PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 7, July 2014, Pages 2303–2311
S 0002-9939(2014)11977-0
Article electronically published on March 28, 2014

FAN TYPE CONDITION AND CHARACTERIZATION OF HAMILTONIAN GRAPHS

KEWEN CHAO, CHUNWEI SONG, AND PING ZHANG

(Communicated by Jim Haglund)

ABSTRACT. Let G be a simple graph of order $n \geq 3$. Ore's classical theorem states that if $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian (*Amer. Math. Monthly* 67 (1960), 55). In 1984, Fan proved that if G is 2-connected and $\max\{d(x), d(y)\} \geq n/2$ for each pair of vertices x, y with distance 2, then G is Hamiltonian. Fan's result is a significant improvement to Ore's theorem, and the condition stated is called Fan's condition. Then in 1987, Benhocine and Wojda showed that if G is a 2-connected graph with independence number $\alpha(G) \leq n/2$, and $\max\{d(x), d(y)\} \geq \frac{n-1}{2}$ for each pair of vertices x, y with distance 2, then G is Hamiltonian with some exceptions: either G is Hamiltonian or G belongs to one of two classes of well characterized graphs. In 2007, Li et al. removed the independence restriction, but they also reversed the Fan type bigger degree lower bound condition to the stronger Ore type degree sum requirement.

In our present work, we drop the independence restriction while keeping Benhocine and Wojda's relaxed Fan type condition to prove that if G is 2-connected and $\max\{d(x), d(y)\} \geq \frac{n-1}{2}$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or G is among certain classes of well characterized graphs. This extends previous results in the literature.

1. Introduction and main results

Throughout this work we consider only simple graphs, i.e., graphs without multi-edges or loops. For a graph G, let $V(G)$ be the vertex set and $E(G)$ be the edge set. The complement graph of G, denoted by G^c, has the same vertex set $V(G^c) = V(G)$, but an edge is in $E(G^c)$ if and only if it is not in $E(G)$. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from the disjoint union $G_1 + G_2$ by adding the edges $\{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$. We use $\mathcal{H}_k$ to represent the collection of all graphs H of order k.

For two vertices u and v, let $d(u, v)$ be the length of a minimum path between vertices u and v in G, or equivalently the distance between u and v. The minimum degree of a graph G is denoted by $\delta(G)$ or simply δ if the graph G under consideration is understood. We also use $u \sim v$ to mean u and v are adjacent and $u \nsim v$ to represent the contrary.

Received by the editors August 26, 2011 and, in revised form, July 26, 2012.
2010 *Mathematics Subject Classification.* Primary 05C45, 05C75; Secondary 05C07, 05C38.
Key words and phrases. Hamiltonian, Fan condition, Benhocine-Wojda condition, Ore condition.
The first author was supported in part by Science Foundation of Hainan Province Grant #10501.
The second author was supported in part by 973 Program Grant #2013CB834201 and NSFC Grant #11101009.

For a subgraph H of G and a vertex u of $V(G)$, let $N_H(u)$ be the set of vertices in H that are adjacent to u, which is called the neighborhood of v in H, and denote the cardinality of $N_H(u)$ by $d_H(u)$. In particular, if $H = G$, then we often omit the subscript H; in this case, $d(u) = |N(u)|$ is the degree of u. In addition, let $G - H$ and $G[S]$ denote the subgraphs of G induced by $V(G) - V(H)$ and S, respectively.

A cycle in a graph G that contains every vertex of G is called a Hamiltonian cycle. A Hamiltonian graph is a graph that contains a Hamiltonian cycle.

We refer to the books [3], [12] and [4] for graph theory notation and terminology not described above. To determine whether a Hamiltonian cycle exists in a given graph, i.e. whether a given graph is a Hamiltonian graph, is NP-complete (see [7] or [11], for instance). For a marvelous survey on the Hamiltonian problem up to 2002, see [8].

In 1952, Dirac provided a nontrivial sufficient condition for a graph to be Hamiltonian, which was probably the first milestone in the area.

Theorem 1.1. (Dirac, 1952 [5]). *If G is a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

In 1960, Ore gave the following refinement to Dirac's theorem.

Theorem 1.2. (Ore [10]). *If G is a graph of order $n \geq 3$ and $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian.*

Since the establishment of the now-classical Theorem 1.2, efforts to improve it have never stopped, although this proves to be an extremely difficult job. One direction is to decrease the lower bound, such as the following result of 1985.

