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WEAK AMENABILITY OF COMMUTATIVE BEURLING ALGEBRAS

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ABSTRACT. For a locally compact Abelian group G and a continuous weight function ω on G we show that the Beurling algebra $L^1(G,\omega)$ is weakly amenable if and only if there is no nontrivial continuous group homomorphism ϕ: $G \to \mathbb{C}$ such that $\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} < \infty$. Let $\widehat{\omega}(t) = \limsup_{s\to\infty} \omega(ts)/\omega(s)$ $(t \in G)$. Then $L^1(G,\omega)$ is 2-weakly amenable if there is a constant $m > 0$ such that $\liminf_{n\to\infty} \frac{\omega(t^n)\widehat{\omega}(t^{-n})}{n} \leq m$ for all $t \in G$.

1. Introduction

Let G be a locally compact group. The integral of a function f on a measurable subset K of G against a fixed left Haar measure will be denoted by $\int_K f dx$. A *weight* on G is a positive valued continuous function ω on G that satisfies $\omega(st) \leq \omega(s)\omega(t)$ for all $s, t \in G$. let $L^1(G)$ and $M(G)$ be, respectively, the usual convolution group algebra and measure algebra of G. Consider

$$L^1(G,\omega) = \{f : f\omega \in L^1(G)\},$$

where $f\omega$ denotes the pointwise product of f and ω. In our discussion, most of time G is fixed, so we will normally write $L^1(\omega)$ for $L^1(G,\omega)$. Equipped with the norm

$$\|f\|_\omega = \int_G |f(t)|\omega(t)dt \quad (f \in L^1(\omega))$$

and with the convolution product, $L^1(\omega)$ is a Banach algebra. When $\omega \equiv 1$ this is just the usual group algebra $L^1(G)$.

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A *derivation* D: $\mathcal{A} \to X$ is a linear mapping from $\mathcal{A}$ into X that satisfies $D(ab) = aD(b) + D(a)b$ $(a, b \in \mathcal{A})$. For each $x \in X$ the mapping $a \mapsto ax - xa$ $(a \in \mathcal{A})$ is a continuous derivation, called an *inner derivation*. The Banach algebra $\mathcal{A}$ is called *amenable* if each continuous derivation from $\mathcal{A}$ into the dual module X^* is inner for every Banach $\mathcal{A}$-bimodule X. The Banach algebra $\mathcal{A}$ is called *weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner, and $\mathcal{A}$ is *n-weakly amenable* for an integer $n > 0$ if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$, the n-th dual of $\mathcal{A}$, is inner. If $\mathcal{A}$ is n-weakly amenable for each $n > 0$, then it is called *permanently*

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weakly amenable. We refer to the monograph [5] for the background and history of these notions.

It is well-known that the group algebra $L^1(G)$ is always weakly amenable [16]. In fact, $L^1(G)$ is permanently weakly amenable for any locally compact group G [4].

For Beurling algebras, N. Grønbæk [12] showed that $L^1(G,\omega)$ is amenable if and only if G is amenable and the function $\omega(t)\omega(t^{-1})$ is bounded on G. Weak amenability of $L^1(\omega)$ was first studied by W.G. Bade, P.C. Curtis and H.G. Dales in [2], where they showed for the additive group $\mathbb{Z}$ of all integers and for the weight $\omega_\alpha(x) = (1+|x|)^\alpha$ on $\mathbb{Z}$ that $L^1(\mathbb{Z},\omega_\alpha)$ is weakly amenable if and only if $0 \le \alpha < \frac{1}{2}$. Following this work, Grønbæk showed in [11] that $L^1(\mathbb{Z},\omega)$ is weakly amenable if and only if $\liminf_{n\to\infty} \frac{\omega(n)\omega(-n)}{n} = 0$. Recently E. Samei [18] (also see [9]) showed that for a commutative group G, if $\liminf \frac{\omega(t^n)\omega(t^{-n})}{n} = 0$ for all $t \in G$, then $L^1(\omega)$ is weakly amenable. For 2-weak amenability H.G. Dales and A. T.-M. Lau showed in [7] that $L^1(\mathbb{Z},\omega_\alpha)$ is 2-weakly amenable if and only if $0 \le \alpha < 1$ and that the same is also true for $L^1(\mathbb{R},\omega_\alpha)$. They conjectured that for an Abelian group G, $L^1(\omega)$ is 2-weakly amenable whenever $\liminf_{n\to\infty} \frac{\omega(t^n)}{n} = 0$ for all $t \in G$, after showing that this is true if ω is almost invariant in the sense that $\lim_{t\to\infty} \sup_{s\in K} |\frac{\omega(st)}{\omega(t)} - 1| = 0$ for each compact set $K \subset G$. The last result was improved in [18], where the almost invariance condition was replaced by the weaker condition that the function $\widehat{\omega}$ defined by $\widehat{\omega}(s) = \limsup_{t\to\infty} \frac{\omega(ts)}{\omega(t)}$ is bounded on G. Related to 2-weak amenability of $L^1(\omega)$, we note that if G is Abelian, $L^1(\omega)$ is semisimple [3], and so, by the Singer-Wermer Theorem [5, 2.7.20], zero is the only continuous derivation on $L^1(\omega)$.

In this paper we study weak amenability and 2-weak amenability for commutative Beurling algebras.

In Section 3 we show that a commutative Beurling algebra $L^1(\omega)$ is weakly amenable if and only if there is no nontrivial continuous group homomorphism ϕ: $G \to \mathbb{C}$ (note that such homomorphisms are called characters in [13, 24.33]) such that $\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} < \infty$. With this characterization we may easily derive some well-known results obtained in [2, 12, 18] on the weak amenability of commutative Beurling algebras. We will also study special cases in the section. For example, we will explore the weak amenability of $L^1(G,\omega)$ when G is the additive group of the real line $\mathbb{R}$, when G is the product group of two or several factors and when $L^1(\omega)$ is the tensor product of two Beurling algebras.

In Section 4 we show that $L^1(\omega)$ is 2-weakly amenable if there is a constant $m > 0$ such that $\liminf_{n\to\infty} \frac{\omega(t^n)\widehat{\omega}(t^{-n})}{n} \le m$ for all $t \in G$. This result covers several known results on the 2-weak amenability of commutative Beurling algebras. We will also give an example of a 2-weakly amenable $L^1(\omega)$ for which $\widehat{\omega}$ is unbounded.

In Section 5 we will discuss some open problems on weak amenability for Beurling algebras.

2. Preliminaries

Given a Banach space X, its dual space will be denoted by X^*. The action of $f \in X^*$ at $x \in X$ will be denoted either by $f(x)$ or by $\langle x, f\rangle$.

