

## AUSLANDER-REITEN THEORY OF SMALL HALF QUANTUM GROUPS

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ABSTRACT. For the small half quantum groups  $u_\zeta(\mathfrak{b})$  and  $u_\zeta(\mathfrak{n})$  we show that the components of the stable Auslander-Reiten quiver containing gradable modules are of the form  $\mathbb{Z}[\mathbb{A}_\infty]$ .

### INTRODUCTION

For a selfinjective algebra the shape of the stable Auslander-Reiten quiver is an invariant of its Morita equivalence class. It is the quiver consisting of the isomorphism classes of indecomposable non-projective modules as vertices and arrows corresponding to irreducible maps. Roughly speaking, a map is called irreducible if it cannot be written as the composition of two non-split maps. By a theorem of Riedtmann a connected component of this quiver always arises from a tree. This tree is called the tree class of the component. In 1995 Erdmann showed (in analogy to a result by Ringel for hereditary algebras) that all components of the stable Auslander-Reiten quiver belonging to a wild block have the same tree class  $\mathbb{A}_\infty$ . She also gave an analogue of this result for wild local restricted enveloping algebras.

In 1990 Lusztig defined a quantum analogue of the restricted enveloping algebra, called the small quantum group  $u_\zeta(\mathfrak{g})$ . Its Borel and nilpotent parts  $u_\zeta(\mathfrak{b})$  and  $u_\zeta(\mathfrak{n})$  were shown to have wild representation type by Feldvoss and Witherspoon for  $\mathfrak{g} \neq \mathfrak{sl}_2$  in [FW09, FW11] (a generalization of a result by Cibils [Cib97]). In this paper we give an analogue of Erdmann's Theorem for the subcategory of restrictions of  $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -modules. This subcategory consists of  $\mathbb{Z}^n$ -gradable modules and can be seen as a quantization of Jantzen's category of  $B_1T$ -modules.

**Main Theorem 1.** *Let  $\mathfrak{g} \neq \mathfrak{sl}_2$ . Let  $\mathcal{C}$  be a component of the stable Auslander-Reiten quiver of  $u_\zeta(\mathfrak{b})$  or  $u_\zeta(\mathfrak{n})$  containing the restriction of a  $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -module. Then  $\mathcal{C}$  has tree class  $\mathbb{A}_\infty$ .*

The main ingredients in the proof are results of Scherotzke on the Auslander-Reiten quiver of a skew group algebra and an analogue of Dade's Lemma for small quantum groups provided by Drupieski.

Our paper is organized as follows: In Section 1 we recall the basic definitions and fix our notation. Section 2 states Kerner and Zacharia's analogue of Webb's Theorem that restricts the tree classes of non-periodic components to Euclidean

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and infinite Dynkin diagrams and describes the periodic components in more detail. Section 3 is concerned with the graded module category and also provides an analogue of Webb's Theorem in this case. Sections 4 and 5 then exclude the Euclidean tree classes and the other infinite Dynkin tree classes, respectively.

## 1. PRELIMINARIES

Let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ . All vector spaces will be assumed to be finite dimensional unless stated otherwise. For a general introduction to Auslander-Reiten theory we refer the reader to [ARS95] or [ASS06]. We denote the syzygy functor by  $\Omega$  and the Auslander-Reiten translation by  $\tau$ . For a Frobenius algebra we denote its Nakayama automorphism by  $\nu$ .

Let  $\mathfrak{g}$  be a finite dimensional simple complex Lie algebra. Denote the corresponding set of roots by  $\Phi$ , a chosen set of simple roots by  $\Pi$ , the corresponding set of positive roots by  $\Phi^+$ , and the Coxeter number by  $h$ . Let  $\ell > 1$  be an odd integer not divisible by three if the corresponding root system is of type  $\mathbb{G}_2$ . Let  $\zeta$  be a primitive  $\ell$ -th root of unity. Let  $U_\zeta(\mathfrak{g})$  be the Lusztig form of the quantized enveloping algebra at the root of unity  $\zeta$  (see e.g. [CP94, (9.2)]). The finite dimensional subalgebra  $u_\zeta(\mathfrak{g}) := \langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \mid \alpha \in \Pi \rangle$  is called the small quantum group (for a general introduction, see e.g. [Dru11]). It has a triangular decomposition  $u_\zeta(\mathfrak{g}) \cong u_\zeta(\mathfrak{n}) \otimes u_\zeta^0(\mathfrak{g}) \otimes u_\zeta(\mathfrak{n}^+)$ , and we denote its Borel part by  $u_\zeta(\mathfrak{b}) := u_\zeta(\mathfrak{n})u_\zeta^0(\mathfrak{g})$ . The nilpotent and the Borel parts are linked via a skew algebra construction  $u_\zeta(\mathfrak{b}) = u_\zeta(\mathfrak{n}) * (\mathbb{Z}/(\ell))^n$ , where  $n = |\Pi|$ . The zero part of the triangular decomposition of  $U_\zeta(\mathfrak{g})$  will be denoted  $U_\zeta^0(\mathfrak{g})$ . Furthermore both  $u_\zeta(\mathfrak{n})$  and  $u_\zeta(\mathfrak{b})$  have only one block.

In [FW09, FW11] Feldvoss and Witherspoon have shown that for  $\mathfrak{g} \neq \mathfrak{sl}_2$  the connected algebras  $u_\zeta(\mathfrak{b})$  and  $u_\zeta(\mathfrak{n})$  are wild. (Note the Erratum [FW13].)

For our considerations the following analogue of Dade's Lemma provided by Drupieski is also essential:

**Proposition 1.1.** *The restriction of a  $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -module  $M$  to  $u_\zeta(\mathfrak{b})$  is projective iff it is projective when restricted to every Nakayama subalgebra  $u_\zeta(f_\alpha) := \langle F_\alpha \rangle$ , where  $\alpha \in \Phi^+$ .*

## 2. WEBB'S THEOREM AND PERIODIC COMPONENTS

In this section we apply a theorem by Kerner and Zacharia to restrict the possible shapes of components to those arising from Euclidean or infinite Dynkin diagrams in the following fashion: For such a diagram  $\Delta$  fix an orientation (for  $\tilde{\mathbb{A}}_n$  a non-oriented cycle). Then the diagram  $\mathbb{Z}[\Delta]$  has vertices  $(i, j)$ , where  $i \in \mathbb{Z}$  and  $j \in \Delta$ , and arrows  $(i, j) \rightarrow (i, j')$  and  $(i-1, j') \rightarrow (i, j)$  for each arrow  $j \rightarrow j'$  in  $\Delta$  and all  $i \in \mathbb{Z}$ .

