

FOLIATIONS ON NON-METRISABLE MANIFOLDS: ABSORPTION BY A CANTOR BLACK HOLE

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ABSTRACT. We investigate contrasting behaviours emerging when studying foliations on non-metrisable manifolds.

1. INTRODUCTION

All of our manifolds are connected, Hausdorff spaces in which each point has a neighbourhood homeomorphic to some Euclidean space \mathbb{R}^n . Recall that there are four 1-manifolds: the circle \mathbb{S}^1 , the real line \mathbb{R} , the *long ray* \mathbb{L}_+ and the *long line* \mathbb{L} . The spaces \mathbb{L}_+ and \mathbb{L} are, respectively, the interior and double of the *closed long ray*, denoted by $\mathbb{L}_{\geq 0}$, which is $\omega_1 \times [0, 1)$ topologised by the lexicographical order, where ω_1 denotes the countable ordinals.

Our primary goal is to study foliations on non-metrisable manifolds. This non-metric shift produces both *regularities* and *anomalies*. First, foliated structures on certain (non-metrisable) manifolds often collapse to a very rigid, ascetic art form, e.g., on the long plane \mathbb{L}^2 , as we shall see. Second, some anomalies may happen, including a codimension-one foliation on a non-metrisable 3-manifold with a single leaf. This is pointed out in Martin Kneser [13] and Hellmuth Kneser [11], mentioned in Haefliger [5] and further popularised by Milnor [14]. See also [3, 4.6] and the picture in [2, Fig. 3]. This example is in fact neither difficult to understand nor truly pathological (granting some familiarity with the Prüfer manifold), yet is in sharp contrast with the metric case, where the set of leaves has the continuum power (cf. [11] and [4]). Does M. Kneser's pathology already occur on surfaces? A negative answer is included in the following theorem, proved in Section 3.

Theorem 1.1. *A dimension-one foliation on a (not necessarily metrisable) manifold of dimension at least 2 has exactly $\mathfrak{c} = \text{card}(\mathbb{R})$ many leaves.*

We pay special attention to foliations on long pipes, in view of the role they play in Nyikos's Bagpipe Theorem [16, Theorem 5.14]. The latter states that every ω -bounded 2-manifold derives from a closed surface by removing finitely many disjoint discs and replacing them by long pipes. Recall that ω -bounded means that every

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countable subset has a compact closure and a *long pipe* is the union of a chain $\langle U_\alpha : \alpha < \omega_1 \rangle$ of open subspaces each homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ such that $\overline{U_\alpha} \subset U_\beta$ and that the frontier of U_α in U_β is homeomorphic to \mathbb{S}^1 when $\alpha < \beta$.

A unifying feature observed in the long pipes and their generalisations which we consider is a product structure in which one factor is a metrisable manifold M and the other is \mathbb{L}_+ . This product structure manifests itself in any foliation yielding an *asymptotic rigidity*; we think of this as a kind of black hole behaviour. We make this more precise in Section 4, where we analyse dimension-one foliations on manifolds of the form $M \times \mathbb{L}_+$. Regarding the simplest long pipe, $\mathbb{S}^1 \times \mathbb{L}_+$, we prove the following in Section 5.

Theorem 1.2. *A dimension-one foliation \mathcal{F} on $\mathbb{S}^1 \times \mathbb{L}_+$ is confined to the following (mutually exclusive) alternatives:*

(i) *either the leaves are frequently horizontal, i.e. the set $C = \{\alpha \in \mathbb{L}_+ : \mathbb{S}^1 \times \{\alpha\} \text{ is a leaf of } \mathcal{F}\}$ is a closed unbounded subset of \mathbb{L}_+ , or*

(ii) *the foliation is ultimately vertical, i.e. there is an $\alpha \in \omega_1$ such that the restricted foliation on $\mathbb{S}^1 \times (\alpha, \omega_1)$ is the trivial product foliation by long rays.*

Picturesquely, the leaves are inclined to fall into the ‘black hole’ in a purely vertical way due to the huge gravitation (case (ii)), yet sometimes manage to resist the attraction by winding fast around it (case (i)). In the metric case, foliations are well understood in codimension-one (at least if smoothness is assumed): existence is systematic in open manifolds even in the topological category (existence of topological Morse functions and Quinn’s smoothing in dimension 4; cf. [2, Thm. 1.4]), whereas in the closed case the Euler characteristic is the unique obstruction (Thurston [19]). (This does not seem to be known in the topological category.)

The hope that non-metrisable (hence open) manifolds might all be foliated is not borne out. Consider the surfaces $\Lambda_{g,n}$ obtained from the compact orientable surface of genus g with n long pipes attached, all homeomorphic to $\mathbb{S}^1 \times \mathbb{L}_+$ (see Figure 3 below). As a corollary to Theorem 1.2 we show the following in Section 5.

Corollary 1.3. *None of the surfaces $\Lambda_{g,n}$ has a foliation except for $\Lambda_{0,2}$, the sphere with two black holes (homeomorphic to $\mathbb{S}^1 \times \mathbb{L}$), and $\Lambda_{1,0}$, the torus.*

A related result is implicit in Nyikos [15, p. 275]. Another of the simplest long pipes is the punctured long plane, $\mathbb{L}^2 - \{\text{pt}\}$. Our effort is concentrated on the behaviour of foliations ‘towards infinity’. Interestingly, $\mathbb{L}^2 - \{\text{pt}\}$, or more generally $\mathbb{L}^2 - K$ for some compactum K , splits naturally into pieces which, while not themselves being products of the form $M \times \mathbb{L}_+$, have enough of the structure of this product for us to be able to apply the results of Section 4. We find six different cases, described in Section 6 (see Figure 5 for a foretaste). Filling the holes in these six structures enables one to classify foliations on the long plane \mathbb{L}^2 . Up to homeomorphism there are only *two* representatives: the trivial foliation and the *broken* foliation, where leaves switch from the vertical to the horizontal mode when crossing the diagonal. Thus, from the foliated viewpoint the Cantor plane \mathbb{L}^2 appears as an extremely ‘rigid’ non-Euclidean crystal when compared to the usual plane \mathbb{R}^2 , whose plasticity permits a menagerie of foliated structures (Kaplan [8], Haefliger-Reeb [6]).