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. The module action of $\mathcal{A}$ on X will be denoted by "$\cdot$". But if no confusion may occur, we will simply write ax or xa instead of $a \cdot x$ or $x \cdot a$ ($a \in \mathcal{A}$, $x \in X$). As is well-known, the dual space X^* of X is a Banach $\mathcal{A}$-bimodule with the natural module actions defined by

$$\langle x, a \cdot f\rangle = \langle xa, f\rangle, \quad \langle x, f \cdot a\rangle = \langle ax, f\rangle$$

for $a \in \mathcal{A}$, $f \in X^*$ and $x \in X$. In particular, $\mathcal{A}^*$ is a Banach $\mathcal{A}$-bimodule. The bidual space $\mathcal{A}^{**}$ of $\mathcal{A}$ may be equipped with two Arens products $\Box$ and $\diamond$, respectively defined by

$$\langle f, u\Box v\rangle = \langle v \cdot f, u\rangle, \quad \langle a, v \cdot f\rangle = \langle fa, v\rangle$$

and

$$\langle f, u \diamond v\rangle = \langle f \cdot u, v\rangle, \quad \langle a, f \cdot u\rangle = \langle af, u\rangle$$

for $u, v \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. With either $\Box$ or $\diamond$ giving the product, $\mathcal{A}^{**}$ becomes a Banach algebra containing $\mathcal{A}$ as a closed subalgebra. For any Banach $\mathcal{A}$-bimodule X, X^* is also a Banach left $(\mathcal{A}^{**}, \Box)$-module and a Banach right $(\mathcal{A}^{**}, \diamond)$-module (but in general it is not an $\mathcal{A}^{**}$-bimodule regardless of whether $\Box$ or $\diamond$ is used for the product of $\mathcal{A}^{**}$). The corresponding module actions are given by

$$\langle x, u \cdot f\rangle = \langle f \cdot x, u\rangle, \text{ where } f \cdot x \in \mathcal{A}^*, \ \langle a, f \cdot x\rangle = \langle xa, f\rangle \quad (a \in \mathcal{A})$$

and

$$\langle x, f \cdot u\rangle = \langle x \cdot f, u\rangle, \text{ where } x \cdot f \in \mathcal{A}^*, \ \langle a, x \cdot f\rangle = \langle ax, f\rangle \quad (a \in \mathcal{A})$$

for $u \in \mathcal{A}^{**}$, $f \in X^*$ and $x \in X$. For any $u \in \mathcal{A}^{**}$ we denote by ℓ_u and r_u respectively the left multiplier operator and the right multiplier operator on X^* defined by $\ell_u(f) = u \cdot f$ and $r_u(f) = f \cdot u$ ($f \in X^*$). If $\mathcal{A}$ has a bounded approximate identity (e_α), we may take a weak* cluster point E of (e_α) in $\mathcal{A}^{**}$. Then ℓ_E and r_E are $\mathcal{A}$-bimodule morphisms on X^*.

Let G be a locally compact group and ω be a weight on it. The dual space of $L^1(\omega)$ may be identified with

$$L^\infty(1/\omega) = L^\infty(G, 1/\omega) = \{f : f/\omega \in L^\infty(G)\}$$

with the norm given by

$$\|f\|_{\sup,1/\omega} = \operatorname*{ess\,sup}_{t\in G} \left|\frac{f(t)}{\omega(t)}\right| \quad (f \in L^\infty(1/\omega)).$$

A function $f \in L^\infty(1/\omega)$ is called *right ω-uniformly continuous* if the mapping $t \mapsto R_t(f)$ is continuous from G into $L^\infty(1/\omega)$, where R_t denotes the right translation by t, i.e. $R_t(f)(s) = f(st)$ ($s \in G$). The space of all right ω-uniformly continuous functions is denoted by $RUC(G, 1/\omega)$ (or abbreviated $RUC(1/\omega)$). It is well-known (see [7, Proposition 7.17] for example) that

$$RUC(1/\omega) = L^1(\omega) \cdot L^\infty(1/\omega).$$

Denote by $C_{00}(G)$ the space of all compactly supported continuous functions on G. The closure of $C_{00}(G)$ in $L^\infty(1/\omega)$ is $C_0(1/\omega)$ which is a Banach $L^1(\omega)$-submodule of $L^\infty(1/\omega)$. The dual space of $C_0(1/\omega)$ is $M(\omega)$, the space of all complex regular Borel measures μ on G that satisfy

$$\|\mu\|_\omega = \int_G \omega(t)d|\mu|(t) < \infty,$$

where $|\mu|$ denotes the total variation measure of μ. $\|\mu\|_\omega$ is indeed the norm of μ in $M(\omega)$. With the convolution product of measures, which is denoted by $*$, $M(\omega)$ is a Banach algebra containing $L^1(\omega)$ as a closed ideal. In fact, $M(\omega)$ is the multiplier algebra of $L^1(\omega)$ [8].

Let X be a Banach space. Denote the space of all bounded linear operators on X by $B(X)$. The *strong operator topology* (or briefly *so*-topology) on $B(X)\times B(X)$ is the topology induced by the family of seminorms $\{p_x : x \in X\}$, where

$$p_x(S,T) = \max\{\|S(x)\|, \|T(x)\|\} \quad (S,T \in B(X))$$

(see [5, page 327]). Indeed, $B(X)$ is a Banach algebra with the operator norm topology and the composition product. So is $B(X)\times B(X)$. As the multiplier algebra of $L^1(\omega)$, $M(\omega)$ is actually regarded as a subalgebra of $B(L^1(\omega))\times B(L^1(\omega))$ with each $\mu \in M(\omega)$ being identified with $(\ell_\mu, r_\mu) \in B(L^1(\omega)) \times B(L^1(\omega))$. See [5] for details.

Lemma 2.1. *Let G be a locally compact group and let ω be a weight on it. Then $lin\{\delta_t : t \in G\}$, the linear space generated by the point measures δ_t $(t \in G)$, is dense in $M(\omega)$ in the so-topology. In particular, for each $h \in L^1(\omega)$ there is a net $(u_\alpha) \subset lin\{\delta_t : t \in G\}$ such that $\|(u_\alpha - h) * a\|_\omega \to 0$ and $\|a * (u_\alpha - h)\|_\omega \to 0$ for all $a \in L^1(\omega)$.*

Proof. Denote $V = lin\{\delta_t : t \in G\}$. It is evident that $V \subset M(\omega)$. Let $\mu \in M(\omega)$. We show that there is a net $(\mu_\alpha) \subset V$ such that $\|\mu_\alpha * a - \mu * a\|_\omega \to 0$ and $\|a * \mu_\alpha - a * \mu\|_\omega \to 0$ for every $a \in L^1(\omega)$.

