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PROJECTIVE VARIETIES COVERED BY ISOTRIVIAL FAMILIES

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Abstract. Let X, Y be projective schemes over a discrete valuation ring R, where Y is generically smooth and $g : X \to Y$ is a surjective R-morphism such that $g_*\mathcal{O}_X = \mathcal{O}_Y$. We show that if the family $X \to Spec(R)$ is isotrivial, then the generic fiber of the family $Y \to Spec(R)$ is isotrivial.

1. Introduction

Let k be a field of characteristic zero and C a smooth projective curve defined over k with function field F.

Definition 1.1. Let S be a scheme over k and $\pi : X \to S$ a flat family of schemes. The family π is called *trivial* if there exists a scheme X_0 defined over k such that $X \cong X_0 \times_k S$, and it is called *isotrivial* if there exists a finite surjective étale extension $S' \to S$ such that $\pi_{S'} : X \times_S S' \to S'$ is trivial.

Let R be the local ring at a closed point $P \in C$. Let X, Y be projective schemes over $Spec(R)$ where Y is generically smooth and $g : X \to Y$ is a surjective R-morphism such that $g_*\mathcal{O}_X = \mathcal{O}_Y$. We show in Theorem 2.7 that if the family $X \to Spec(R)$ is isotrivial, then the generic fiber of the family $Y \to Spec(R)$ is isotrivial.

Sketch of proof. The deformations of Y are controlled by its differentials. We study the deformations locally, i.e. over the discrete valuation ring R, and show that the fundamental exact sequence associated to the diagram

$$X \to Y \xrightarrow{p} Spec(R) \to Spec(k),$$

$$0 \to p^*\Omega_{Spec(R)/Spec(k)} \to \Omega_{Y/Spec(k)} \to \Omega_{Y/Spec(R)} \to 0 \tag{1}$$

is split exact and consequently the deformations of Y are governed by those of X. Then we consider an infinitesimal deformation of $Y \to Spec(R)$ over the henselization of R (denoted by $\tilde{R}$) and show that the sequence above remains split exact at every level of the deformation. Finally we use a result of Greenberg to pass from $\tilde{R}$ to R.

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The proof can probably be adapted as is to the positive characteristic case if we assume the extension F/k is separable. As an application we shall use this result in a forthcoming paper with Lucien Szpiro on the parametrization of points of canonical height zero of an algebraic dynamical system.

In [2], ([3], Thm. 3.3, p. 702) Chatzidakis and Hrushovski answer the same question using model theoretic methods. Their methods are intrinsically birational; thus they have a slightly less precise conclusion. They assume the extension F/k is regular. If the ground field k is perfect and Y is one dimensional, then their result extends to positive characteristic.

Notation. Throughout this paper k denotes a field of characteristic zero and C a smooth projective curve over k with function field F. Given a scheme X over C we denote its generic fiber by X_F.

2. Descent

In this section we assume that k is algebraically closed. The proofs work without this hypothesis with some minor modifications. Let P be a closed point on the curve C and $R = \mathcal{O}_{C,P}$, the local ring at P. For ease of notation we denote $Spec(R)$ and $Spec(k)$ by R and k respectively in the sheaves of differentials.

Proposition 2.1. *Let*

$$X \xrightarrow{g} Y \xrightarrow{p} Spec(R) \to Spec(k)$$

be morphisms of schemes where X, Y are projective, $X \to Spec(R)$ is an isotrivial family, Y is reduced and $g : X \to Y$ is a surjective R-morphism such that $g_*\mathcal{O}_X = \mathcal{O}_Y$. Then the sequence of differentials on Y,

$$p^*\Omega_{R/k} \to \Omega_{Y/k} \to \Omega_{Y/R} \to 0, \tag{2}$$

is split exact.