That all components have a particular shape mainly follows from the fact that  $u_\zeta(\mathfrak{b})$  is an (fg)-Hopf algebra in the following sense:

A finite dimensional Hopf algebra  $A$  is called an (fg)-Hopf algebra if the even cohomology ring  $H^{ev}(A, k)$  is finitely generated and each  $\text{Ext}^\bullet(M, N)$  is finitely generated as a module for the even cohomology ring (via cup product).

For general theory on (fg)-Hopf algebras we refer the reader to [FW09], [Kül12] and also to the group algebra case [Ben91]. Their definition is designed to formulate a theory of support varietes that has essential features arising in the context of

finite groups. To every module  $M$  one can associate a variety, the variety associated to the ideal  $\ker \Phi_M$ , where  $\Phi_M : H^{ev}(A, k) \rightarrow \text{Ext}^\bullet(M, M)$  is the map induced by tensoring with  $M$ . This so-called support variety  $\mathcal{V}_A(M)$  detects certain properties of the module  $M$ . Recall that a component  $\mathcal{C}$  of the stable Auslander–Reiten quiver is called **periodic** if there exist  $M \in \mathcal{C}$  and  $m \in \mathbb{N}$  such that  $\tau^m M \cong M$ .

**Theorem 2.1.** *The non-periodic components  $\mathcal{C}$  for  $u_\zeta(\mathfrak{b})$  are of the form  $\mathbb{Z}[\Delta]$ , where  $\Delta$  is a Euclidean or infinite Dynkin diagram.*

*Proof.* The result follows from the fact that  $u_\zeta(\mathfrak{b})$  satisfies (fg) (see e.g. [Dru10, Theorem 6.2.6]). This in particular implies that the complexity of each module is finite. Hence the result follows from [KZ11, Main Theorem].  $\square$

For the algebras  $u_\zeta(\mathfrak{n})$  this also holds since projective modules stay projective under restriction.

**Theorem 2.2.** *The non-periodic components for the algebra  $u_\zeta(\mathfrak{n})$  are of the form  $\mathbb{Z}[\Delta]$ , where  $\Delta$  is a Euclidean or infinite Dynkin diagram.*

*Proof.* The result follows from the fact that the restriction of a projective  $u_\zeta(\mathfrak{b})$ -module is projective for  $u_\zeta(\mathfrak{n})$  by [Dru11, Remark 3.4]: For  $M \in \text{mod } u_\zeta(\mathfrak{n})$  take a minimal  $u_\zeta(\mathfrak{b})$ -projective resolution  $P_*$  of the induced module  $u_\zeta(\mathfrak{b}) \otimes_{u_\zeta(\mathfrak{n})} M$ . The rate of growth of this projective resolution is finite, as  $u_\zeta(\mathfrak{b})$  is a finite dimensional Hopf algebra satisfying (fg). If we restrict this resolution to  $u_\zeta(\mathfrak{n})$  we will get a projective resolution of  $M$  which has finite rate of growth, as  $M$  is a direct summand of the restriction of  $u_\zeta(\mathfrak{b}) \otimes_{u_\zeta(\mathfrak{n})} M$  by [RR85, Proposition 1.8]. Therefore the complexity of  $M$  is finite. So the result again follows from [KZ11, Main Theorem].  $\square$

We proceed by narrowing down the possibilities of periodic components that by a classical result of Happel, Preiser and Ringel are always finite or infinite tubes (i.e. of the form  $\mathbb{Z}[\mathbb{A}_\infty]/\tau^m$  for some  $m \in \mathbb{N}$ ). The case of finite components does not occur in our context since finite components correspond to representation-finite blocks by a classical theorem of Auslander; these do not appear as shown in [FW09, FW11]. Here we need to restrict our parameters. Recall that for a root system  $\Phi$  an integer  $\ell$  is called **good** for  $\Phi$  if  $\ell \geq 3$  for types  $\mathbb{B}_n, \mathbb{C}_n$  and  $\mathbb{D}_n, \ell \geq 5$  for types  $\mathbb{E}_6, \mathbb{E}_7$  and  $\mathbb{G}_2$  and  $\ell \geq 7$  for  $\mathbb{E}_8$ . Otherwise it is called **bad**.

**Proposition 2.3.** *For char  $k = 0$  let  $\ell$  be good for  $\Phi$  and  $\ell > 3$  for types  $\mathbb{B}$  and  $\mathbb{C}$  and  $\ell \nmid n + 1$  for type  $\mathbb{A}_n$  and  $\ell \neq 9$  for  $\mathbb{E}_6$ . For char  $k = p > 0$  let  $p$  be good for  $\Phi$  and  $\ell \geq h$ . Then the  $\Omega$ -period of every periodic module for  $u_\zeta(\mathfrak{n})$  divides 2 while the  $\tau$ -period is 1. Furthermore any  $u_\zeta(\mathfrak{b})$ -module of complexity 1 is  $\Omega_{u_\zeta(\mathfrak{b})}$ - and  $\tau_{u_\zeta(\mathfrak{b})}$ -periodic, and the  $\Omega_{u_\zeta(\mathfrak{b})}$ -period of a module  $M$  divides  $2\ell$  while the  $\tau$ -period divides  $\ell$ .*

*Proof.* In order to prove the claimed periodicity of  $\Omega_{u_\zeta(\mathfrak{n})}$  and  $\tau_{u_\zeta(\mathfrak{n})}$  we apply [EHT<sup>+</sup>04, Proposition 5.4] to the subalgebra  $\tilde{H} := HH^0(u_\zeta(\mathfrak{n})) \cdot H^{ev}(u_\zeta(\mathfrak{n}), k)^{\mathbb{Z}/(\ell)^n}$  of the Hochschild cohomology  $HH^\bullet(u_\zeta(\mathfrak{n}))$  (compare [FW11, Section 5]). As  $H^{ev}(u_\zeta(\mathfrak{n}), k)^{\mathbb{Z}/(\ell)^n} \cong H^{ev}(u_\zeta(\mathfrak{b}), k)$  as algebras, the finite generation of  $\tilde{H}$  in degree 2 follows from [Dru11, Theorem 5.13]. The arguments of [FW11, Section 5] carry through in our slightly more general setting to show the appropriate (Fg)-conditions for Hochschild cohomology.