By [5, Proposition 3.3.41(i)] V is dense in $M(G)$ in the *so*-topology. Since $u := \omega\mu \in M(G)$, there is a net $(u_\alpha) \subset V$ such that

$$\|u_\alpha * g - u * g\|_1 \to 0 \text{ and } \|g * u_\alpha - g * u\|_1 \to 0$$

for every $g \in L^1(G)$. Let $\mu_\alpha = \frac{1}{\omega} u_\alpha$ and $g = \omega a$. Then μ_α still belongs to V and $g \in L^1(G)$. We have

$$\|\mu_\alpha * a - \mu * a\|_\omega \le \|u_\alpha * g - u * g\|_1 \to 0$$

and

$$\|a * \mu_\alpha - a * \mu\|_\omega \le \|g * u_\alpha - g * u\|_1 \to 0.$$

Thus, $\mu \in$ *so*-cl(V). This is true for every $\mu \in M(\omega)$. The proof is complete. $\square$

3. Weak amenability

We denote the additive group of complex numbers (with the usual metric topology) by $\mathbb{C}$ and denote by $\mathbb{R}$ the closed subgroup of $\mathbb{C}$ consisting of all real numbers.

Theorem 3.1. *Let G be a locally compact Abelian group and ω be a weight on G. Then $L^1(\omega)$ is weakly amenable if and only if there exists no nontrivial continuous group homomorphism ϕ: $G \to \mathbb{C}$ such that*

$$\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} < \infty. \tag{3.1}$$

Proof. If $L^1(\omega)$ is not weakly amenable, then there is a nonzero continuous derivation D: $L^1(\omega) \to L^\infty(1/\omega)$. It is standard (see [15] for example) that one can extend D to a derivation, still denoted by D, from $M(\omega)$ to $L^\infty(1/\omega)$. Define $\Delta(t) = \delta_{t^{-1}} \cdot D(\delta_t)$ $(t \in G)$. Then Δ satisfies

$$\begin{aligned}\Delta(t_1t_2) = \delta_{t_2^{-1}t_1^{-1}} \cdot D(\delta_{t_1}\delta_{t_2}) &= \delta_{t_1^{-1}} \cdot D(\delta_{t_1}) + \delta_{t_2^{-1}} \cdot D(\delta_{t_2}) \\ &= \Delta(t_1) + \Delta(t_2)\end{aligned} \tag{3.2}$$

for $t_1, t_2 \in G$, and $\Delta(e) = 0$, where e is the unit of G.

We note that D is *so*-weak* continuous. In fact, since $L^1(\omega)$ has a bounded approximate identity, by Cohen's Factorization Theorem every $f \in L^1(\omega)$ may be written as $f = f_1 * f_2$ for some $f_1, f_2 \in L^1(\omega)$. So, if $\mu_\alpha \overset{so}{\to} \mu$ in $M(\omega)$, then

$$\begin{aligned}\lim_\alpha \langle f, D(\mu_\alpha)\rangle &= \lim_\alpha \langle f_1, D(f_2 * \mu_\alpha)\rangle - \lim_\alpha \langle \mu_\alpha * f_1, D(f_2)\rangle \\ &= \langle f_1, D(f_2 * \mu)\rangle - \langle \mu * f_1, D(f_2)\rangle = \langle f, D(\mu)\rangle.\end{aligned}$$

This clarifies the *so*-weak* continuity of D. Since $span\{\delta_t : t \in G\}$ is dense in $M(\omega)$ in the *so*-topology (Lemma 2.1), Δ is a nontrivial mapping from G to $L^\infty(1/\omega)$. So there is $h \in L^1(\omega)$ such that $\phi(t) = \langle h, \Delta(t)\rangle$ is a nontrivial complex valued function on G. By (3.2) the function ϕ is clearly a group homomorphism from G to $\mathbb{C}$. It is also continuous. To see this (again due to Cohen's Factorization Theorem) we write $h = h_1 * h_2$ for some $h_1, h_2 \in L^1(\omega)$. Then

$$\phi(t) = \langle h_1, D(h_2)\rangle - \langle \delta_t * h_1, D(h_2 * \delta_{t^{-1}})\rangle,$$

which is clearly continuous in t. Moreover,

$$|\phi(t)| \le (\|D\|\|h\|_\omega)\omega(t)\omega(t^{-1}) \quad (t \in G).$$

Thus ϕ: $G \to \mathbb{C}$ is a nontrivial continuous group homomorphism and it satisfies (3.1).

For the converse, we assume ϕ: $G \to \mathbb{C}$ is a continuous nontrivial group homomorphism that satisfies

$$\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} \le m$$

for some $m < \infty$. Fix a compact neighborhood B of e in G. For each $h \in L^1(\omega)$ we define

$$D(h)(t) = \int_B \phi(t^{-1}\xi)h(t^{-1}\xi)d\xi \quad (t \in G). \tag{3.3}$$

It is standard to check that $D(h)(t)$ is continuous (and hence is measurable) on G. Since

$$\left|\frac{D(h)(t)}{\omega(t)}\right| \le m \int_B \omega(\xi^{-1})\omega(t^{-1}\xi)|h(t^{-1}\xi)|d\xi \le ml\|h\|_\omega$$

for all $t \in G$, we derive $D(h) \in L^\infty(1/\omega)$ for each $h \in L^1(\omega)$, where

$$l = \sup\{\omega(s^{-1}) : s \in B\}.$$

The mapping $h \mapsto D(h)$ is clearly a nonzero bounded linear mapping from $L^1(\omega)$ to $L^\infty(1/\omega)$. We show it is indeed a derivation. Let $a, b \in L^1(\omega)$. Then

$$\begin{aligned}D(a * b)(t) &= \int_B \phi(t^{-1}\xi) \int_G a(s)b(s^{-1}t^{-1}\xi)dsd\xi \\ &= \int_B \int_G a(s)\left(\phi(s^{-1}t^{-1}\xi) + \phi(s)\right)b(s^{-1}t^{-1}\xi)dsd\xi \\ &= \int_G a(s)D(b)(ts)ds + \int_G D(a)(st)b(s)ds \\ &= [a \cdot D(b) + D(a) \cdot b](t) \quad (t \in G).\end{aligned}$$

In the above computation we have used the Fubini theorem to exchange the order of integrals. We can do this because B is compact and the supports of a and b are σ-finite. Therefore $D(a * b) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in L^1(\omega)$; i.e.