Proof. After a quasi-finite unramified base change $Spec(R') \to Spec(R)$ we may assume that the family $X_{R'} \to Spec(R')$ is trivial. Slightly abusing the notation we shall say the family $X \to Spec(R)$ is trivial. It follows that the sequence of differentials on X,

$$0 \to g^*p^*\Omega_{R/k} \to \Omega_{X/k} \to \Omega_{X/R} \to 0, \tag{3}$$

is split exact. Since pullbacks preserve right exactness, the sequence of differentials on Y pulled back to X along g,

$$g^*p^*\Omega_{R/k} \to g^*\Omega_{Y/k} \to g^*\Omega_{Y/R} \to 0, \tag{4}$$

is exact. The morphism $g : X \to Y$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} & & g^*p^*\Omega_{R/k} & \longrightarrow & g^*\Omega_{Y/k} & \longrightarrow & g^*\Omega_{Y/R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & g^*p^*\Omega_{R/k} & \longrightarrow & \Omega_{X/k} & \longrightarrow & \Omega_{X/R} & \longrightarrow & 0 \end{array} \tag{5}$$

It follows that (4) is split exact. Since g_* preserves direct sums, we have

$$g_*g^*\Omega_{Y/k} \cong g_*g^*\Omega_{Y/R} \oplus g_*g^*p^*\Omega_{R/k}.$$

The natural map $\Omega_{Y/k} \to g_*g^*\Omega_{Y/k}$ induces the following commutative diagram:

$$\begin{tikzcd}
 & p^*\Omega_{R/k} \arrow[r] \arrow[d] & \Omega_{Y/k} \arrow[r] \arrow[d] & \Omega_{Y/R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_*g^*p^*\Omega_{R/k} \arrow[r] & g_*g^*\Omega_{Y/k} \arrow[r] & g_*g^*\Omega_{Y/R} \arrow[r] & 0
\end{tikzcd}$$

Note that the bottom row is split exact, $p^*\Omega_{R/k} \cong \mathcal{O}_Y$ and $g_*g^*p^*\Omega_{R/k} \cong g_*\mathcal{O}_X$. By assumption $g_*\mathcal{O}_X = \mathcal{O}_Y$; thus the top row is split exact. $\square$

Remark. The condition $g_*\mathcal{O}_X = \mathcal{O}_Y$ implies that g has connected fibers ([7], Cor. 11.3, p. 279). If g is finite, then the condition $g_*\mathcal{O}_X = \mathcal{O}_Y$ implies that $\deg(g) = 1$; hence g is birational. Moreover if Y is normal, using Zariski's Main Theorem ([8], p. 209) we conclude that g is an isomorphism.

We now consider an infinitesimal deformation of Y over the henselian discrete valuation ring, denoted $\tilde{R}$, and proceed to show that the family $Y_F \to Spec(F)$ is isotrivial. Before we proceed we need the following definitions:

Definition 2.2. Let S be a smooth scheme of finite type over k and $f : X \to S$ a morphism of schemes. If f is smooth, then the sequence

$$0 \to f^*\Omega_{S/k} \to \Omega_{X/k} \to \Omega_{X/S} \to 0 \tag{6}$$

is exact. This extension is non-trivial in general and is given by a class $c \in Ext^1(\Omega_{X/S}, f^*\Omega_{S/k})$. Since $\Omega_{X/S}$ is locally free, one has

$$Ext^1_{\mathcal{O}_X}(\Omega_{X/S}, f^*\Omega_{S/k}) \cong Ext^1_{\mathcal{O}_X}(\mathcal{O}_X, T_{X/S} \otimes f^*\Omega_{S/k}) \cong H^1(X, T_{X/S} \otimes f^*\Omega_{S/k}).$$

The image of c by the canonical map

$$\begin{array}{rc} H^1(X, T_{X/S} \otimes f^*\Omega_{S/k}) \to & H^0(S, R^1f_*(T_{X/S} \otimes f^*\Omega_{S/k})) \\ & || \\ & H^0(S, R^1f_*T_{X/S} \otimes \Omega_{S/k}) \end{array}$$

is called the *Kodaira-Spencer class of* X/S. One can view this class as a morphism also, i.e. the Kodaira-Spencer morphism

$$\kappa_{X/S} : T_S \to R^1f_*T_{X/S}.$$

The fiber $(\kappa_{X/S})_s = \kappa_s : T_{S,s} \to H^1(X_s, T_{X_s})$ is the Kodaira-Spencer map at $s \in S$.

The Kodaira-Spencer map at s measures how X_s deforms in the family X/S in the neighborhood of s ([1], p. 165).