Note that Section 2 and Section 3 can be read independently of each other.

2. PREPARATORY RESULTS

Definition 2.1. Call a topological space X *squat* provided that every continuous map $f: \mathbb{L}_+ \rightarrow X$ is *eventually constant*; i.e. there are $\beta \in \mathbb{L}_+$ and $x \in X$ so that $f(\alpha) = x$ for each $\alpha \geq \beta$.

Our first lemma generalises the well-known fact that \mathbb{R} is squat (cf. [12, Satz 3] or [16, Lemma 3.4 (iii)]) and implies that all metrisable manifolds are squat.

Lemma 2.2. *A first countable, Lindelöf and Hausdorff space is squat.*

Proof. We prove this using the graph of $f: \mathbb{L}_+ \rightarrow X$ and thus work in $\mathbb{L}_+ \times X$. The graph Γ_f of f is closed (X is Hausdorff) and \mathbb{L}_+ -unbounded (i.e. its projection on the \mathbb{L}_+ -factor is unbounded). We shall use the following:

Sublemma. *Let X be a space as in Lemma 2.2 and let $C \subset \mathbb{L}_+ \times X$ be closed and \mathbb{L}_+ -unbounded. Then there is $x \in X$ so that $C \cap (\mathbb{L}_+ \times \{x\})$ is \mathbb{L}_+ -unbounded.*

We apply this to $C = \Gamma_f$. Since the \mathbb{L}_+ -projection of $\Gamma_f \cap (\mathbb{L}_+ \times \{x\})$ is nothing but the fibre $f^{-1}(x)$, we conclude that the latter is a closed unbounded set.

Let $(V_n)_{n \in \mathbb{N}}$ be a countable fundamental system of open neighbourhoods of x so that $\bigcap_n V_n = \{x\}$, and consider the closed subsets $f^{-1}(X - V_n) \subset \mathbb{L}_+$. The latter are disjoint from $f^{-1}(x)$ and therefore bounded (recall that two closed unbounded subsets of \mathbb{L}_+ always intersect). Hence $f^{-1}(X - \{x\}) = \bigcup_{n \in \mathbb{N}} f^{-1}(X - V_n)$ is bounded as well; beyond this bound f can take only the value x . This completes the proof of the lemma. □

Proof of the Sublemma. If not, then for all $x \in X$, $C \cap (\mathbb{L}_+ \times \{x\})$ is \mathbb{L}_+ -bounded. So there is a $\beta_x \in \mathbb{L}_+$ such that $[\beta_x, \omega_1) \times \{x\}$ does not intersect C . Fix some $x \in X$. Then there is an n so that the ‘thickening’ $[\beta_x, \omega_1) \times V_n$ still does not meet C . If not, construct a sequence $\langle x_n \rangle$ by choosing points $x_n \in ([\beta_x, \omega_1) \times V_n) \cap C$ which, owing to the sequential compactness of $[\beta_x, \omega_1)$, can be assumed to be convergent (extracting a subsequence if necessary). The limiting point x_ω would belong to $([\beta_x, \omega_1) \times \{x\}) \cap C$, a contradiction.

Now let x vary and denote the V_n above more accurately by $V_{n(x)}^x$. The $(V_{n(x)}^x)_{x \in X}$ form an open cover of X . By Lindelöfness we may extract a countable subcover corresponding to some countable subset N of X . Then $\beta = \sup_{x \in N} \beta_x$ is an \mathbb{L}_+ -bound for C . This contradiction proves the Sublemma. □

The proof of the following result is found in [2, Lemma 4.5].

Lemma 2.3 (Tube Lemma). *Suppose L is a leaf of a dimension-one foliation \mathcal{F} on a manifold and that $e: [0, 2] \rightarrow L$ is an embedding. Then there is a foliated chart (U, φ) such that $e([0, 2]) \subset U$.*

3. FOLIATIONS OF DIMENSION-ONE HAVE MANY LEAVES

In this section we show that the anomaly of a single leaf filling up the whole manifold cannot occur if the ambient dimension is only 2. The reason behind this well-behaviour lies in the one-dimensionality of the leaves, particularly the fact that 1-manifolds are completely classified.

Proof of Theorem 1.1. Let $\mathcal{L}(\mathcal{F})$ denote the set of leaves of the dimension-one foliation \mathcal{F} on the n -manifold M and set $\lambda = \text{card} \mathcal{L}(\mathcal{F})$: to show $\lambda = \mathfrak{c}$ (if $n \geq 2$).

The obvious surjection $M \rightarrow \mathcal{L}(\mathcal{F})$ shows that $\lambda \leq \mathfrak{c}$ because the cardinality of non-trivial connected, Hausdorff manifolds is always that of the continuum (see [18, Problem 8, pp. A-15–A-16] or [16, Theorem 2.9]).

Let $\varphi: U \rightarrow \mathbb{R}^n$ be a foliated chart for \mathcal{F} with $\varphi(U) = \mathbb{R}^n$ so that $P_y = \varphi^{-1}(\mathbb{R} \times \{y\})$ with $y \in \mathbb{R}^{n-1}$ are the corresponding plaques. One has an ‘integration’ map $\mathcal{P} := \{P_y\}_{y \in \mathbb{R}^{n-1}} \rightarrow \mathcal{L}(\mathcal{F})$ taking each plaque to its leaf extension.

It suffices to show that each leaf of \mathcal{F} contains only countably many plaques of \mathcal{P} . Indeed, in that case one can find an injection $\mathcal{P} \hookrightarrow \mathcal{L}(\mathcal{F}) \times \mathbb{N}$. Since $n \geq 2$, this gives $\mathfrak{c} = \text{card}(\mathcal{P}) \leq \lambda \cdot \omega$, and hence $\lambda \geq \mathfrak{c}$.