For the second part we use skew group algebra arguments: It follows from [Sch09, Lemma 5.3(i)] that if  $M, N$  are indecomposable  $u_\zeta(\mathfrak{b})$ -modules with  $M|_{u_\zeta(\mathfrak{n})} \cong N|_{u_\zeta(\mathfrak{n})}$ , then  $M \cong N \otimes k_\lambda$ . Furthermore we have  $\Omega_{u_\zeta(\mathfrak{b})}^2(M)|_{u_\zeta(\mathfrak{n})} \cong \Omega_{u_\zeta(\mathfrak{n})}^2(M|_{u_\zeta(\mathfrak{n})})$ . This follows from the fact that  $(\text{top}_{u_\zeta(\mathfrak{b})} M)|_{u_\zeta(\mathfrak{n})} = \text{top}_{u_\zeta(\mathfrak{n})} M|_{u_\zeta(\mathfrak{n})}$ , as all simple (respectively indecomposable projective)  $u_\zeta(\mathfrak{b})$ -modules have the same dimension (see e.g. [Sch09, Lemma 5.1]). Indeed let  $t := \dim \text{top}_{u_\zeta(\mathfrak{b})} M$ ; then restricting  $\bigoplus_{j=1}^t P(k_{\lambda_j}) \twoheadrightarrow M$  to  $u_\zeta(\mathfrak{n})$  yields  $u_\zeta(\mathfrak{n})^t \twoheadrightarrow M$ , hence  $t \geq \dim \text{top}_{u_\zeta(\mathfrak{n})} M$ . On the other hand we have  $M \twoheadrightarrow \bigoplus_{j=1}^t k_{\lambda_j}$ , and restricting yields  $M|_{u_\zeta(\mathfrak{n})} \twoheadrightarrow k^t$ , whence  $t \leq \dim \text{top}_{u_\zeta(\mathfrak{n})} M$ .

Thus as  $M|_{u_\zeta(\mathfrak{n})} \cong \Omega_{u_\zeta(\mathfrak{n})}^2(M|_{u_\zeta(\mathfrak{n})}) \cong \Omega_{u_\zeta(\mathfrak{b})}^2(M)|_{u_\zeta(\mathfrak{n})}$  we have that  $\Omega_{u_\zeta(\mathfrak{b})}^2(M) \cong M \otimes k_\lambda$  for some  $\lambda \in \mathbb{Z}/(\ell)^n$ . As the order of any element of  $\mathbb{Z}/(\ell)^n$  divides  $\ell$ , it now follows that the order of  $\Omega$  divides  $2\ell$ .

By a similar reasoning as in [Kül12, Section 5] one can show that  $u_\zeta(\mathfrak{b})$  is a  $\gamma$ -Frobenius extension of  $u_\zeta(f_\beta) := \langle F_\beta \rangle$  for some automorphism  $\gamma$ ; i.e.  $u_\zeta(\mathfrak{b})$  is free over  $u_\zeta(f_\beta)$  and there is an isomorphism of  $u_\zeta(\mathfrak{b})$ - $u_\zeta(f_\beta)$ -bimodules  $u_\zeta(\mathfrak{b}) \cong \text{Hom}_{u_\zeta(f_\beta)}(u_\zeta(\mathfrak{b}), u_\zeta(f_\beta)^{(\gamma)})$ . Here  $u_\zeta(f_\beta)^{(\gamma)}$  is the module defined via  $u \cdot v := \gamma(u)v$  for all  $u \in u_\zeta(\mathfrak{b}), v \in u_\zeta(f_\beta)$ . As the latter is a Frobenius algebra, it follows from [BF93, Proposition 1.3] that  $u_\zeta(\mathfrak{b})$  is also a Frobenius algebra. From this construction we get the Frobenius homomorphism induced by  $F^{\ell-1}K^{\ell-1} \mapsto 1$ . Now the commutation relations from [dCK90, Proposition 1.7] in the graded algebra yield a Nakayama automorphism with  $K_\alpha \mapsto \zeta^{-\sum_{\beta \in \Phi^+} \langle \alpha, \beta \rangle} K_\alpha$  and  $F_\beta \mapsto \zeta^{\sum_{\alpha' \in \Pi} \langle \alpha', \beta \rangle} F_\beta$ . As  $\zeta$  is an  $\ell$ -th root of unity, its order divides  $\ell$ . As for Frobenius algebras we have  $\tau \cong \Omega^2 \circ \nu$ ; it follows that the  $\tau_{u_\zeta(\mathfrak{b})}$ -period divides  $\ell$ .  $\square$

*Remark 2.4.* The bound of two for the  $\Omega_{u_\zeta(\mathfrak{n})}$ -period is sharp, e.g. for  $\mathfrak{g} = \mathfrak{sl}_2$ . For the Borel part the bound on the  $\tau$ -period is sharp, e.g. for  $\mathfrak{g} = \mathfrak{sl}_2$ .

### 3. GRADED MODULES

We proceed by studying the  $\mathbb{Z}^n$ -graded modules for the small half quantum groups.

**Lemma 3.1.** *Let  $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$ .*

- (i) *The category of finite dimensional modules over  $u_\zeta(L)U_\zeta^0(\mathfrak{g})$  is a sum of blocks for the category  $\text{mod } u_\zeta(L)\#U_\zeta^0(\mathfrak{g})$ .*
- (ii) *The category of finite dimensional  $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ -modules has almost split sequences.*
- (iii) *The canonical restriction functor  $\text{mod } u_\zeta(L)U_\zeta^0(\mathfrak{g}) \rightarrow \text{mod } u_\zeta(L)$  sends indecomposables to indecomposables and almost split sequences to almost split sequences.*
- (iv) *The restriction functor induces a homomorphism  $\mathcal{F} : \Gamma_s(u_\zeta(L)U_\zeta^0(\mathfrak{g})) \rightarrow \Gamma_s(u_\zeta(L))$  of stable translation quivers and components are mapped to components via this functor.*

*Proof.* The result follows as in [Kül12, Section 5].  $\square$

An analogue of Webb’s Theorem also holds for the category of graded modules for small quantum groups and their half analogues by constructing a subadditive function on the components. Our reasoning is similar to [Far05] for restricted enveloping algebras.