Definition 2.3 ([4], p. 255)**.** A local ring A is *henselian* if every finite A-algebra B is a product of local rings. We define the *henselization* of A to be a pair $(\tilde{A}, i)$, where $\tilde{A}$ is a local henselian ring and $i : A \to \tilde{A}$ is a local homomorphism such that for any local henselian ring B and any local homomorphism $u : A \to B$ there exists a unique local homomorphism $\tilde{u} : \tilde{A} \to B$ such that $u = \tilde{u} \circ i$.

From here on we assume that Y is generically smooth. Let R be the local ring at $P \in C$, $\tilde{R}$ its henselization, and $\tilde{\mathfrak{m}}$ the maximal ideal of $\tilde{R}$. Define $R_n = \tilde{R}/\tilde{\mathfrak{m}}^{n+1}$ for each $n \geq 0$. There are natural maps $Spec(\tilde{R}) \to Spec(R)$ and $Spec(R_{n-1}) \to Spec(R_n)$ induced by the projections $R_n \to R_{n-1}$ for $n \geq 1$. Define $\tilde{Y} := Y \times_R Spec(\tilde{R})$, $Y_n := \tilde{Y} \times_R Spec(R_n)$ for each $n \geq 0$. We have the following commutative diagram of schemes:

$$
\begin{tikzcd}
Y_F \arrow[r] \arrow[d] & Y \arrow[d, "p"] & \tilde{Y} \ldots \arrow[l] \arrow[d] & Y_n \ldots \arrow[l] \arrow[d, "p_n"] & Y_0 \arrow[l] \arrow[d, "p_0"] \\
Spec(F) \arrow[r] & Spec(R) & Spec(\tilde{R}) \ldots \arrow[l] & Spec(R_n) \ldots \arrow[l] & Spec(k) \arrow[l]
\end{tikzcd} \tag{7}
$$

Proposition 2.4. *For each $n \geq 0$ the sequence of differentials associated to $Y_n \to Spec(R_n)$, i.e.*

$$0 \to p_n^* \Omega_{R_n/k} \to \Omega_{Y_n/k} \to \Omega_{Y_n/R_n} \to 0, \tag{8}$$

is split exact. Moreover, $Y_n \to Spec(R_n)$ is trivial.

Proof. Pulling back (2) along the natural map $Spec(R_n) \to Spec(R)$ we get the sequence (8). Since pullbacks preserve direct sums, the sequence (8) is split exact; i.e. the Kodaira-Spencer class of $Y_n/Spec(R_n)$ is trivial. In other words, $Y_n \to Spec(R_n)$ is trivial. It follows that $Y_n \cong Y_0 \times_k Spec(R_n)$ for each $n \geq 0$. □

Definition 2.5. If V, W and T are S-schemes, an *S-isomorphism from V to W* parametrized by T will mean a T-isomorphism from $V \times_S T \to W \times_S T$. The set of all such isomorphisms will be denoted by $\underline{Isom}_S(V, W)(T)$.

The association $T \mapsto \underline{Isom}_S(V, W)(T)$ defines a contravariant functor

$$\underline{Isom}_S(V, W) : (Sch/S)^\circ \to (Sets).$$

The functor $\underline{Isom}_S(V, W)$ is representable whenever V, W are flat and projective over S. For a proof of the representability of the $\underline{Isom}$ functor we refer the reader to ([5] pp. 132-133). We denote the scheme representing the functor $\underline{Isom}_S(V, W)$ by $Isom_S(V, W)$.

To conclude that the family $Y_F \to Spec(F)$ is isotrivial we need the following result of Greenberg:

Theorem 2.6. *Let $\tilde{R}$ be a henselian discrete valuation ring, with t the generator of the maximal ideal. Let $\tilde{Z}$ be a scheme of finite type over $\tilde{R}$. Then $\tilde{Z}$ has a point in $\tilde{R}$ if and only if $\tilde{Z}$ has a point in $\tilde{R}/t^n$ for every $n \geq 1$.*

Proof. [6], Corollary 2. □

Theorem 2.7. *Let X, Y be projective schemes over a discrete valuation ring R where $X \to Spec(R)$ is an isotrivial family, Y is generically smooth and $g : X \to Y$ is a surjective R-morphism such that $g_* \mathcal{O}_X = \mathcal{O}_Y$. Then $Y \to Spec(R)$ is generically isotrivial, i.e.*

$$Y_{F'} \cong Y_0 \times_k Spec(F'),$$

where F' is a finite extension of F.