Suppose for a contradiction that there is a leaf L containing uncountably many plaques of \mathcal{P} . In view of the classification of 1-manifolds the leaf L is either \mathbb{L} or \mathbb{L}_+ , because the two separable manifolds \mathbb{S}^1 and \mathbb{R} cannot contain uncountably many pairwise disjoint open sets.

First, assume $L \approx \mathbb{L}$. The uncountable subset $\{y \in \mathbb{R}^{n-1} : P_y \subset L\}$ of \mathbb{R}^{n-1} has a *condensation point*, i.e. a point of the set which is the limit point of a non-stationary sequence of points in the set. Hence, one finds in L a point $x \in U$ which is the limit of a converging sequence $\langle x_n \rangle$ of points of U , none of which belongs to the plaque through x . Since the long line \mathbb{L} is sequentially compact, taking a subsequence if necessary, we may assume that $\langle x_n \rangle$ converges also in the leaf topology on L (say to \tilde{x}). Note that $\tilde{x} \neq x$, because the plaque through x does not contain any member of the sequence $\langle x_n \rangle$. Since the leaf topology on M is a refinement of its usual topology, it follows that $\langle x_n \rangle$ converges to \tilde{x} as well in the usual topology on M . This violates the uniqueness of the limit in Hausdorff spaces.

Finally, if $L \approx \mathbb{L}_+$, one finds a point $p \in L$ not in any of the plaques of U lying in L . The short side of $L - \{p\}$ can absorb at most countably many plaques, and arguing as before one can contradict the assumption that there are uncountably many plaques in the long side of $L - \{p\}$ (think of this as closed by adding p to make it sequentially compact). \square

The same argument shows that if all leaves of a codimension > 0 foliation are sequentially compact, then there are \mathfrak{c} many leaves. Note also that sequentially compact leaves (e.g., \mathbb{L}) are embedded (for the leaf inclusion is a closed map).

4. BLACK HOLES

In this section we use the concept of a squat manifold to analyse the asymptotic behaviour of a dimension-one foliation on a product manifold $M \times \mathbb{L}_+$, provided that M is both squat and separable. Basically, squatness forces an individual ‘long’ leaf to move vertically inside the product, while separability enables one to extend this individual verticality to a collective one for the foliation.

Call a 1-manifold *long* if it is non-metrisable (so is either the long ray or the long line), and *short* otherwise.

Theorem 4.1. *Suppose that M is a squat, separable manifold and that \mathcal{F} is a dimension-one foliation on $M \times \mathbb{L}_+$ having at least one long leaf. Then there is $\alpha \in \mathbb{L}_+$ so that \mathcal{F} restricted to $M \times (\alpha, \omega_1)$ is the trivial product foliation by long rays.*

Proof. The proof breaks into three steps.

Step 1. If L is a long leaf of \mathcal{F} , then there is $(x, \alpha) \in M \times \mathbb{L}_+$ such that $L \supset \{x\} \times [\alpha, \omega_1)$. (Say in this case that the foliation is vertical above the point (x, α) .)

Let $i: \mathbb{L}_+ \rightarrow L$ be an embedding. As M is squat the M -coordinate of i is eventually constant, say equal to x after some $\beta \in \mathbb{L}_+$. Next, the \mathbb{L}_+ -coordinate of i cannot be bounded, for if it were, it would be contained in a homeomorph of \mathbb{R} , which is squat, so the second coordinate would be eventually constant, violating the injectivity of i . It follows that $i([\beta, \omega_1)) = \{x\} \times [\alpha, \omega_1)$, where α is the \mathbb{L}_+ -coordinate of $i(\beta)$. This completes Step 1.

Step 2. Let

$$A = \{x \in M : \text{there is } \alpha \in \mathbb{L}_+ \text{ so that } \{x\} \times [\alpha, \omega_1) \text{ lies in a single leaf of } \mathcal{F}\}.$$

Then we claim that $A = M$.

As M is connected, it suffices to show that (i) $A \neq \emptyset$; (ii) A is open; (iii) A is closed.

(i) $A \neq \emptyset$; this follows from the assumption that \mathcal{F} has a long leaf via Step 1.

(ii) A is open. Let $x \in A$, so there is an $\alpha \in \mathbb{L}_+$ so that the foliation \mathcal{F} is vertical above the point (x, α) . Since x is a point in a manifold M we can fix a countable fundamental system of neighbourhoods $(V_n)_{n \in \mathbb{N}}$. By applying Lemma 2.3 to the arc $\{x\} \times [\alpha, \beta]$ for varying $\beta \in \omega_1$ greater than α , we see that for each such β there is an $n \in \mathbb{N}$ such that “every leaf through $V_n \times \{\alpha\}$ crosses $M \times \{\beta\}$ ”. Call this last (italicised) statement $S(n, \beta)$ and let $S = \{(n, \beta) \in \mathbb{N} \times \omega_1 : S(n, \beta) \text{ is true}\}$. By the argument above the set S is uncountable; hence there is an $n \in \mathbb{N}$ so that $S \cap (\{n\} \times \omega_1)$ is uncountable. This means that each leaf through $V_n \times \{\alpha\}$ crosses $M \times \{\beta\}$ for uncountably many $\beta > \alpha$. In particular each such leaf is long.

By squateness of the base M , each long leaf of \mathcal{F} stabilises and becomes purely vertical above some height $\alpha \in \mathbb{L}_+$; i.e. the leaf intersects $M \times [\alpha, \omega_1)$ in one or two vertical segments (depending on whether the leaf is a long ray or a long line).