**Theorem 3.2.** *The tree classes of the components of the stable Auslander-Reiten quiver of  $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ , where  $L \in \{\mathfrak{g}, \mathfrak{b}, \mathfrak{n}\}$ , are Euclidean diagrams or infinite Dynkin diagrams.*

*Proof.* Let  $\Theta$  be a non-periodic component of the stable Auslander-Reiten quiver of the category of finite dimensional  $u_\zeta(L)U^0(\mathfrak{g})$ . Let  $[M'] \in \Theta$ . By [Dru11, Theorem 4.3] and the foregoing lemma there exists  $\alpha \in \Phi^+$  such that the restriction  $\mathcal{F}(M')|_{u_\zeta(f_\alpha)}$  is not injective. Consider the induced module  $M_\alpha := u_\zeta(L) \otimes_{u_\zeta(f_\alpha)} k$ . The function  $d_\alpha : \Theta \rightarrow \mathbb{N}_0, [M] \mapsto \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M))$  is a subadditive  $\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}$ -invariant function on  $\Theta$ : We have that  $\mathcal{F}(\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M)) \cong \tau_{u_\zeta(L)}(\mathcal{F}(M)) \cong \Omega_{u_\zeta(L)}^2(\mathcal{F}(M)^{(\nu)})$  by the foregoing lemma,  $M_\alpha^{(\nu)} \cong M_\alpha$  since  $\nu|_{u_\zeta(f_\alpha)} = \text{id}$  ( $\nu \otimes \text{id}$  provides an isomorphism), and  $\Omega_{u_\zeta(L)}^2(M_\alpha) \cong M_\alpha \oplus P$  for a projective module  $P$  by inducing a projective  $u_\zeta(f_\alpha)$ -resolution of  $k$  to  $u_\zeta(L)$ . Therefore we obtain:

$$\begin{aligned} d_\alpha([\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M)]) &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(\tau_{u_\zeta(L)U_\zeta^0(\mathfrak{g})}(M))) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(\Omega_{u_\zeta(L)}^2(M_\alpha), \Omega_{u_\zeta(L)}^2(\mathcal{F}(M)^{(\nu)})) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M)^{(\nu)}) \\ &= \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha^{(\nu)}, \mathcal{F}(M)^{(\nu)}) = d_\alpha([M]); \end{aligned}$$

i.e.  $d_\alpha$  is  $\tau$ -invariant. As  $\Theta$  is non-periodic we have that the  $\tau$ -periodic module  $M_\alpha$  does not belong to  $\Theta \cup \Omega(\Theta)$ . Thus we can apply [ES92, Lemma 3.2] to  $\Omega M_\alpha$  to get that  $d_\alpha$  is a subadditive  $\tau$ -periodic function on  $\Theta$ .

It remains to prove that  $d_\alpha \neq 0$ . By [Far, Proposition 1.1(1)] it suffices to prove that  $d_\alpha$  is non-zero at one point of  $\Theta$ . Since we have that  $u_\zeta(L) : u_\zeta(f_\alpha)$  is a Frobenius extension of the second kind we conclude that

$$d_\alpha([M']) = \dim \text{Ext}_{u_\zeta(L)}^1(M_\alpha, \mathcal{F}(M')) = \dim \text{Ext}_{u_\zeta(f_\alpha)}^1(k, \mathcal{F}(M')|_{u_\zeta(f_\alpha)}) \neq 0.$$

Thus the statement follows. □

*Remark 3.3.* This provides a different way to prove the result for components of  $u_\zeta(L)$  consisting of gradable modules: The function which is obtained by removing the forgetful functor  $\mathcal{F}$  in the foregoing proposition provides a subadditive  $\tau_{u_\zeta(L)}$ -invariant function for every component of the stable Auslander-Reiten quiver of  $u_\zeta(L)$  containing gradable modules. The foregoing proof applies verbatim (replacing  $u_\zeta(L)U_\zeta^0(\mathfrak{g})$  with  $u_\zeta(L)$  and removing  $\mathcal{F}$  everywhere). Thus this provides us with a subadditive  $\tau$ -invariant function on  $\Theta$ , a component of  $u_\zeta(L)U_\zeta^0(\mathfrak{g})$ , that stays subadditive  $\tau$ -invariant upon restriction. This way one can get information on  $\Theta$  or  $\mathcal{F}(\Theta)$  by observing the other.

#### 4. EUCLIDEAN COMPONENTS

In this section we exclude components of the form  $\mathbb{Z}[\Delta]$ , where  $\Delta$  is a Euclidean diagram. Our approach relies on results by Scheretzke [Sch09], who corrected and generalized results of Farnsteiner for restricted enveloping algebras [Far99]. To apply it we need the following statement:

**Proposition 4.1.** *For  $\text{char } k = 0$  let  $\ell$  be good for  $\Phi$  and  $\ell > 3$  for types  $\mathbb{B}$  and  $\mathbb{C}$  and  $\ell \nmid n + 1$  for type  $\mathbb{A}_n$  and  $\ell \neq 9$  for  $\mathbb{E}_6$ . For  $\text{char } k = p > 0$  let  $p$  be good for  $\Phi$  and  $\ell \geq h$ . If  $\mathfrak{b} \neq \mathfrak{b}_{\text{sl}_2}$  there is a non-periodic module of length 3 for  $u_\zeta(\mathfrak{b})$ .*

*Proof.* Let  $\mathcal{S}$  be a set of representatives for the isoclasses of the simple  $u_\zeta(\mathfrak{b})$ -modules. As  $u_\zeta(\mathfrak{b})$  is not a Nakayama algebra, it follows from [Hup81, Theorem 9] that there is a simple module  $S$  such that

$$\sum_{[T] \in \mathcal{S}} \dim \text{Ext}_{u_\zeta(\mathfrak{b})}^1(T, S) \geq 2.$$

Let  $P$  be the injective hull of  $S$ . By the definition of  $\text{Ext}^1$  and  $\text{soc}^2$  we immediately obtain

$$2 \leq \sum_{[T] \in \mathcal{S}_{u_\zeta(\mathfrak{b})}} \dim \text{Hom}_{u_\zeta(\mathfrak{b})}(T, \text{soc}^2(P)/\text{soc}(P)),$$

so that  $l(\text{soc}^2(P)/\text{soc} P) \geq 2$  and  $l(\text{soc}^2(P)) \geq 3$ .