$D: L^1(\omega) \to L^\infty(1/\omega)$ is a nonzero continuous derivation. Thus $L^1(\omega)$ is not weakly amenable. □

Remark 3.2. If G is an IN group, we take any compact neighborhood B of e in G such that $sBs^{-1} = B$ for all $s \in G$. Then the argument for the necessity part of Theorem 3.1 may be adapted to show the following: If $L^1(\omega)$ is weakly amenable, then there is no continuous group homomorphism ϕ: $G \to \mathbb{C}$ such that ϕ is not trivial on B and such that (3.1) holds. To see this we first note $\int_B f(s\xi)d\xi = \int_B f(\xi s)d\xi$ for $f \in L^1(G,\omega)$ and $s \in G$. This property ensures that the mapping D defined by (3.3) is still a continuous derivation, assuming the above ϕ exists. If D is inner, then it must be trivial at all h belonging to the center $ZL^1(\omega)$ of $L^1(\omega)$. However, $h_\phi := \overline{\phi}\chi_B \in ZL^1(\omega)$ and

$$D(h_\phi)(t) = \int_{B\cap tB} |\phi(t^{-1}\xi)|^2 d\xi = \int_{B\cap t^{-1}B} |\phi(\xi)|^2 d\xi$$

is nontrivial if ϕ is not trivial on B, where $\overline{\phi}$ is the conjugate of ϕ and χ_B is the characteristic function of B. So D is not inner and thus $L^1(\omega)$ is not weakly amenable.

Remark 3.3. For a discrete Abelian group G, Theorem 3.1 was obtained by Grønbæk in [11]. As indicated there, when $G = \mathbb{Z}$ (the discrete additive group of all integers), all group homomorphisms from $\mathbb{Z}$ to $\mathbb{C}$ are of the form $\phi(n) = nc_0$ ($n \in \mathbb{Z}$, $c_0 \in \mathbb{C}$). Therefore, for any weight ω on $\mathbb{Z}$, $\ell^1(\mathbb{Z},\omega)$ is weakly amenable if and only if $\sup_{n\in\mathbb{N}} \frac{n}{\omega(n)\omega(-n)} = \infty$ or, equivalently, $\inf_{n\in\mathbb{N}} \frac{\omega(n)\omega(-n)}{n} = 0$.

The above argument certainly works also for $G = \mathbb{R}$. But we have more to say for $\mathbb{R}$ later in Corollary 3.7.

Remark 3.4. Since a commutative Banach algebra is permanently weakly amenable if and only if it is weakly amenable [6], the condition in Theorem 3.1 is also a necessary and sufficient condition for $L^1(\omega)$ to be permanently weakly amenable.

When one applies Theorem 3.1, it suffices to consider only real valued group homomorphisms. Precisely we have the following theorem.

Theorem 3.5. *Let G be a locally compact Abelian group and ω be a weight on G. Then $L^1(\omega)$ is weakly amenable if and only if there exists no nontrivial continuous group homomorphism ϕ: $G \to \mathbb{R}$ such that* (3.1) *holds.*

Proof. The necessity is trivial. For the sufficiency, suppose that $L^1(\omega)$ is not weakly amenable. Then, by Theorem 3.1, there is a continuous complex valued nonzero homomorphism ϕ such that (3.1) holds. The real part ϕ_r and the imaginary part ϕ_i of ϕ are both still continuous group homomorphisms, they satisfy the same inequality (3.1), and they are real valued. If $\phi \neq 0$, then at least one of ϕ_r and ϕ_i is nonzero. So there exists a nontrivial continuous real valued group homomorphism such that (3.1) holds. □

We now consider some special cases to illustrate how Theorem 3.1 applies.

Corollary 3.6 ([18]). *Let G be a locally compact Abelian group and ω be a weight on G. If for each $t \in G$*

$$\inf_{n\in\mathbb{N}} \frac{\omega(t^n)\omega(t^{-n})}{n} = 0, \tag{3.4}$$

then $L^1(\omega)$ is weakly amenable.

Proof. Let ϕ: $G \to \mathbb{R}$ be any nontrivial group homomorphism and let $s \in G$ be such that $\phi(s) \neq 0$. We have $\phi(s^n) = n\phi(s)$ $(n \in \mathbb{N})$. If (3.4) holds for $t = s$, then

$$\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{|\phi(s^n)|}{\omega(s^n)\omega(s^{-n})} = \sup_{n\in\mathbb{N}} \frac{|\phi(s)|n}{\omega(s^n)\omega(s^{-n})} = \infty.$$

So (3.1) does not hold for any such nonzero homomorphism ϕ. By Theorem 3.5, $L^1(\omega)$ is weakly amenable. □

Corollary 3.7. *Let ω be a weight on $\mathbb{R}$. Then the following statements are equivalent:*

(1) *The Beurling algebra $L^1(\mathbb{R}, \omega)$ is weakly amenable.*
(2) $\limsup_{t\to\infty} \frac{|\phi(t)|}{\omega(t)\omega(-t)} = \infty$ *for each nonzero continuous group homomorphism ϕ: $\mathbb{R} \to \mathbb{C}$.*
(3) $\liminf_{t\to\infty} \frac{\omega(t)\omega(-t)}{|t|} = 0$.
(4) $\liminf_{n\to\infty} \frac{\omega(nt)\omega(-nt)}{n} = 0$ *for all $t \in \mathbb{R}$.*
(5) $\liminf_{n\to\infty} \frac{\omega(n)\omega(-n)}{n} = 0$.
(6) *There is $t_0 \in \mathbb{R}$ such that $t_0 \neq 0$ and* $\liminf_{n\to\infty} \frac{\omega(nt_0)\omega(-nt_0)}{n} = 0$.

Proof. The equivalence of (1) and (2) follows straightforwardly from Theorem 3.1.

(2)⇒(3): Simply consider $\phi(t) = t$ $(t \in \mathbb{R})$. We see immediately that (3) holds if (2) is true.

(3)⇒(2): Given a nonzero continuous group homomorphism ϕ: $\mathbb{R} \to \mathbb{C}$, it is a well-known fact that there is $z_0 \in \mathbb{C}$, $z_0 \neq 0$, such that $\phi(t) = tz_0$. Thus,

$$\frac{|\phi(t)|}{\omega(t)\omega(-t)} = |z_0| \frac{|t|}{\omega(t)\omega(-t)}.$$

This relation shows that (2) is the case if (3) holds.