Take a countable dense $D \subset V_n \times \{\alpha\}$. Each leaf L_d through the point $d \in D$ is long, and thus there is a height $\alpha_d \in \mathbb{L}_+$ above which L_d is purely vertical. Consider $\alpha_D = \sup_{d \in D} \alpha_d \in \mathbb{L}_+$. Apply Lemma 2.3 to $\{x\} \times [\alpha, \alpha_D]$ to produce a foliated chart (U, φ) with $U \supset \{x\} \times [\alpha, \alpha_D]$. Looking through the chart one obtains a pair of neighbourhoods N, N' of $\varphi(x, \alpha)$ in $\varphi((V_n \times \{\alpha\}) \cap U)$ respectively of $\varphi(x, \alpha_D)$ in $\varphi((M \times \{\alpha_D\}) \cap U)$ related by a homeomorphism $h: N \rightarrow N'$ which is just propagation along the vertical straight lines (see Figure 1). Let $\Delta := \varphi(D \cap U) \cap N$: by construction the foliation \mathcal{F} is vertical above $\varphi^{-1}(h(\Delta))$. Since $\varphi^{-1}(h(\Delta))$ is dense in $\varphi^{-1}(N')$, it follows that \mathcal{F} is vertical above the neighbourhood $\varphi^{-1}(N')$; hence the M -projection of $\varphi^{-1}(N')$ is a neighbourhood of x contained in A .

(iii) A is closed. Since M is first countable, it suffices to show that A is sequentially closed. Let $\langle x_n \rangle$ be a sequence in A converging to x . For each n there is $\alpha_n \in \mathbb{L}_+$ so that \mathcal{F} is vertical above (x_n, α_n) . We may assume that the sequence $\langle \alpha_n \rangle$ is increasing. Let $\alpha = \lim_{n \rightarrow \infty} \alpha_n \in \mathbb{L}_+$. Using foliated charts centred at various points (x, β) with $\beta \geq \alpha$, it follows that the foliation is vertical above the point (x, α) , hence $x \in A$.

Step 3. \mathcal{F} is ultimately vertical. Since M is separable, it admits a countable dense subset D . For each $d \in D$ there is by Step 2 an $\alpha_d \in \mathbb{L}_+$ such that \mathcal{F} is vertical above the point (d, α_d) . Take $\alpha = \sup_{d \in D} \alpha_d$; then \mathcal{F} is vertical above the subset $D \times \{\alpha\}$, hence vertical above $M \times \{\alpha\}$. □

Proposition 4.2. *Let $M = \bigcup_{\alpha \in \omega_1} U_\alpha$ be a manifold where each U_α is separable and open, $\overline{U_\alpha} \subset U_\beta$ whenever $\alpha < \beta$, and $U_\lambda = \bigcup_{\alpha < \lambda} U_\alpha$ whenever λ is a limit*

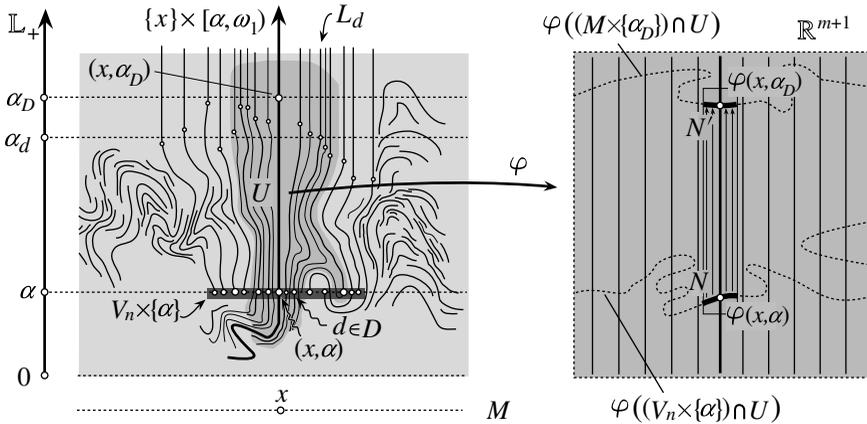


FIGURE 1. Applying the tube lemma around a vertical leaf

ordinal. Suppose that \mathcal{F} is a one-dimensional foliation on M , all of whose leaves are metrisable. Then $C = \{\alpha \in \omega_1 : U_\alpha \text{ is saturated by } \mathcal{F}\}$ is a closed unbounded subset of ω_1 . In particular, for each $\alpha \in C$, the set $\overline{U_\alpha} - U_\alpha$ is saturated by \mathcal{F} .

Proof. A connected metrisable manifold is Lindelöf; hence each leaf of \mathcal{F} is contained in some U_α for $\alpha \in \omega_1$ (the leaf is then said to be *bounded by* α). We show that C is unbounded. Construct an increasing sequence $\langle \alpha_n \rangle$ in ω_1 as follows. Let $\alpha_0 \in \omega_1$ be arbitrary. Now suppose α_n is given.

Let $D_\alpha \subset U_\alpha$ be a countable dense subset and consider the leaves of \mathcal{F} which pass through points of D_{α_n} . Because each leaf is bounded, collectively they all are bounded; say $\alpha_{n+1} > \alpha_n$ is such that each of these leaves lies in $U_{\alpha_{n+1}}$. We claim that $L \subset \overline{U_{\alpha_{n+1}}}$ for each leaf L of \mathcal{F} for which $L \cap U_{\alpha_n} \neq \emptyset$.

Suppose that L is a leaf with $L \cap U_{\alpha_n} \neq \emptyset$ and let $e: [0, 1] \rightarrow L$ be any embedding so that $e(0) \in U_{\alpha_n}$. To show that $L \subset \overline{U_{\alpha_{n+1}}}$ it suffices to show that $e(1) \in \overline{U_{\alpha_{n+1}}}$, because the arcwise-connectivity of L allows the end-point $e(1)$ to reach any point of L . By Lemma 2.3 there is a foliated chart (U, φ) so that $e([0, 1]) \subset U$. Choose $\langle x_n \rangle$ as a sequence in D_{α_n} converging to $e(0)$. Since U is open, the sequence is eventually in U , and via the chart we construct a sequence $\langle y_n \rangle$ converging to $e(1)$ by setting $\varphi^{-1}(L_n \cap H) = \{y_n\}$, where L_n is the line that is the image under φ of the leaf through x_n and H is the orthogonal hyperplane through $\varphi(e(1))$. By construction y_n belongs to the same leaf as x_n , so each y_n is in $U_{\alpha_{n+1}}$, and thus the limit $e(1)$ belongs to $\overline{U_{\alpha_{n+1}}}$.