Denote by  $\pi : P \rightarrow P/\text{soc}(P)$  the natural projection. Then  $\pi$  induces a surjection  $\text{soc}^2(P) \twoheadrightarrow \text{soc}(P/\text{soc}(P))$  and  $\pi(\text{rad } u_\zeta(\mathfrak{b}) \text{ soc}^2(P)) = 0$ . Thus  $\text{rad } u_\zeta(\mathfrak{b}) \text{ soc}^2(P) = \text{soc}(P)$ . Now let  $X \subset \text{soc}^2(P)$  be a submodule of length 3. Then  $\text{soc } X = \text{soc } P$  and  $\text{rad } u_\zeta(\mathfrak{b}) X \subseteq \text{rad } u_\zeta(\mathfrak{b}) \text{ soc}^2(P) = \text{soc}(X)$ , so that  $\text{rad } u_\zeta(\mathfrak{b}) X = \text{soc } X$ . Thus  $\text{top } X = X/\text{soc } X$  has length 2. Suppose that  $X$  was periodic. Then by the proof of Proposition 2.3,  $\dim \Omega_{u_\zeta(\mathfrak{b})}^2 X = \dim X$ . As  $P$  is the injective hull of  $\text{soc } X$  there exist a projective module  $Q$  and an exact sequence

$$0 \rightarrow \Omega^2 X \rightarrow P \rightarrow Q \rightarrow X \rightarrow 0.$$

Therefore  $\dim P = \dim Q$ , so that  $Q$  is indecomposable, as all projective indecomposable modules have the same dimension by [Dru09, p. 85]. But as  $u_\zeta(\mathfrak{b})$  is selfinjective this would mean that  $\text{top } X$  is irreducible and therefore has length 1, a contradiction.  $\square$

**Theorem 4.2.** *For the stable Auslander-Reiten quivers of  $u_\zeta(\mathfrak{b})$  and  $u_\zeta(\mathfrak{n})$ , if a component is isomorphic to  $\mathbb{Z}[\Delta]$  with  $\Delta$  Euclidean, then  $\Delta$  is  $\tilde{\mathbb{D}}_n$ , where  $n > 5$ . If we further impose that  $\ell > h$ , then also these components cannot occur.*

*Proof.* As  $(\mathbb{Z}/\ell\mathbb{Z})^n$  is isomorphic to its character group we can define an action of  $(\mathbb{Z}/\ell\mathbb{Z})^n$  on  $u_\zeta(\mathfrak{b})$  by setting  $g(u * h) = u * \chi_{g^{-1}}(h)h$  for all  $u \in u_\zeta(\mathfrak{b})$  and all  $g, h \in (\mathbb{Z}/\ell\mathbb{Z})^n$ . With this action  $(\mathbb{Z}/\ell\mathbb{Z})^n$  is a subgroup of  $\text{Aut}(u_\zeta(\mathfrak{b}))$ . We have that  $(\mathbb{Z}/\ell)^n$  acts transitively on the simple  $u_\zeta(\mathfrak{b})$ -modules and all modules of complexity 1 are  $\Omega$ - and  $\tau$ -periodic, as  $u_\zeta(\mathfrak{b})$  is an (fg)-Hopf algebra and by Proposition 2.3. By a result of Scherotzke [Sch09, Theorem 3.3, Theorem 3.7] in this situation the components of Euclidean tree class of  $u_\zeta(\mathfrak{b})$  can only be of the form  $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$  or  $\mathbb{Z}[\tilde{\mathbb{D}}_n]$ , where  $n > 5$ . In the case of  $\mathbb{Z}[\tilde{\mathbb{A}}_{12}]$  the projective module would have dimension 4 by a result of Erdmann [Erd90, Theorem IV.3.8.3], which is not possible, as the dimension of each projective module is divisible by  $\ell$ . But since by the foregoing proposition there is a non-periodic module of length 3, [Sch09, Theorem 3.11] tells us that the remaining case of tree class  $\tilde{\mathbb{D}}_n$  cannot occur because in this case the projective module would have dimension 8.

For  $u_\zeta(\mathfrak{n})$  note that  $u_\zeta(\mathfrak{b})$  is a skew group algebra of  $u_\zeta(\mathfrak{n})$ . Therefore also  $u_\zeta(\mathfrak{n})$  satisfies the conditions of [Sch09, Theorem 3.3, Theorem 3.7] for  $G = \{e\}$  by the fact that  $u_\zeta(\mathfrak{n})$  has only one simple module. Thus as  $u_\zeta(\mathfrak{b})$  has no components of tree class  $\tilde{\mathbb{D}}_n$  or  $\tilde{\mathbb{A}}_{12}$ , also  $u_\zeta(\mathfrak{n})$  has no components of tree class  $\tilde{\mathbb{D}}_n$  or  $\tilde{\mathbb{A}}_{12}$  by [Sch09, Theorem 5.15].

It remains to consider the case of a component  $\Theta$  of type  $\mathbb{Z}[\tilde{\mathbb{A}}_n]$ . For  $u_\zeta(\mathfrak{b})$  let  $G = (\mathbb{Z}/\ell)^n$  and for  $u_\zeta(\mathfrak{n})$  let  $G = \{e\}$ . Let  $U = u_\zeta(\mathfrak{b})$ , respectively  $U = u_\zeta(\mathfrak{n})$ . Such

a component is attached to a principal indecomposable module (see e.g. [KZ11]). For  $\lambda \in G$  let  $k_\lambda$  be the simple module corresponding to  $\lambda$ . Since the principal indecomposable modules are all related via tensoring with a  $k_\lambda$  for some  $\lambda \in G$ , the same holds true for these components. Since  $\Omega\Theta \cong \Theta$  there exists  $\lambda \in G$  such that the automorphism  $\varphi : \Gamma_s(U) \rightarrow \Gamma_s(U), [M] \mapsto [\Omega(M \otimes k_\lambda)]$  satisfies  $\varphi(\Theta) = \Theta$ . Note that  $(\varphi|_\Theta)^{2\ell} = \Omega^{2\ell} = \tau^\ell$ . In [Far99, Lemma 2.2] Farnsteiner has computed the automorphism group of  $\mathbb{Z}[\tilde{A}_n]$ . It is  $\{\tau^q \circ \alpha^r | q \in \mathbb{Z}, 0 \leq r \leq n - 1\}$ , where  $\alpha$  is defined via  $\alpha(s, [i]) = (s, [i + 1])$  with the vertex set of  $\mathbb{Z}[\tilde{A}_n]$  identified with  $\mathbb{Z} \times \mathbb{Z}/(n)$  and  $\tau(s, [i]) = (s - 1, [i])$ . In this notation we write  $\varphi|_\Theta = \tau^j \circ \alpha^r$  and obtain  $\tau^\ell = \tau^{2\ell j} \circ \alpha^{2\ell r}$ , whence  $\tau^{\ell(2j-1)} = \alpha^{-2\ell r}$ , a contradiction.  $\square$

### 5. INFINITE DYNKIN TREE CLASS

In the last section we have seen that Euclidean tree classes do not occur for components of the stable Auslander-Reiten quiver containing  $u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g})$ -modules. In this section we want to exclude two of the remaining possible tree classes so that only one possible tree class remains. Our proof follows the strategy of Erdmann for local restricted enveloping algebras in [Erd96]. A similar strategy has also been used by Bergh and Erdmann for quantum complete intersections; see [BE11].