(3)⇒(4): If $t = 0$, the limit in (4) is trivially true. If $t \neq 0$, without loss of generality, we may assume $t = t_0 > 0$. If (3) holds, then there is a positive sequence $(t_i) \subset \mathbb{R}$ such that $t_i \to \infty$ and

$$\lim_{i\to\infty} \frac{\omega(t_i)\omega(-t_i)}{t_i} = 0.$$

Take $n_i \in \mathbb{N}$ and $0 \leq s_i < t_0$ such that $t_i = n_i t_0 + s_i$. We have

$$\frac{\omega(t_i)\omega(-t_i)}{t_i} \geq \frac{1}{(t_0 + s_i/n_i)\omega(s_i)\omega(-s_i)} \frac{\omega(n_i t_0)\omega(-n_i t_0)}{n_i}.$$

Since $0 \leq s_i < t_0$, $n_i \to \infty$ and $\frac{1}{(t_0+s_i/n_i)\omega(s_i)\omega(-s_i)}$ is uniformly bounded away from 0 as $i \to \infty$. The above inequality leads to

$$\lim_{i\to\infty} \frac{\omega(n_i t_0)\omega(-n_i t_0)}{n_i} = 0.$$

This shows that (4) holds when $t = t_0$ for all $t_0 > 0$. Therefore it holds for all $t \in \mathbb{R}$.

(4)⇒ (5), (5)⇒(6) and (6)⇒(3) are trivial. The proof is complete. □

Let H and R be two locally compact Abelian groups. We consider the product group $H \times R = \{(s,t) : s \in H, t \in R\}$. With the product topology it is a commutative locally compact group. We may regard H and R as closed subgroups of $H \times R$, identifying s with (s, e_R) and t with (e_H, t) for $s \in H$ and $t \in R$, where e_H

and e_R are identities of H and R respectively. Let ω be a weight on $H \times R$. Then $\omega_H = \omega|_H$ and $\omega_R = \omega|_R$ are weights on H and R respectively. Following [12] we denote the symmetrization of ω by Ω, that is, $\Omega(s,t) = \omega(s,t)\omega(s^{-1},t^{-1})$ ($s \in H$, $t \in R$).

Theorem 3.8. *If both $L^1(H,\omega_H)$ and $L^1(R,\omega_R)$ are weakly amenable, then so is $L^1(H \times R, \omega)$. Conversely, the algebra $L^1(H \times R, \omega)$ is not weakly amenable in either of the following conditions:*

(1) *$L^1(H,\omega_H)$ is not weakly amenable and* $\sup_{(s,t)\in H\times R} \frac{\Omega(s,e_R)}{\Omega(s,t)} < \infty$.

(2) *$L^1(R,\omega_R)$ is not weakly amenable and* $\sup_{(s,t)\in H\times R} \frac{\Omega(e_H,t)}{\Omega(s,t)} < \infty$.

Proof. If ϕ: $H \times R \to \mathbb{C}$ is a nonzero continuous group homomorphism, then either $\phi|_H$ or $\phi|_R$ is nonzero. If (3.1) holds for $G = H \times R$, then it holds for $G = H$ and for $G = R$. Thus at least one of $L^1(H,\omega_H)$ and $L^1(R,\omega_R)$ is not weakly amenable if $L^1(H \times R, \omega)$ is not weakly amenable. This shows the first assertion of the theorem.

For the second assertion, suppose that (1) is the case (the proof for the other case is similar). Then there is nonzero continuous group homomorphism ϕ: $G = H \to \mathbb{C}$ such that (3.1) holds. Let $\phi'(s,t) = \phi(s)$ ($s \in H$, $t \in R$). ϕ' is a nonzero continuous group homomorphism from $H \times R$ to $\mathbb{C}$ and

$$\frac{|\phi'(s,t)|}{\Omega(s,t)} = \frac{|\phi(s)|}{\Omega(s,e_R)}\frac{\Omega(s,e_R)}{\Omega(s,t)} \le l\frac{|\phi(s)|}{\omega_H(s)\omega_H(s^{-1})},$$

where l is a constant such that $\sup_{(s,t)\in H\times R} \frac{\Omega(s,e_R)}{\Omega(s,t)} \le l$. So (3.1) holds for ϕ' and $G = H \times R$. Therefore, $L^1(H \times R, \omega)$ is not weakly amenable from Theorem 3.1. $\square$

Example. Consider the polynomial weight $\omega(s,t) = (1+|s|+|t|)^\alpha$ on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ ($\alpha > 0$). Then $\omega_H = \omega_R = \omega_\alpha$. From Corollary 3.7 we see $L^1(\mathbb{R},\omega_\alpha)$ is weakly amenable if and only if $\alpha < 1/2$. Since $\omega(s,0) \le \omega(s,t)$ for $s,t \in \mathbb{R}$, we have $\Omega(s,0)/\Omega(s,t) \le 1$. Therefore, the inequality in (1) (and (2)) of Theorem 3.8 holds. From Theorem 3.8 we immediately derive that $L^1(\mathbb{R}^2,\omega)$ is weakly amenable if and only if $\alpha < 1/2$.

We now discuss some consequences of Theorem 3.8.

Corollary 3.9. *Let ω be a weight on the additive group $\mathbb{R}^n$. Denote*

$$e_i = (0,\cdots,1,0,\cdots,0)$$

($i = 1,2,\cdots,n$), where 1 appears only at the i-th coordinate, and let ω_i be the weight on $\mathbb{R}$ defined by $\omega_i(t) = \omega(te_i)$ ($t \in \mathbb{R}$). If

$$\liminf_{n\to\infty} \frac{\omega_i(n)\omega_i(-n)}{n} = 0$$

for all $i = 1,2,\cdots,n$, then $L^1(\mathbb{R}^n,\omega)$ is weakly amenable.

Proof. From Corollary 3.7(5) $L^1(\mathbb{R},\omega_i)$ is weakly amenable for each i. Then Theorem 3.8 applies. $\square$

Corollary 3.10. *Let G_1 and G_2 be two locally compact Abelian groups and let ω_1 and ω_2 be weights on them, respectively. Then the (projective) tensor product algebra $L^1(G_1,\omega_1)\hat{\otimes}L^1(G_2,\omega_2)$ is weakly amenable if and only if both $L^1(G_1,\omega_1)$ and $L^1(G_2,\omega_2)$ are weakly amenable.*

Proof. $L^1(G_1,\omega_1)\hat{\otimes}L^1(G_2,\omega_2)\cong L^1(G_1\times G_2,\omega_1\times\omega_2)$. For $H=G_1$, $R=G_2$ and $\omega=\omega_1\times\omega_2$ the inequalities in (1) and (2) of Theorem 3.8 evidently hold. Thus, the result follows from Theorem 3.8. □

Corollary 3.11. *Let $G_1, G_2, \cdots, G_k$ be k locally compact Abelian groups. Suppose that ω is a weight on $G_1\times G_2\times\cdots\times G_k$ and there is a constant $r>0$ such that for each $i=1,2,\cdots,k$,*

$$\omega(T_i)\le r\omega(t_1,t_2,\cdots,t_k)\quad (t_j\in G_j,\, j=1,2,\cdots,k),$$

where T_i represents the element of $G_1\times G_2\times\cdots\times G_k$ whose i-th coordinate is t_i and each of the other coordinates is the unit element of the corresponding component group. Then $L^1(G_1\times G_2\times\cdots\times G_k,\omega)$ is weakly amenable if and only if all $L^1(G_i,\omega_{G_i})$ $(i=1,2,\cdots,k)$ are weakly amenable.