Now let $\alpha = \lim \alpha_n$. Then $\alpha \in C$, because if L is any leaf meeting $\overline{U_\alpha}$, then L meets U_{α_n} for some n (since α is a limit ordinal) and hence lies in $\overline{U_{\alpha_{n+1}}} \subset U_\alpha$, so U_α is saturated. That C is closed follows from the fact that a union of \mathcal{F} -saturated subsets is saturated. \square

5. BLACK HOLE CONSEQUENCES

In this section we complete our analysis of foliations on the simplest long pipe, $\mathbb{S}^1 \times \mathbb{L}_+$. We then exhibit a family of surfaces lacking foliations, but when punctured admitting foliations. This is followed by a look at dimension-one foliations on $\mathbb{S}^2 \times \mathbb{L}_+$.

Proof of Theorem 1.2. If there are only short leaves, then by applying Proposition 4.2 to $\mathbb{S}^1 \times \mathbb{L}_+ = \bigcup_{\alpha \in \omega_1} \mathbb{S}^1 \times (0, \alpha)$ the situation described in (i) holds. In Proposition 4.2 we only obtained closedness of $C \cap \omega_1$, yet C is clearly closed in \mathbb{L}_+ . On the other hand, if there is a long leaf, then situation (ii) follows from Theorem 4.1. \square

Remark 5.1. In situation (ii) of Theorem 1.2 it is possible to have a bounded collection of circular leaves running around the cylinder. The situation described in (i) is ‘sharp’ in the sense that one cannot expect all leaves to be ultimately circular. Consider indeed the *Kneser foliation* on $\mathbb{S}^1 \times [0, 1]$, namely the unique foliation without circular leaves except the two boundaries which is transverse to the foliation by intervals. A transfinite gluing of such foliated annuli produces a foliation on $\mathbb{S}^1 \times \mathbb{L}_+$ whose set C is exactly ω_1 . In contrast, the *Reeb foliation* on the annulus develops ‘singularities’ when reaching a limit ordinal. See Figure 2.

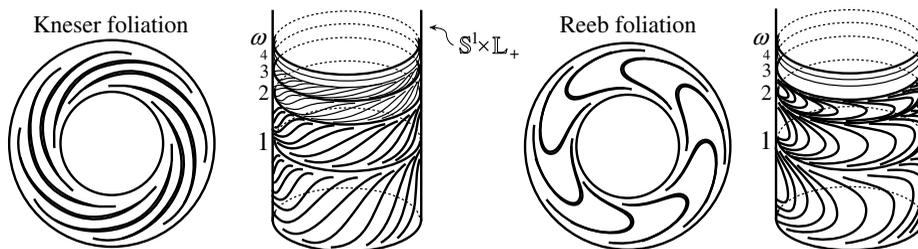


FIGURE 2. Transfinite gluing of Kneser, impossible with Reeb

As an application of Theorem 1.2 we now describe a family of open surfaces lacking foliations. Start with $\Sigma_{g,n}$ a genus g (orientable) surface with n boundaries, and attach n long cylinders $\mathbb{S}^1 \times \mathbb{L}_{\geq 0}$ (see Figure 3). Call the resulting surface $\Lambda_{g,n}$ the *genus g surface with n black holes*.

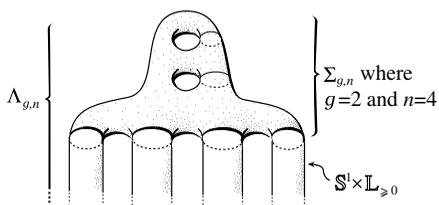


FIGURE 3. Genus g surface with n black holes

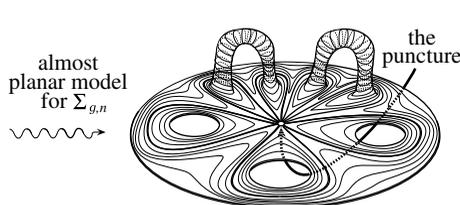


FIGURE 4. Foliating the punctured surface

Proof of Corollary 1.3. Assume that $\Lambda_{g,n}$ has a foliation. Then $\Lambda_{g,n} - \Sigma_{g,n}$ splits into n tubes $\mathbb{S}^1 \times \mathbb{L}_+$. By Theorem 1.2 each of those tubes contains a circle either tangent or transverse to the foliation. Cutting the surface $\Lambda_{g,n}$ along those n circles and discarding the non-metric tubes leads to a bordered surface homeomorphic to $\Sigma_{g,n}$ with a foliation well-behaved on the boundary. Doubling yields a foliation on the double $2\Sigma_{g,n}$ which has genus $2g+(n-1)$. This is possible only if $2g+(n-1) = 1$ (cf. Memento 5.2), which has only the two solutions $(g, n) = (0, 2)$ and $(1, 0)$. \square

Memento 5.2. The fact that a closed foliated surface has Euler characteristic $\chi = 0$ is due to Poincaré [17, pp. 203–8]. Alternatively one can use Lefschetz, as in the proof of Corollary 5.3 below. Another simple proof is Kneser’s [9], [10], which we sketch. Triangulate the surface so that each simplex is transversely foliated. Then each 2-simplex determines a distinguished vertex through which the leaf enters the triangle (hence also a second triangle). Thus, $2\sigma_0 = \sigma_2$, where σ_i is the number of i -simplices. Furthermore $2\sigma_1 = 3\sigma_2$ (each triangle has three edges, and each edge lies on two triangles). Hence, $\chi = \sigma_0 - \sigma_1 + \sigma_2 = \frac{1}{2}\sigma_2 - \frac{3}{2}\sigma_2 + \sigma_2 = 0$, as desired.

A glance at Figure 4 shows that after a single puncture the surfaces $\Lambda_{g,n}$ all permit a foliation. Is there a surface lacking a foliation even after puncturing? (We believe the answer to be positive; cf. a subsequent paper by the authors.)