Proof. Observe that $\tilde{Y}$ and $Y_0 \times_k Spec(\tilde{R})$ are flat, projective over $Spec(\tilde{R})$. Let $\underline{Isom}_{\tilde{R}}(\tilde{Y}, Y_0 \times_k Spec(\tilde{R}))(T)$ be the set of isomorphisms from

$$\tilde{Y} \times_{\tilde{R}} T \to (Y_0 \times_k Spec(\tilde{R})) \times_{\tilde{R}} T.$$

Let $f \in Isom_{\tilde{R}}(\tilde{Y}, Y_0 \times_k Spec(\tilde{R}))(k)$ and Γ_f denote the graph of f. Let L be a very ample line bundle of Y and $\tilde{L}$ the pullback of L to Y along the morphism $\tilde{Y} \to Y$. Let $P(t)$ be the Hilbert polynomial of $(\Gamma_f)_k$, the special fiber of Γ_f, with respect to $\tilde{L}$. Then the functor

$$\underline{Isom}_{\tilde{R}}(\tilde{Y}, Y_0 \times_k Spec(\tilde{R})) \cap \underline{Hilb}^{P(t)}_{\tilde{Y} \times_{\tilde{R}} (Y_0 \times_k Spec(\tilde{R}))}$$

is representable and the scheme representing it, denoted by Z', is of finite type. For

$$Z = Isom_R(Y, Y_0 \times Spec(R)) \cap Hilb^{P(t)}_{Y \times_R (Y_0 \times_k Spec(R))}$$

we have $Z' = Z \times_R Spec(\tilde{R})$. By Proposition 2.4, $Y_n \cong Y_0 \times_k Spec(R_n)$ for each $n \geq 0$. Thus Z' has an R_n-point for every $n \geq 0$. By the previous theorem Z' has a $\tilde{R}$-point, i.e. $\tilde{Y} \cong Y_0 \times_k Spec(\tilde{R})$. Note that $\tilde{R}$ is a limit of etale covers of R so there exists an etale cover R' of R such that $Y_{R'} \cong Y_0 \times_k Spec(R')$. Thus F' (the quotient field of R') satisfies the requirements of the theorem. $\square$

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References

[1] José Bertin, Jean-Pierre Demailly, Luc Illusie, and Chris Peters, *Introduction to Hodge theory*, SMF/AMS Texts and Monographs, vol. 8, American Mathematical Society, Providence, RI, 2002. Translated from the 1996 French original by James Lewis and Peters. MR1924513 (2003g:14009)

[2] Zoé Chatzidakis and Ehud Hrushovski, *Difference fields and descent in algebraic dynamics. I*, J. Inst. Math. Jussieu **7** (2008), no. 4, 653–686, DOI 10.1017/S1474748008000273. MR2469450 (2009m:03051a)

[3] Zoé Chatzidakis and Ehud Hrushovski, *Difference fields and descent in algebraic dynamics. II*, J. Inst. Math. Jussieu **7** (2008), no. 4, 687–704, DOI 10.1017/S1474748008000170. MR2469451 (2009m:03051b)

[4] Jean Alexandre Dieudonné, *A panorama of pure mathematics*, As seen by N. Bourbaki. Translated from the French by I. G. Macdonald. Pure and Applied Mathematics, vol. 97, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982. MR662823 (83e:00003)

[5] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Grothendieck's FGA explained, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005. MR2222646 (2007f:14001)

[6] Marvin J. Greenberg, *Rational points in Henselian discrete valuation rings*, Inst. Hautes Études Sci. Publ. Math. **31** (1966), 59–64. MR0207700 (34 #7515)

[7] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977. MR0463157 (57 #3116)

[8] David Mumford, *The red book of varieties and schemes*, Second, expanded edition, includes the Michigan lectures (1974) on curves and their Jacobians, with contributions by Enrico Arbarello, Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, 1999. MR1748380 (2001b:14001)

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