**Proposition 5.1.** *Let char  $k$  be odd or zero and good for  $\Phi$ . Let  $L \in \{\mathfrak{b}, \mathfrak{n}\}$ . Assume  $\ell \geq h$  and  $\mathfrak{g} \neq \mathfrak{sl}_2$ . If  $M$  is a projective module or a periodic  $u_\zeta(L)$ -module, which is also a  $u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g})$ -module, then  $\dim M \equiv 0 \pmod{\ell}$ .*

*Proof.* If  $M$  is projective, it stays projective when restricting to  $u_\zeta(f_\alpha)$  by [Dru10, Theorem 4.3]. Hence its dimension is divisible by  $\ell$ .

Now let  $M$  be periodic and indecomposable; then we first show that  $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$  is a line (the proof is an adaption of [Ben91, Proposition 5.10.2]). Since  $M$  is periodic there exists a minimal  $n$  such that there is an isomorphism  $x : \Omega^n M \rightarrow M$ . One can regard  $x$  as an element of  $\text{Ext}^n(M, M)$ . Its powers with respect to the cup product  $x^q$  are also isomorphisms in the stable category as  $\Omega$  is a functor. Thus we have that  $\text{Ext}^{nq}(M, M) = x^q \cdot \underline{\text{End}}(M)$ . Now we look at  $\Phi_M : H^{ev}(u_\zeta(\mathfrak{b}), k) \rightarrow \text{Ext}^\bullet(M, M)$ . An adaption of the argument of [S-A04, Theorem 1.7] (see [KPhd, Proposition 3.1.7]) shows that its image lies in the centre. Let  $y \in H^{2nq}(u_\zeta(\mathfrak{b}), k)$ ; then  $\Phi_M(y) = x^{2q} \cdot f$ , where  $f \in \underline{\text{End}}(M)$ . We claim that  $xf = fx$ . Indeed we have  $x^{2q}(xf - fx) = 0$  by associativity of the cup product and since  $x^{2q}f$  lies in the centre of  $\text{Ext}^\bullet(M, M)$ . But since  $x^{2q}$  is an isomorphism and  $\Omega$  sends isomorphisms to isomorphisms, we have that  $fx = xf$ . Hence  $\Phi_M(y^m) = x^{2qm} f^m$ . Since  $M$  is not projective there is  $y_0$  of minimal degree such that the corresponding  $f$  is invertible (otherwise its support variety would be zero). We claim that  $H^{ev}(u_\zeta(\mathfrak{b}), k) / \ker \Phi_M \cong k[y_0] \oplus (\text{nilpotents})$ . Indeed let  $y : \Omega^m M \rightarrow M$  be in the image of  $\Phi_M$  for some  $m \notin n\mathbb{Z}$ ; then  $y^n = x^q \cdot g$  for some  $q \in \mathbb{N}$  and some  $g \in \underline{\text{End}}(M)$ . If  $g$  were invertible, then  $y$  also would be, a contradiction to the minimality of  $n$ . Otherwise  $g$  is nilpotent since  $\text{End}(M)$  is local, and hence (as  $y \in Z(\text{Ext}^\bullet(M, M))$  commutes with  $x^q$ )  $y$  is nilpotent as well. For the degrees  $n\mathbb{Z}$  we have  $H^{2nqm}(u_\zeta(\mathfrak{b}), k) / \ker \Phi_M \cong x^{2qm} f^m \underline{\text{End}}(M) \cong (x^{2qm} f^m) \cdot k \oplus (\text{nilpotents})$  since  $\text{End}(M)$  is a local ring with residue class field  $k$ . In conclusion we have that the support variety being the maximal ideal spectrum of  $H^{ev}(u_\zeta(\mathfrak{b}), k) / \ker \Phi_M$  is a line.

Note that by a generalization of the theorem of Ginzburg and Kumar we have that the support variety of a  $u_\zeta(\mathfrak{b})$ -module identifies with a conical subvariety of  $\mathfrak{n}$  (for our restrictions on the parameter see [Dru11, Theorem 5.2]). Thus there exists  $\alpha \in \Phi^+$  such that  $f_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$ . Hence by [Dru10, Corollary 5.13] we have that  $M|_{u_\zeta(f_\alpha)}$  is projective, hence  $\dim M \equiv 0 \pmod{\ell}$ .  $\square$

**Theorem 5.2.** *Let  $\text{char } k$  be odd or zero and good for  $\Phi$ . Assume  $\ell \geq h$ . The category of  $u_\zeta(\mathfrak{n})$ -modules that are also  $u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g})$ -modules does not contain any components of type  $\mathbb{Z}[\mathbb{A}_\infty]$  or type  $\mathbb{Z}[\mathbb{D}_\infty]$ .*

*Proof.* By [Far00, Lemma 2.5], which is valid in the context of selfinjective algebras, if we have a component  $\Theta$  of type  $\mathbb{A}_\infty$  or  $\mathbb{D}_\infty$ , then there exists an irreducible map  $\psi$  corresponding to an arrow in  $\Theta$  such that  $\psi$  and  $\Omega^{-1}\psi$  are surjective or  $\psi$  and  $\Omega\psi$  are injective.