When the base manifold M lacks a foliation of dimension 1, then $M \times \mathbb{L}_+$ behaves more cannibalistically, forcing the leaves to fall into the hole in a vertical way:

Corollary 5.3. *Let \mathcal{F} be a dimension-one foliation on $\mathbb{S}^2 \times \mathbb{L}_+$. Then there is $\alpha \in \mathbb{L}_+$ so that \mathcal{F} restricted to $\mathbb{S}^2 \times (\alpha, \omega_1)$ is the trivial product foliation by long rays. The same conclusion holds when \mathbb{S}^2 is replaced by any closed (topological) manifold with non-vanishing Euler characteristic.*

Proof. By Theorem 4.1 it suffices to show that there is a long leaf. If not, then by Proposition 4.2 applied to $\bigcup_{\alpha \in \omega_1} \mathbb{S}^2 \times (0, \alpha)$ there would be $\alpha \in \omega_1$ such that \mathcal{F} restricts to a dimension-one foliation on $\mathbb{S}^2 \times \{\alpha\}$. However \mathbb{S}^2 cannot be foliated. The generalisation uses the fact that a closed manifold M with $\chi(M) \neq 0$ lacks a dimension-one foliation (apply Whitney [20] to create a compatible flow and Lefschetz to generate a fixed point, passing first to a double cover if the foliation is not orientable). \square

6. FOLIATING LARGE SUBREGIONS OF \mathbb{L}^2

First, we describe the asymptotic behaviour of foliations on the long plane, possibly punctured by the removal of a compact subset. Up to rigid motions, only six possible pictures emerge, as depicted in Figure 5. The key idea in detecting these six patterns is to cut \mathbb{L}^2 not along the axes but along the two diagonals. Doing so yields quadrants which, while not being themselves products, can be filled by strips such as $(-\alpha, \alpha) \times (\alpha, \omega_1)$ having a ‘squat-long’ product decomposition to subordinate their foliation theory to the methods of Section 4. In the special case where the puncture is just a single point, we have foliated a second long pipe.

Next, we investigate how the free regions of Figure 5 may be foliated, to conclude that \mathbb{L}^2 has *only* two foliations up to homeomorphism and six up to isotopy.

Proposition 6.1. *Let K be a compact subset of \mathbb{L}^2 and suppose that \mathcal{F} is a foliation on $\mathbb{L}^2 - K$. Within the quadrant $Q = \{(x, y) \in \mathbb{L}^2 : -y < x < y\}$ exactly one of the following two sets must be closed and unbounded in \mathbb{L}_+ :*

- $\{\alpha \in \mathbb{L}_+ : Q \cap ([-\alpha, \alpha] \times \{\alpha\}) \text{ is part of a leaf of } \mathcal{F}\}$;
- $\{\alpha \in \mathbb{L}_+ : \{x\} \times [\alpha, \omega_1) \text{ is part of a leaf of } \mathcal{F} \text{ for each } x \in [-\alpha, \alpha]\}$.

Proof. Either no leaf of \mathcal{F} meets Q in an unbounded set, leading to the first option, or there is a leaf whose intersection with Q is unbounded, leading to the second option. The first case follows by applying Proposition 4.2 to $Q = \bigcup_{\alpha > 0} (Q \cap (\mathbb{L} \times (0, \alpha)))$, noting that bounded leaves are Lindelöf, hence metrisable.

We address unboundedness in the second case. Let L be an unbounded leaf. Suppose $\alpha_0 \in \omega_1$. Construct an increasing sequence $\langle \alpha_n \rangle$ as follows. Any leaf unbounded in Q must have a bounded first coordinate (otherwise by a ‘leapfrog’ argument it meets the boundary of Q in a closed unbounded set) and hence, by Lemma 2.2, eventually has a constant first coordinate. Thus we may assume that α_0 is big enough that $L \subset Q \cap ((-\alpha_0, \alpha_0) \times \mathbb{L}_+)$. Given α_n , the manifold $(-\alpha_n, \alpha_n)$ satisfies the hypotheses of the manifold M in Theorem 4.1. Hence there is $\alpha_{n+1} > \alpha_n$ such that $\mathcal{F}(-\alpha_n, \alpha_n) \times (\alpha_{n+1}, \omega_1)$ is the trivial product foliation by long rays. Letting $\alpha = \lim \alpha_n$, we find that $\alpha > \alpha_0$ and $\{x\} \times [\alpha, \omega_1)$ is part of a leaf of \mathcal{F} for each $x \in (-\alpha, \alpha)$, hence for each $x \in [-\alpha, \alpha]$. Closedness is routine. \square

Figure 5 illustrates the six cases respectively of the following theorem. Arrow-heads indicate that the leaf is long in the direction of the arrow.

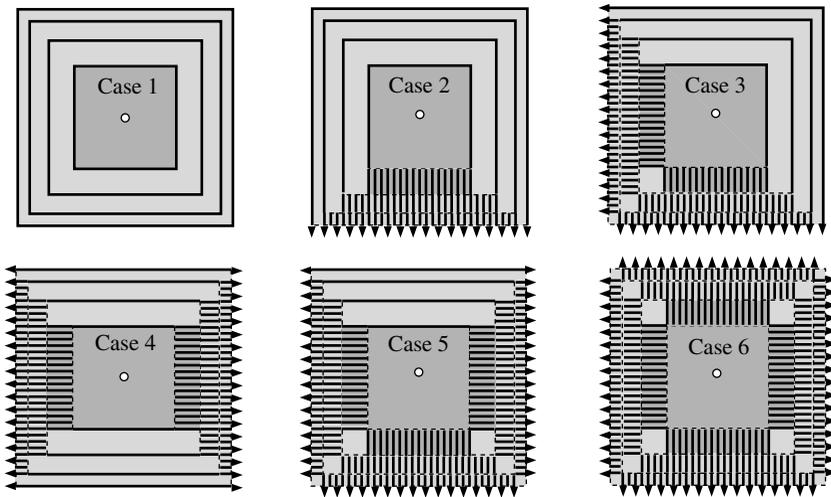


FIGURE 5. Foliating $\mathbb{L}^2 - \{pt\}$

Theorem 6.2. *Let K be a compact subset of \mathbb{L}^2 and suppose that \mathcal{F} is a foliation on $\mathbb{L}^2 - K$. Then there is a closed unbounded set $C \subset \mathbb{L}_+$ so that, up to rotation of the axes, leaves of \mathcal{F} take one of the following forms for each $\alpha \in C$:*