Proof. Simply apply induction and use Theorem 3.8. □

If G is a compactly generated locally compact Abelian group, then it is topologically isomorphic with $\mathbb{R}^p\times\mathbb{Z}^q\times F$ for some integers $p\ge 0, q\ge 0$ and some compact group F. We may write such a group as $G=G_1\times G_2\times\cdots\times G_k\times F$, where G_i is either $\mathbb{R}$ or $\mathbb{Z}$ $(i=1,\cdots,k)$. Denote $G_i^+=\{t\in G_i : t\ge 0\}$. We note that for any compact group F, $L^1(F,\omega)$ is isomorphic with $L^1(F)$ and hence is weakly amenable for any weight ω.

Corollary 3.12. *Let $G=G_1\times G_2\times\cdots\times G_k\times F$ be a locally compact, compact generated Abelian group, where G_i is either $\mathbb{R}$ or $\mathbb{Z}$ $(i=1,\cdots,k)$ and F is a compact group. Let ω be a weight on G which can be written in the form*

$$\omega(t_1,t_2,\cdots,t_k,s)=w(|t_1|,|t_2|,\cdots,|t_k|,s),\quad ((t_1,t_2,\cdots,t_k,s)\in G),$$

where $w(x_1,x_2,\cdots,x_k,s)$ is a function on $G_1^+\times G_2^+\times\cdots\times G_k^+\times F$ which is increasing in each x_i $(i=1,2,\cdots,k)$. Then $L^1(G,\omega)$ is weakly amenable if and only if all $L^1(G_i,\omega_{G_i})$ $(i=1,2,\cdots,k)$ are weakly amenable.

Proof. It is readily checked that the condition of Corollary 3.11 is fulfilled with $r=\sup_{s\in F}\omega(0,0,\cdots,0,s)$. □

Remark 3.13. In particular, if ω is a polynomial weight, then Corollary 3.12 gives [18, Theorem 7.1(i)].

4. 2-WEAK AMENABILITY

Let G be a locally compact group and let ω be a weight on G. Define

$$\widehat{\omega}(t)=\limsup_{s\to\infty}\frac{\omega(ts)}{\omega(s)}:=\inf_K\sup_{s\in G\setminus K}\frac{\omega(ts)}{\omega(s)},$$

where the infimum is taken over all compact subsets of G. The function $\widehat{\omega}$ is not guaranteed to be continuous although it is indeed submultiplicative. It is not even clear whether $\widehat{\omega}$ is a measurable function on G. But we will not use the measurability of $\widehat{\omega}$ in our argument.

Theorem 4.1. *Let G be a locally compact Abelian group and let ω be a weight on G. If there is a constant $m>0$ such that*

$$\liminf_{n\to\infty}\frac{\omega(t^n)\widehat{\omega}(t^{-n})}{n}\le m\quad (t\in G),\tag{4.1}$$

then $L^1(\omega)$ is 2-weakly amenable.

Proof. As in [7], we use $\mathcal{A}_\omega$ to denote $L^1(\omega)$. Suppose that D: $\mathcal{A}_\omega \to \mathcal{A}_\omega{}^{**}$ is a continuous derivation. We aim to show that $D = 0$.

It was proved in [7] that, as a consequence of the Singer-Wermer theorem, $D(L^1(\omega)) \subset C_0(1/\omega)^\perp$. Here $C_0(1/\omega)$ is regarded as a closed submodule of the $\mathcal{A}_\omega$-bimodule $L^\infty(1/\omega)$. Denote the quotient module $L^\infty(1/\omega)/C_0(1/\omega)$ by X. Then $C_0(1/\omega)^\perp \cong X^*$ as a Banach $\mathcal{A}_\omega$-bimodules. Let (e_α) be a bounded approximate identity of $\mathcal{A}_\omega$ and let E be a weak* cluster point of (e_α) in $\mathcal{A}_\omega{}^{**}$. We then have

$$X^* = r_E(X^*) \oplus (I - r_E)(X^*),$$

where I denotes the identity operator on X^*. It is evident that $r_E(X^*) \cong (\mathcal{A}_\omega X)^*$ and $(I - r_E)(X^*) \cong (X/\mathcal{A}_\omega X)^*$ as Banach $\mathcal{A}_\omega$-bimodules (see the proof of [15, Proposition 1.8]). With this module decomposition the derivation D may be written as the sum of two continuous derivations: $D_1 = r_E \circ D$: $\mathcal{A}_\omega \to r_E(X^*)$ and $D_2 = (I - r_E) \circ D$: $\mathcal{A}_\omega \to (I - r_E)(X^*)$. Since the $\mathcal{A}_\omega$-module actions on $X/(\mathcal{A}_\omega X)$ are trivial, $D_2 = 0$ according to [15, Proposition 1.5]. Therefore $D = D_1$, that is, D: $\mathcal{A}_\omega \to r_E(X^*)$. Note that X is a symmetric $\mathcal{A}_\omega$-bimodule. So $\mathcal{A}_\omega X = \mathcal{A}_\omega X \mathcal{A}_\omega$ is a neo-unital $\mathcal{A}_\omega$-bimodule. As is well-known, we may extend $\mathcal{A}_\omega$-module actions on $\mathcal{A}_\omega X$ to $M(\omega)$-bimodule actions so that $\mathcal{A}_\omega X$ becomes a unital (symmetric) $M(\omega)$-bimodule. Moreover, D may be uniquely extended to a continuous derivation from $M(\omega)$ to $r_E(X^*)$. Thus, for each $t \in G$, $D(\delta_t)$ is well-defined in this sense. We show $D(\delta_t) = 0$ for all $t \in G$.