Without loss of generality consider the case where they are surjective; otherwise dual arguments yield the result. Hence there is a non-split sequence which is not almost split  $0 \rightarrow M \rightarrow E \xrightarrow{\psi} N \rightarrow 0$ . Let  $\alpha$  be such that  $M|_{u_\zeta(f_\alpha)}$  is not projective (this is possible by [Dru10, Theorem 4.3]). Let  $M|_{u_\zeta(f_\alpha)} \cong M_1^{n_1} \oplus \dots \oplus M_\ell^{n_\ell}$ , where  $M_i$  is the indecomposable  $u_\zeta(f_\alpha)$ -module of dimension  $i$  and  $n_i \geq 0$ . Then  $\sum_{i=1}^{\ell-1} n_i \neq 0$  as  $M$  is not projective. Let  $V_i := u_\zeta(\mathfrak{n}) \otimes_{u_\zeta(f_\alpha)} M_i$ . Induction of the sequence  $0 \rightarrow M_{\ell-i} \rightarrow u_\zeta(f_\alpha) \rightarrow M_i \rightarrow 0$  yields the sequence  $0 \rightarrow V_{\ell-i} \rightarrow u_\zeta(\mathfrak{n}) \rightarrow V_i \rightarrow 0$  since  $u_\zeta(\mathfrak{n})$  is free over  $u_\zeta(f_\alpha)$ . Therefore  $V_i$  has simple top and socle (since  $u_\zeta(\mathfrak{n})$  is local and selfinjective) and  $V_i$  is  $\Omega$ -periodic.

We will show next that  $\text{soc } M$  is also simple. We do this by applying [Erd95, Proposition 1.5] to the  $\Omega$ -periodic module  $V_1$ . Hence we have either an embedding  $M \rightarrow V_1$  or every map  $M \rightarrow \Omega^{-1}(V_1)$  which does not factor through a projective module is a monomorphism. If we have an embedding  $M \rightarrow V_1$ , since  $\text{soc } V_1$  is simple, so is  $\text{soc } M$ . We have the following chain of isomorphisms since  $u_\zeta(\mathfrak{n}) : u_\zeta(f_\alpha)$  is a  $\gamma$ -Frobenius extension:

$$\underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_i) \cong \text{Ext}_{u_\zeta(\mathfrak{n})}^1(M, V_i) \cong \text{Ext}_{u_\zeta(f_\alpha)}^1(M, k),$$

which is non-zero since  $M|_{u_\zeta(f_\alpha)}$  is non-projective. Hence if we take a non-zero representative this has to be a monomorphism. Moreover  $\text{soc } \Omega^{-1}V_i \cong \text{soc } V_{\ell-i}$  is simple and so is  $\text{soc } M$ . Concluding and taking into account that  $u_\zeta(\mathfrak{n})$  is local we have a minimal injective resolution of the form  $0 \rightarrow M \rightarrow u_\zeta(\mathfrak{n}) \rightarrow u_\zeta(\mathfrak{n}) \rightarrow \dots$

Hence the complexity of  $M$  is  $\leq 1$  and therefore  $M$  is periodic; in particular  $\dim M \equiv 0 \pmod{\ell}$  by the foregoing proposition.

Now  $\dim M + \dim \Omega^{-1}M = \dim u_\zeta(\mathfrak{n})$  by the minimal injective resolution. We may assume that  $\dim M \geq \frac{1}{2} \dim u_\zeta(\mathfrak{n})$ ; otherwise replace  $M$  by  $\Omega^{-1}M$ , which is possible since by the choice of  $\psi$  we can do the same argument with  $\Omega^{-1}\psi$ . Now

$$\dim V_1 = \frac{1}{\ell} \dim u_\zeta(\mathfrak{n}) < \frac{1}{2} \dim u_\zeta(\mathfrak{n}) \leq \dim M.$$

So there is no monomorphism  $M \rightarrow V_1$ . Therefore every homomorphism  $M \rightarrow \Omega^{-1}V$  that does not factor through a projective module is injective by [Erd95, Proposition 1.5]. By the above argument we have that  $\dim \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_1) \geq \sum_{i=1}^{\ell-1} n_i$ . This must be at least 2; otherwise we would have that  $M|_{u_\zeta(f_\alpha)}$  has a unique non-projective summand and the dimension of  $M$  would not be divisible by  $\ell$ .

Hence there are  $\phi_1, \phi_2 \in \text{Hom}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V)$  with  $[\phi_1], [\phi_2] \in \underline{\text{Hom}}_{u_\zeta(\mathfrak{n})}(M, \Omega^{-1}V_1)$  linearly independent. By the above reasoning,  $\phi_1$  and  $\phi_2$  must be monomorphisms.

We know that  $\text{soc } M$  is simple; hence there is some  $c \in k$  such that  $\phi_1 - c\phi_2$  is not a monomorphism, and again by [Erd95, Proposition 1.5] we have that  $\phi_1 - c\phi_2$  factors through a projective module, a contradiction.  $\square$

As a corollary we also get the corresponding statement for the Borel part:

**Theorem 5.3.** *The category of gradable  $u_\zeta(\mathfrak{b})$ -modules does not contain any components of type  $\mathbb{Z}[\mathbb{A}_\infty]$  or type  $\mathbb{Z}[\mathbb{D}_\infty]$ .*

*Proof.* In the following commutative diagram indecomposable modules are mapped to indecomposable modules, and Auslander-Reiten sequences are mapped to Auslander-Reiten sequences by the vertical arrows by Lemma 3.1, therefore also by the horizontal arrow:

$$\begin{array}{ccc} \text{mod } u_\zeta(\mathfrak{b})U_\zeta^0(\mathfrak{g}) & \xlongequal{\quad} & \text{mod } u_\zeta(\mathfrak{n})U_\zeta^0(\mathfrak{g}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \text{mod } u_\zeta(\mathfrak{b}) & \xrightarrow{\text{res}} & \text{mod } u_\zeta(\mathfrak{n}) \end{array}$$

Thus the existence of an Auslander-Reiten sequence with three indecomposable direct summands would be preserved, and therefore the non-existence of such sequences for  $u_\zeta(\mathfrak{n})$  implies the same for  $u_\zeta(\mathfrak{b})$ ; i.e. there are no components of type  $\mathbb{Z}[\mathbb{D}_\infty]$ . Furthermore, as for every component  $\Theta$  containing a gradable module, there is an Auslander-Reiten sequence in  $\Theta$  with indecomposable middle term for  $u_\zeta(\mathfrak{n})$ , the same has to hold for  $u_\zeta(\mathfrak{b})$ , and thus the case of components of type  $\mathbb{Z}[\mathbb{A}_\infty]$  can be excluded.  $\square$

*Remark 5.4.* The same argument also works in the case of gradable modules for the restricted enveloping algebra of a Borel subalgebra.