- (1) $(\{\pm\alpha\} \times [-\alpha, \alpha]) \cup ([-\alpha, \alpha] \times \{\pm\alpha\})$;
- (2) $(\{\pm\alpha\} \times (-\omega_1, \alpha]) \cup ([-\alpha, \alpha] \times \{\alpha\})$;
- (3) $(\{\alpha\} \times (-\omega_1, \alpha]) \cup ((-\omega_1, \alpha] \times \{\alpha\})$ and $(\{-\alpha\} \times (-\omega_1, -\alpha]) \cup ((-\omega_1, -\alpha] \times \{-\alpha\})$;
- (4) $(-\omega_1, \omega_1) \times \{\alpha\}$ and $(-\omega_1, \omega_1) \times \{-\alpha\}$;
- (5) $(-\omega_1, \omega_1) \times \{\alpha\}$, $((-\omega_1, -\alpha] \times \{-\alpha\}) \cup (\{-\alpha\} \times (-\omega_1, -\alpha])$ and $([\alpha, \omega_1) \times \{-\alpha\}) \cup (\{\alpha\} \times (-\omega_1, -\alpha])$;
- (6) $((-\omega_1, -\alpha] \times \{\alpha\}) \cup (\{-\alpha\} \times [\alpha, \omega_1))$, $([\alpha, \omega_1) \times \{\alpha\}) \cup (\{\alpha\} \times [\alpha, \omega_1))$, $((-\omega_1, -\alpha] \times \{-\alpha\}) \cup (\{-\alpha\} \times (-\omega_1, -\alpha])$ and $([\alpha, \omega_1) \times \{-\alpha\}) \cup (\{\alpha\} \times (-\omega_1, -\alpha])$.

Further, where there are unbounded leaves as described above, then C may be chosen so that for any $\alpha \in C$, sets of the form $\{x\} \times [\alpha, \omega_1)$, or appropriate variants of them with coordinates interchanged or multiplied by -1 , will lie entirely in one leaf of \mathcal{F} whenever $x \in [-\alpha, \alpha]$.

Proof. It is a matter of piecing together the four quadrants asymptotically foliated according to the two options of Proposition 6.1, while arranging a consistent patchwork along the four semi-diagonals using the fact that the intersection of finitely many closed unbounded sets is again closed and unbounded. \square

In the first case of Theorem 6.2 we do not have complete freedom to foliate the concentric annuli. As in Figure 2, there are essentially two ways to foliate an annulus with real lines and two circles so that the boundaries are leaves (cf. [7, pp. 57–59] or [10]). Because the set referred to in Case 1 (cf. Figure 5) is closed and unbounded, it follows that outside any given square we can observe only finitely many Reeb annuli (there may, of course, be infinitely many converging towards the puncture).

We now address the question: how may the regions in which the foliation is not prescribed by Theorem 6.2 be filled? The central regions, shaded in Figure 5, fall into six types denoted by C_i , $i = 1, \dots, 6$, respectively. Each C_i is understood as being partially foliated along its boundary (compare also Figure 6, where those subregions are depicted in a thick pen stroke). Besides, the peripheral subregions (cf. again Figure 5) are of two types: annuli in Case 1 and certain *road tracks* homeomorphic to the subregion C_4 (in all other cases). Note that C_3 reduces to C_4 (retract slightly the boundary prescription to liberate the corner of C_3). A crucial observation is that a road track admits a unique extension of its foliated boundary data to the whole interior (cf. Lemma 6.4 below due to Kerékjártó and Whitney).

Proposition 6.3. *None of the regions C_i for $2 \leq i \leq 6$, when punctured by a singleton, admits a foliated extension of its boundary data with a circle leaf.*

Proof. If there is a circle leaf, it must enclose the puncture in its interior (otherwise via Schoenflies we foliate the 2-disc). Removing from C_i the interior of this circle leaf, we obtain a compact bordered surface W_i . Plumbing two copies of W_i along the foliated boundary-data (see Figure 6) annihilates the mixed transverse-tangential behaviour along the boundary to render it purely tangential. This ‘plumbed-double’ is in each case a disc with a certain number of holes (say g_i). Counting holes in Figure 6, we find $g_2 = 2$, $g_3 = 3$, $g_4 = 3$, $g_5 = 4$ and $g_6 = 5$, but never 1, the only admissible value in view of the Euler-Poincaré obstruction. \square

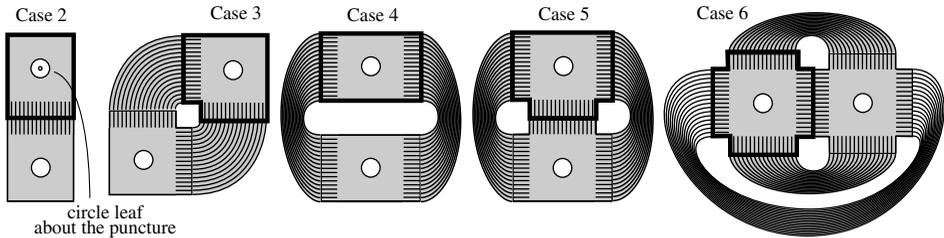


FIGURE 6. Different types of plumblings (Case 6 recalls Victor Hugo’s Gwynplain, alias Bessel-Hagen by Kerékjártó)

This describes the possible leaf types occurring for the sets C_i when punctured by a singleton: *For $i = 1$, only S^1 and \mathbb{R} can occur, while for $2 \leq i \leq 6$ only the circle is precluded.* Of course this is still far from a complete classification of

the foliations on $\mathbb{L}^2 - \{0\}$, where the main complication arises from the possible occurrences of real leaves (in the form of ‘petals’ about the puncture). Petals can be arranged into ‘flowers’ with finitely many petals or even countably many petals (of shrinking sizes). Next, one can nest many (non-nested) petals inside a given petal, and also plug a Reeb component between two nested petals, etc. Hence any classification looks quite complicated, yet is akin to Kaplan’s classification of foliations on the plane by means of chordal systems or via non-Hausdorff, second countable, simply-connected 1-manifolds (Haefliger-Reeb’s viewpoint).

