$ by the standard Chern-Weil theory, i.e. $\left[\frac{\sqrt{-1}}{2\pi}\Phi_K^{i,*}F_i\right] = c_{1,i}^F \in H_F^2(K;\mathbb{Z}) \subset H_F^2(K;\mathbb{R})$. By (5), $c_{1,i}^F = \left[\frac{\sqrt{-1}}{2\pi}\Phi_K^{i,*}F_i\right]$ converges to $c_{1,\infty}^F = \left[\frac{\sqrt{-1}}{2\pi}F_\infty\right]$ in $H_F^2(K;\mathbb{R})$, which implies that by passing to a subsequence, $c_{1,i}^F = c_{1,\infty}^F$ for $i \gg 1$. Since there are only finite elements in $H_T^2(K;\mathbb{Z})$, by passing to a subsequence we obtain that $c_{1,i}^T \equiv c_{1,\infty}^T$ in $H_T^2(K;\mathbb{Z})$. Hence there is a $U(1)$-bundle $\mathcal{L}_{K,\infty}$ on K such that the first Chern class $c_1(\mathcal{L}_{K,\infty}) = c_{1,\infty}^F + c_{1,\infty}^T$, $\mathcal{L}_{K,\infty} = \Phi_K^{i,-1}\mathcal{L}_i$ and F_∞ is the curvature of a Yang-Mills connection on $\mathcal{L}_{K,\infty}$. Now we take a sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset M_\infty^o$ with $\bigcup K_a = M_\infty^o$. By the standard diagonal argument, there is a $U(1)$-bundle $\mathcal{L}_\infty$ over M_∞^o such that F_∞ is the curvature of a Yang-Mills connection A_∞ and, for $i \gg 1$,

$$\mathcal{L}_\infty = \Phi_{K_a}^{i,-1}\mathcal{L}_i.$$

The remainder of the proof is the same as arguments in Theorem C in [1] and Theorem 2.6 in [2]. For the convenience of readers we sketch the proof here and refer to the two papers for more details.

Let $M_\infty = M_\infty^o \cup \{p_k\}_{k=1}^N$ be the metric completion of M_∞^o. By using a blow up argument in [1], M_∞ is a connected orbifold with a finite number of singular points $\{p_k\}_{k=1}^N \subset M_\infty$, and the metric can be extended to a C^0 orbifold Riemannian metric on M_∞. Furthermore, each singular point p_k has a neighborhood homeomorphic to the cone $C\left(S^{n-1}/\Gamma_k\right)$, with Γ_k a finite subgroup of $O(n)$. □

From Theorem 1.1, one can see that orbifolds with orbifold metrics solving Einstein Yang-Mills equations on regular parts appear naturally as limits of sequences of EYM systems. We would like to construct such orbifolds which do not admit any Einstein orbifold metrics. First, let's recall an example of EYM systems from [16]. Let g_1 be the standard metric with Gaussian curvature 1 on S^2, g_2 be the standard metric with Gaussian curvature -1 on a surface H with genus $\mathfrak{g}$ bigger than 1, and ω_1 (resp. ω_2) be the volume form of g_1 (resp. g_2). If US^2 and UH denote the unit tangent bundles of S^2 and H respectively, then $F = \pi_1^*\omega_1 + \pi_2^*\omega_2$ is the curvature of $\mathcal{L} = \pi_1^{-1}US^2 \otimes \pi_2^{-1}UH$ on $M = S^2 \times H$, where π_1 and π_2 are standard projections from M to S^2 and H. For any $\lambda < 0$, the Riemannian metric $g = A_\lambda \pi_1^* g_1 + B_\lambda \pi_2^* g_2$ and F solve the Einstein Yang-Mills equations (1); i.e. $(M, \mathcal{L}, g, F, \lambda)$ is an EYM system, where $A_\lambda = \frac{1-\sqrt{1-2\lambda}}{2\lambda}$ and $B_\lambda = \frac{-1-\sqrt{1-2\lambda}}{2\lambda}$.

Now we assume that H is a hyperelliptic Riemann surface; i.e. H admits a conformal involution with $2\mathfrak{g}+2$ fixed points. Consider the involution ι of M obtained as the product of a 180° rotation of S^2 around an axis and the hyperelliptic involution of H. It is clear that the $\mathbb{Z}_2$-action induced by the involution preserves $\mathcal{L}$, g and F, i.e. $\iota^{-1}\mathcal{L} \cong \mathcal{L}$, $\iota^* g = g$ and $\iota^* F = F$. Thus the orbifold $M/\mathbb{Z}_2$ has $4\mathfrak{g}+4$ singular points, g induces an orbifold Riemannian metric on $M/\mathbb{Z}_2$, and $(\mathcal{L}, g, F, \lambda)$ induces an EYM system on the regular part of $M/\mathbb{Z}_2$. We claim that

$M/\mathbb{Z}_2$ wouldn't admit any orbifold Einstein metric. Note that the orbifold Euler characteristic $\chi_{orb}(M/\mathbb{Z}_2) = \frac{1}{2}\chi(M) = 2 - 2\mathfrak{g} < 0$. However, if there is an orbifold Einstein metric g' on $M/\mathbb{Z}_2$, then

$$\chi_{orb}(M/\mathbb{Z}_2) = \frac{1}{8\pi^2}\int_{M/\mathbb{Z}_2} |\mathrm{Rm}(g')|^2 d\mu \geq 0,$$

by (6.2) in [1], which is a contradiction.

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References

[1] Michael T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. **2** (1989), no. 3, 455–490, DOI 10.2307/1990939. MR999661 (90g:53052)

[2] Michael T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445, DOI 10.1007/BF01233434. MR1074481 (92c:53024)

[3] Michael T. Anderson, *Einstein metrics and metrics with bounds on Ricci curvature*, Vols. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 443–452. MR1403944 (97g:53046)

[4] Shigetoshi Bando, Atsushi Kasue, and Hiraku Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), no. 2, 313–349, DOI 10.1007/BF01389045. MR1001844 (90c:53098)

[5] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR867684 (88f:53087)

[6] Jeff Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970), 61–74. MR0263092 (41 #7697)

[7] Jeff Cheeger and Gang Tian, *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 487–525 (electronic), DOI 10.1090/S0894-0347-05-00511-4. MR2188134 (2006i:53042)

[8] Shiing Shen Chern, *Circle bundles*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977, pp. 114–131. MR0461353 (57 #1338)

[9] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)]; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates. MR1699320 (2000d:53065)

[10] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin, 1977. MR0473443 (57 #13109)

[11] R. E. Greene and H. Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math. **131** (1988), no. 1, 119–141. MR917868 (89g:53063)

[12] Nigel Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry **9** (1974), 435–441. MR0350657 (50 #3149)

[13] Claude LeBrun, *The Einstein-Maxwell equations, extremal Kähler metrics, and Seiberg-Witten theory*, The many facets of geometry, Oxford Univ. Press, Oxford, 2010, pp. 17–33, DOI 10.1093/acprof:oso/9780199534920.003.0003. MR2681684 (2011h:53098)

[14] Claude LeBrun, *Einstein metrics, four-manifolds, and differential topology*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, pp. 235–255. MR2039991 (2005g:53078)

[15] P. Peterson, *Riemannian Geometry,* Science Press (2007).

[16] Jeffrey D. Streets, *Ricci Yang-Mills flow*, ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–Duke University. MR2709943

[17] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172, DOI 10.1007/BF01231499. MR1055713 (91d:32042)
[18] Andrea Nicole Young, *Modified Ricci flow on a principal bundle*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Texas at Austin. MR2712036

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