Note that $\mathcal{A}_\omega X = RUC(1/\omega)/C_0(1/\omega)$. Given any $x \in \mathcal{A}_\omega X$, any compact set $K \subset G$ and any $\varepsilon > 0$, there is $f \in RUC(1/\omega)$ such that

$$x = [f],\ f|_K = 0 \text{ and } \|f\|_{\sup,1/\omega} \le \|x\| + \varepsilon,$$

where $[f] = f + C_0(1/\omega)$ represents the coset of f modulo $C_0(1/\omega)$. In fact, from the definition of the quotient norm one may choose $h \in RUC(1/\omega)$ such that $x = [h]$ and $\|h\|_{\sup,1/\omega} \le \|x\| + \varepsilon$. On the other hand, there is $f_0 \in C_{00}(G)$ such that $0 \le f_0 \le 1$, $f_0(t) = 1$ for $t \in K$. We take $f = (1 - f_0)h$. Then $[f] = [h] = x$ since $f_0 h \in C_{00}(G)$. It is easily seen that $\|f\|_{\sup,1/\omega} \le \|h\|_{\sup,1/\omega} \le \|x\| + \varepsilon$. Then for $t \in G$ we have

$$\begin{aligned}\|x \cdot \delta_t\| &\le \|f \cdot \delta_t\|_{\sup,1/\omega} = \sup_{s\in G} \left|\frac{f(ts)}{\omega(s)}\right| \\ &\le \|f\|_{\sup,1/\omega} \sup_{s \in G\setminus(t^{-1}K)} \frac{\omega(ts)}{\omega(s)} \le (\|x\| + \varepsilon) \sup_{s \in G\setminus(t^{-1}K)} \frac{\omega(ts)}{\omega(s)}.\end{aligned}$$

Since $\varepsilon > 0$ and $K \subset G$ were arbitrarily given, we derive

$$\|x \cdot \delta_t\| \le \|x\|\widehat{\omega}(t) \quad (x \in \mathcal{A}_\omega X,\ t \in G).$$

Then for $\Phi \in (\mathcal{A}_\omega X)^*$,

$$|\langle x, \delta_t \cdot \Phi\rangle| = |\langle x \cdot \delta_t, \Phi\rangle| \le \|x\|\widehat{\omega}(t)\|\Phi\| \quad (x \in \mathcal{A}_\omega X).$$

This implies that $\|\delta_t \cdot \Phi\| \le \widehat{\omega}(t)\|\Phi\|$ ($\Phi \in (\mathcal{A}_\omega X)^*$, $t \in G$).

As G is commutative, for each integer n we have

$$\delta_{t^{-n}} \cdot D(\delta_{t^n}) = n\delta_{t^{-1}} \cdot D(\delta_t) \quad (t \in G).$$

The above discussion allows us to estimate the norm as follows:

$$\|\delta_{t^{-n}} \cdot D(\delta_{t^n})\| \le \widehat{\omega}(t^{-n})\|D(\delta_{t^n})\| \le \omega(t^n)\widehat{\omega}(t^{-n})\|D\|.$$

We then have

$$\|\delta_{t^{-1}} \cdot D(\delta_t)\| = \frac{1}{n}\|\delta_{t^{-n}} \cdot D(\delta_{t^n})\| \leq \frac{\omega(t^n)\widehat{\omega}(t^{-n})}{n}\|D\|.$$

From the hypothesis we immediately obtain

$$\|\delta_{t^{-1}} \cdot D(\delta_t)\| \leq m\|D\| \quad (t \in G).$$

Let $\Delta(t) = \delta_{t^{-1}} \cdot D(\delta_t)$ $(t \in G)$, and let $B = \Delta(G)$. Then B is a bounded subset of $r_E(X^*)$. However, similar to the counterpart that we showed in the proof of Theorem 3.1, $\Delta(t_1 t_2) = \Delta(t_1) + \Delta(t_2)$ for all $t_1, t_2 \in G$. As a consequence, one sees easily via induction that $\Delta(t^k) = k\Delta(t)$ $(t \in G)$. Therefore

$$\|\Delta(t)\| = \frac{1}{k}\|\Delta(t^k)\| \leq \frac{m\|D\|}{k}$$

for all integers $k > 0$. This shows that $\Delta(t) = 0$. Hence $D(\delta_t) = 0$ for all $t \in G$. This implies that $D(u) = 0$ for $u \in span\{\delta_t : t \in G\}$. As a continuous derivation from $\mathcal{A}_\omega$ to $(\mathcal{A}_\omega X)^*$, D is *so*-weak* continuous. Since $span\{\delta_t : t \in G\}$ is dense in $M(\omega)$ in the *so*-topology (Lemma 2.1), we finally get $D(u) = 0$ for all $u \in M(\omega)$. So $D = 0$. This shows that $L^1(\omega)$ is 2-weakly amenable.

$\square$

Example. Consider the additive group $\mathbb{Z}^2$ and the weight ω on it defined by

$$\omega(s,t) = (1 + |s| + |t|)^\alpha (1 + |s + t|)^\beta,$$

where $\alpha \geq 0$ and $\beta \geq 0$. Then $\widehat{\omega}(s,t) = (1 + |s+t|)^\beta$ which is unbounded if $\beta > 0$. However, it is readily seen that $\lim \frac{\omega(ns,nt)\widehat{\omega}(-ns,-nt)}{n} = 0$ when $\alpha + 2\beta < 1$. So $\ell^1(\mathbb{Z}^2, \omega)$ is 2-weakly amenable if $\alpha + 2\beta < 1$ due to Theorem 4.1.

In general, when a weight is the product of a polynomial weight of order less than 1 and some other weight which does not increase "too fast", the corresponding Beurling algebra will be 2-weakly amenable. Precisely we have the following.

Corollary 4.2. *Let $G = \mathbb{R}^m \times \mathbb{Z}^k$, where $m, k \geq 0$ are integers. Let*

$$\omega(\mathbf{s}, \mathbf{t}) = (1 + |\mathbf{s}| + |\mathbf{t}|)^\alpha \omega_0(\mathbf{s}, \mathbf{t}) \quad (\mathbf{s} \in \mathbb{R}^m, \mathbf{t} \in \mathbb{Z}^k),$$

where $0 \leq \alpha < 1$, $|\mathbf{s}|$ and $|\mathbf{t}|$ denote the Euclidean norm of $\mathbf{s}$ and $\mathbf{t}$, and ω_0 is a weight on G satisfying

$$\liminf_{n\to\infty} \frac{\omega_0(n\mathbf{s}, n\mathbf{t})}{n^{(1-\alpha)/2}} = 0 \quad (\mathbf{s} \in \mathbb{R}^m, \mathbf{t} \in \mathbb{Z}^k).$$

Then $L^1(G, \omega)$ is 2-weakly amenable.

Proof. It is readily seen that $\widehat{\omega} = \widehat{\omega}_0 \leq \omega_0$. So

$$\omega(\mathbf{s}, \mathbf{t})\widehat{\omega}(-\mathbf{s}, -\mathbf{t}) \leq (1 + |\mathbf{s}| + |\mathbf{t}|)^\alpha \omega_0(\mathbf{s}, \mathbf{t})\omega_0(-\mathbf{s}, -\mathbf{t}).$$

Therefore

$$\liminf_{n\to\infty} \frac{\omega(n\mathbf{s}, n\mathbf{t})\widehat{\omega}(-n\mathbf{s}, -n\mathbf{t})}{n} = 0$$

for all $\mathbf{s} \in \mathbb{R}^m, \mathbf{t} \in \mathbb{Z}^k$. By Theorem 4.1, $L^1(G, \omega)$ is 2-weakly amenable. $\square$

Remark 4.3. The product weights discussed in [7, page 168] clearly satisfy the condition of the above corollary. So the 2-weak amenability of the corresponding Beurling algebras also follows from this corollary.