The plumbing trick also allows one to understand the unpunctured case completely. First we present a lemma.

Lemma 6.4. *Let \mathcal{F} be a foliation on the square $S = [0, 1]^2$ extending the following boundary data: the two horizontal sides $[0, 1] \times \{0, 1\}$ of the square are leaves and the foliation is horizontal on thin strips $([0, \varepsilon] \cup [1 - \varepsilon, 1]) \times [0, 1]$ along the vertical sides (for some small $\varepsilon > 0$). Then there is a self-homeomorphism $h: S \rightarrow S$ such that the push-forward foliation $h_*\mathcal{F}$ becomes the (straight) horizontal foliation.*

Proof. The proof breaks into three steps.

Step 1. (Analysis of the possible leaf types via Poincaré-Bendixson.) Each leaf L of \mathcal{F} is an arc with extremities lying on the opposite sides $\Sigma_0 = \{0\} \times [0, 1]$ and $\Sigma_1 = \{1\} \times [0, 1]$ of the square. Since S is metric, an argument (of Chevalley-Haefliger, [4]) shows that the leaf L with the leaf topology remains second countable. Hence L is one of the four possible metric bordered 1-manifolds: namely \mathbb{S}^1 , \mathbb{R} , $[0, \infty)$ or $[0, 1]$. A circle leaf cannot occur (otherwise Schoenflies gives a foliation on the 2-disc). The two cases \mathbb{R} , $[0, \infty)$ are precluded by the Poincaré-Bendixson theorem (restrict the foliation to the interior of the square $(0, 1)^2$ and note that the unbordered-side of L cannot escape to infinity). Thus, the only possible leaf-type is $[0, 1]$. Such a leaf is forced to traverse the square, for if it turns back to the starting side, it cuts out a portion of S , which, when doubled, yields a foliated 2-disc.

Step 2. (Synchronising a Whitney flow.) To eliminate the transversal behaviour of \mathcal{F} along the vertical sides of S , extend the square S to the (infinite) strip $X = \mathbb{R} \times [0, 1]$ and propagate the foliation horizontally. Since X is simply-connected, the extended foliation \mathcal{F}_∞ on X is orientable, hence admits a compatible flow $\psi: \mathbb{R} \times X \rightarrow X$ (Whitney [20]). According to Step 1 (reversing time if necessary), each point s of Σ_0 is carried by the flow ψ to a point of Σ_1 after the elapsing of a certain amount of time $\tau(s)$ (which depends continuously on s). Via the time-change $\varphi(t, x) = \psi(\tau(s(x))t, x)$, where $s(x)$ denotes the unique point of the leaf through $x \in X$ lying on Σ_0 (existence guaranteed by Step 1), we get a new flow φ for which the required time to traverse from Σ_0 to Σ_1 is identically equal to 1.

Step 3. (The synchronised flow φ induces a global straightening to the horizontal foliation.) Restrict $\varphi: \mathbb{R} \times X \rightarrow X$ to $[0, 1] \times \Sigma_0$ to obtain the map $g: [0, 1] \times \Sigma_0 \rightarrow S$. It is easy to check that g is bijective. When (t, s) moves horizontally in the square $[0, 1] \times \Sigma_0$ (i.e. s fixed, t variable), the point $g(t, s)$ describes a specific leaf of \mathcal{F} . Hence the inverse homeomorphism $h = g^{-1}$ takes the foliation \mathcal{F} to the horizontal (straight) foliation on the square. □

Corollary 6.5. *Up to homeomorphism there are only two foliations on the long plane \mathbb{L}^2 given by the rectilinear models extending Cases (3) and (4) of Theorem 6.2. Up to isotopy there are only six foliations (four of the ‘broken type’ corresponding to Case (3) and two which are the product foliations).*

Proof. Apply Theorem 6.2 and consider the subregions C_i without punctures. The plumbed doubles of C_2 , C_5 and C_6 have non-zero Euler characteristic, so do not allow the corresponding foliations to extend. The case C_1 does not lead to a foliation on \mathbb{L}^2 , for it involves a foliated disc. Each of the remaining two cases, C_3 and C_4 , admits a unique foliated extension according to Lemma 6.4, which applies also to all the peripheral regions arising in Cases 3 and 4 of Figure 5. The classification up to isotopy follows from the ‘super-rigidity’ of the group of self-homeomorphisms of \mathbb{L}^2 isotopic to the identity map, [1, Theorem 1.1]. \square

The method of this section may be adapted to other situations. As an example, \mathbb{L}^2 is really just obtained by sewing eight copies of the octant $\{(x, y) \in \mathbb{L}^2 : 0 \leq y \leq x\}$ in a judicious way. We may sew together any finite number of such octants similarly and puncture the outcome to get many more long pipes and related manifolds. We may also wish to analyse dimension-one foliations on \mathbb{L}^n for $n \geq 3$.

Often foliations may also serve as a medium to distinguish similar looking manifolds undistinguished by algebraic invariants (homology or the fundamental group).

Corollary 6.6. *The rectangular quadrant $\mathbf{Q} = \mathbb{L}_{\geq 0} \times \mathbb{L}_{\geq 0}$ has a foliation tangent to the boundary, whereas the rhombic quadrant $\overline{\mathbf{Q}} = \{(x, y) \in \mathbb{L}^2 : -y \leq x \leq y\}$ does not. In particular the quadrants \mathbf{Q} and $\overline{\mathbf{Q}}$ are not homeomorphic.*

Proof. First, \mathbf{Q} has such a foliation by broken long lines $([\alpha, \omega_1] \times \{\alpha\}) \cup (\{\alpha\} \times [\alpha, \omega_1])$ with $\alpha \in \mathbb{L}_{\geq 0}$. In contrast, by Proposition 6.1 any foliation on $\overline{\mathbf{Q}}$ is either asymptotically horizontal or vertical. In both cases a singularity is created, where a horizontal (resp. vertical) leaf meets the boundary leaf. \square

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