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REFERENCES

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995. MR1314422 (96c:16015)

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. MR2197389 (2006j:16020)

[BE11] Petter Andreas Bergh and Karin Erdmann, *The stable Auslander-Reiten quiver of a quantum complete intersection*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 79–90, DOI 10.1112/blms/bdq081. MR2765552 (2012c:16049)

- [Ben91] D. J. Benson, *Representations and cohomology. II. Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, Cambridge, 1991. MR1156302 (93g:20099)
- [BF93] Allen D. Bell and Rolf Farnsteiner, *On the theory of Frobenius extensions and its application to Lie superalgebras*, Trans. Amer. Math. Soc. **335** (1993), no. 1, 407–424, DOI 10.2307/2154275. MR1097163 (93c:17049)
- [Cib97] Claude Cibils, *Half-quantum groups at roots of unity, path algebras, and representation type*, Internat. Math. Res. Notices **12** (1997), 541–553, DOI 10.1155/S1073792897000366. MR1456563 (98j:17009)
- [CP94] Vyjayanthi Chari and Andrew Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. MR1300632 (95j:17010)
- [dCK90] Corrado De Concini and Victor G. Kac, *Representations of quantum groups at roots of 1*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, pp. 471–506. MR1103601 (92g:17012)
- [Dru09] Christopher Martin Drupieski, *Cohomology of Frobenius-Lusztig kernels of quantized enveloping algebras*, Ph.D. thesis, University of Virginia, 2009.
- [Dru10] Christopher M. Drupieski, *On injective modules and support varieties for the small quantum group*, Int. Math. Res. Not. IMRN **10** (2011), 2263–2294, DOI 10.1093/imrn/rnq156. MR2806565 (2012e:17030)
- [Dru11] Christopher M. Drupieski, *Representations and cohomology for Frobenius-Lusztig kernels*, J. Pure Appl. Algebra **215** (2011), no. 6, 1473–1491, DOI 10.1016/j.jpaa.2010.09.006. MR2769244 (2012b:17030)
- [EHT<sup>+</sup>04] Karin Erdmann, Miles Holloway, Rachel Taillefer, Nicole Snashall, and Øyvind Solberg, *Support varieties for selfinjective algebras*, K-Theory **33** (2004), no. 1, 67–87, DOI 10.1007/s10977-004-0838-7. MR2199789 (2007f:16014)
- [Erd90] Karin Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin, 1990. MR1064107 (91c:20016)
- [Erd95] Karin Erdmann, *On Auslander-Reiten components for group algebras*, J. Pure Appl. Algebra **104** (1995), no. 2, 149–160, DOI 10.1016/0022-4049(94)00133-7. MR1360172 (97g:16026)
- [Erd96] Karin Erdmann, *The Auslander-Reiten quiver of restricted enveloping algebras*, Representation theory of algebras (Cocoyoc, 1994), CMS Conf. Proc., vol. 18, American Mathematical Society, Providence, RI, 1996, pp. 201–214. MR1388051 (97e:16035)
- [ES92] Karin Erdmann and Andrzej Skowroński, *On Auslander-Reiten components of blocks and self-injective biserial algebras*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 165–189, DOI 10.2307/2154159. MR1144759 (93b:16022)
- [Far] Rolf Farnsteiner, *Jordan types for indecomposable modules of finite group schemes*, preprint, arXiv:0910.3093.
- [Far99] Rolf Farnsteiner, *On Auslander-Reiten quivers of enveloping algebras of restricted Lie algebras*, Math. Nachr. **202** (1999), 43–66, DOI 10.1002/mana.19992020105. MR1694715 (2001j:17039)
- [Far00] Rolf Farnsteiner, *Auslander-Reiten components for Lie algebras of reductive groups*, Adv. Math. **155** (2000), no. 1, 49–83, DOI 10.1006/aima.2000.1925. MR1789848 (2001k:16026)
- [Far05] Rolf Farnsteiner, *Auslander-Reiten components for  $G_1T$ -modules*, J. Algebra Appl. **4** (2005), no. 6, 739–759, DOI 10.1142/S0219498805001502. MR2192153 (2006i:16026)
- [FW09] Jörg Feldvoss and Sarah Witherspoon, *Support varieties and representation type of small quantum groups*, Int. Math. Res. Not. IMRN **7** (2010), 1346–1362, DOI 10.1093/imrn/rnp189. MR2609022 (2011d:16045)
- [FW11] Jörg Feldvoss and Sarah Witherspoon, *Support varieties and representation type of self-injective algebras*, Homology Homotopy Appl. **13** (2011), no. 2, 197–215, DOI 10.4310/HHA.2011.v13.n2.a13. MR2854335 (2012m:16020)
- [FW13] Jörg Feldvoss and Sarah Witherspoon, *Erratum to “Support varieties and representation type of small quantum groups”*, arXiv: 1308.4928, August 2013.
- [Hup81] Lillian E. Peters Hupert, *Homological characteristics of pro-uniserial rings*, J. Algebra **69** (1981), no. 1, 43–66, DOI 10.1016/0021-8693(81)90126-5. MR613856 (83b:16018b)

- [KZ11] Otto Kerner and Dan Zacharia, *Auslander-Reiten theory for modules of finite complexity over self-injective algebras*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 44–56, DOI 10.1112/blms/bdq082. MR2765548 (2012a:16031)
- [Kül12] Julian Külshammer, *Auslander-Reiten theory of Frobenius-Lusztig kernels*, Bull. London Math. Soc. **45** (2013), 1285–1300, DOI: 10.1112/blms/bdt064.
- [KPhd] ———, *Representation type and Auslander-Reiten theory of Frobenius-Lusztig kernels*, Ph.D. thesis, University of Kiel, 2012.
- [RR85] Idun Reiten and Christine Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra **92** (1985), no. 1, 224–282, DOI 10.1016/0021-8693(85)90156-5. MR772481 (86k:16024)
- [Sch09] Sarah Scherotzke, *Euclidean components for a class of self-injective algebras*, Colloq. Math. **115** (2009), no. 2, 219–245, DOI 10.4064/cm115-2-7. MR2491746 (2010c:16023)
- [S-A04] Mariano Suárez-Álvarez, *The Hilton-Heckmann argument for the anti-commutativity of cup products*, Proc. Amer. Math. Soc. **132** (2004), no. 8, 2241–2246 (electronic), DOI 10.1090/S0002-9939-04-07409-X. MR2052399 (2005a:18016)

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