5. Further comments on open problems

1. Weak amenability of non-Abelian Beurling algebras.

   When G is not Abelian and ω is not trivial, except for the necessary condition discussed in Remark 3.2 for IN groups, weak amenability of $L^1(G, \omega)$ is completely unknown. In fact, to the author's knowledge, thus far in the literature there is not even an example of weakly amenable non-Abelian Beurling algebras with a non-trivial weight.
2. 2-Weak amenability of Beurling algebras.

   2-weak amenability of $L^1(G)$ is closely related to the well-known derivation problem for $L^1(G)$ which asks whether every continuous derivation from $L^1(G)$ into $M(G)$ is inner. The problem has recently been solved affirmatively in general by V. Losert [17]. The derivation problem for a Beurling algebra $L^1(G, \omega)$ is still open and seems not to be approachable by the method of Losert. In general, we would like to know when $L^1(G, \omega)$ is 2-weakly amenable. For Abelian groups G, after our Theorem 4.1 we would like to know whether condition (4.1) is also necessary for $L^1(G, \omega)$ to be 2-weakly amenable.
3. Weak amenability of the center algebra of a non-abelian Beurling algebra.

   The center $ZL^1(G, \omega)$ of $L^1(G, \omega)$ is an Abelian Banach subalgebra of $L^1(G, \omega)$. It is well-known that $ZL^1(G, \omega)$ is not trivial if and only if G is an IN group. Since $ZL^1(G, \omega) = L^1(G, \omega)$ when G is abelian, studying weak amenability of $ZL^1(G, \omega)$ may be regarded as a natural extension to the investigation of weak amenability for Abelian Beurling algebras. Even for $\omega \equiv 1$, we do not know a full answer to whether $ZL^1(G, \omega)$ (abbreviated $ZL^1(G)$) is weakly amenable. However, it was shown in [1, Theorem 2.4] that $ZL^1(G)$ is hyper-Tauberian if each conjugacy class of G is relatively compact. Consequently, $ZL^1(G)$ is weakly amenable for this type of group G [1, Theorem 0.2(i)]. In particular, $ZL^1(G)$ is weakly amenable if G is compact. We conclude this paper with a simple proof of this last fact. The proof is based on the famous Peter-Weyl theorem.

Proposition 5.1. *For every compact group G, $ZL^1(G)$ is weakly amenable.*

Proof. It is a simple fact that if an Abelian Banach algebra $\mathcal{A}$ contains a subset E of mutually annihilating idempotents (that is, $e^2 = e$ for all $e \in E$, and $e_1 e_2 = 0$ if $e_1, e_2 \in E$ and $e_1 \neq e_2$) and if $span(E)$ is dense in $\mathcal{A}$, then $\mathcal{A}$ is weakly amenable. Indeed, when G is compact, $ZL^1(G)$ has such a subset $E = \{d_\pi \chi_\pi : \pi \in \widehat{G}\}$ (see [14, Section 27]), where $\widehat{G}$ is the dual object of G, d_π is the dimension of the associated irreducible unitary representation π, and χ_π is the character of the representation π. Therefore, $ZL^1(G)$ is weakly amenable if G is compact. □

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References

[1] A. Azimifard, E. Samei, and N. Spronk, *Amenability properties of the centres of group algebras*, J. Funct. Anal. **256** (2009), no. 5, 1544–1564, DOI 10.1016/j.jfa.2008.11.026. MR2490229 (2010e:43002)

[2] W. G. Bade, P. C. Curtis Jr., and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. (3) **55** (1987), no. 2, 359–377, DOI 10.1093/plms/s3-55_2.359. MR896225 (88f:46098)

[3] S. J. Bhatt and H. V. Dedania, *A Beurling algebra is semisimple: an elementary proof*, Bull. Austral. Math. Soc. **66** (2002), no. 1, 91–93, DOI 10.1017/S0004972700020700. MR1922610 (2004d:43004)

[4] Y. Choi, F. Ghahramani, and Y. Zhang, *Approximate and pseudo-amenability of various classes of Banach algebras*, J. Funct. Anal. **256** (2009), no. 10, 3158–3191, DOI 10.1016/j.jfa.2009.02.012. MR2504522 (2010f:46072)

[5] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, New Series, vol. 24, The Clarendon Press, Oxford University Press, New York, 2000. MR1816726 (2002e:46001)

[6] H. G. Dales, F. Ghahramani, and N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), no. 1, 19–54. MR1489459 (99g:46064)

[7] H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **177** (2005), no. 836, vi+191. MR2155972 (2006k:43002)

[8] G. I. Gaudry, *Multipliers of weighted Lebesgue and measure spaces*, Proc. London Math. Soc. (3) **19** (1969), 327–340. MR0243000 (39 #4325)

[9] F. Ghahramani and G. Zabandan, *2-weak amenability of a Beurling algebra and amenability of its second dual*, Int. J. Pure Appl. Math. **16** (2004), no. 1, 75–86. MR2095738

[10] N. Grønbæk, *Commutative Banach algebras, module derivations, and semigroups*, J. London Math. Soc. (2) **40** (1989), no. 1, 137–157, DOI 10.1112/jlms/s2-40.1.137. MR1028919 (91b:46045)

[11] N. Groenbaek, *A characterization of weakly amenable Banach algebras*, Studia Math. **94** (1989), no. 2, 149–162. MR1025743 (92a:46055)

[12] N. Grønbæk, *Amenability of weighted convolution algebras on locally compact groups*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 765–775, DOI 10.2307/2001264. MR962282 (90j:43003)

[13] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979. MR551496 (81k:43001)

[14] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*, Die Grundlehren der mathematischen Wissenschaften, Band 152, Springer-Verlag, New York, 1970. MR0262773 (41 #7378)

[15] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, No. 127, American Mathematical Society, Providence, R.I., 1972. MR0374934 (51 #11130)

[16] B. E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. **23** (1991), no. 3, 281–284, DOI 10.1112/blms/23.3.281. MR1123339 (92k:43004)

[17] V. Losert, *The derivation problem for group algebras*, Ann. of Math. (2) **168** (2008), 221–246. MR2415402 (2009j:43007)

[18] E. Samei, *Weak amenability and 2-weak amenability of Beurling algebras*, J. Math. Anal. Appl. **346** (2008), no. 2, 451–467, DOI 10.1016/j.jmaa.2008.05.085. MR2431541 (2009j:46119)

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