Theorem 1.3. (Ainouche and Christofides [1, Corollary 5]). *If G is a graph of order $n \geq 3$ and $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then either G is Hamiltonian or $G \in \mathcal{H}_{(n-1)/2} \vee K^c_{(n+1)/2}$, where in the second case $\mathcal{H}_{(n-1)/2}$ denotes the collection of all graphs with order $\frac{n-1}{2}$ and n is certainly odd.*

Another direction led to G. Fan's condition and his theorem, a breakthrough made in 1984. Clearly, being Hamiltonian implies 2-connectedness.

Theorem 1.4. (Fan, 1984 [6]). *If G is a 2-connected graph of order $n \geq 3$ and $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for each pair of vertices x, y with distance 2, then G is Hamiltonian.*

Note that Ore's condition actually implies 2-connectivity (without assuming the Hamiltonian cycle), while the lower bound of degree sums as described in Theorem 1.3 does not necessarily require it.

Then in 1987 Benhocine and Wojda extended Theorem 1.4 by slightly (but not that easily!) decreasing its lower bound requirement.

Theorem 1.5. (Benhocine and Wojda, 1987 [2]). *If G is a 2-connected graph of order $n \geq 3$ with independence number $\alpha(G) \leq n/2$, and $\max\{d(x), d(y)\} \geq \frac{n-1}{2}$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in \mathcal{G}_n$ or $G = H_9$, where $\mathcal{G}_n$ is a class of well characterized graphs and H_9 is a specific graph of order 9.*

In 2007, Li et al. removed the independence restriction, but in considering the 2-distance pairs they reversed the Fan type bigger degree requirement to the Ore type degree sum requirement.

Theorem 1.6. (Li, Li and Feng, 2007 [9]). *If G is a 2-connected graph of order $n \geq 3$, and $d(x) + d(y) \geq n - 1$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or G belongs to an exceptional class of graphs $\mathcal{L}_n$.*

In this present paper, we drop the independence restriction as stated in Theorem 1.5 while assuming the slightly relaxed Fan type condition to give a Hamiltonian characterization, therefore improving the results mentioned above in some sense.

Theorem 1.7. (Main theorem). *If G is a 2-connected graph of order $n \geq 3$, and $\max\{d(x), d(y)\} \geq \frac{n-1}{2}$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in \mathcal{G}_n^*$ or $G \in \mathcal{J}_n$ or $G = H_9$, where $\mathcal{G}_n^*$ and $\mathcal{J}_n$ are two classes of well characterized graphs and H_9 is the graph prescribed in Theorem 1.5.*

2. Proof of the main theorem

Note that to remove the limit on the independence number as described in Theorem 1.5, it suffices to deal with the case $\alpha(G) = \lfloor \frac{n}{2} \rfloor + 1$ only. This is because with the conditions given, the independence number actually cannot be bigger. Indeed, if there is any independence set of order $m > \lfloor \frac{n}{2} \rfloor + 1$, call it I, then there must be some vertex v of $G[V - I]$ having at least two neighbors $x, y \in I$ (given the connectivity and respective cardinalities). So $d(x, y) = 2$ and thus $\frac{n-1}{2} \leq \max\{d(x), d(y)\}$, but clearly $\max\{d(x), d(y)\} \leq n - m < n - \lfloor \frac{n}{2} \rfloor - 1 \leq \frac{n-1}{2}$. Nonetheless, in general we do not consider independence in our treatment, unless occasionally for presentational necessities. Our principal technique is to investigate certain pairs of vertices, where each vertex has degree $\geq \frac{n-1}{2}$ and has distance 2 to some other vertex with insufficient degree, but the distance between the two vertices in the studied pair is not necessarily 2, and to study their degree sum in the specific configuration. With the pair cleverly chosen at different stages, this method proves to be effective. Another basis of our results is a lemma in Fan's 1984 paper.

Lemma 2.1 ([6]). *Let G be a 2-connected graph of order $n \geq 3$ and $\max\{d(x), d(y)\} \geq k/2$ for each pair of vertices x, y with distance 2, where k is any positive integer between 3 and n. Then the circumference of G is at least k.*

The proof of our main theorem is achieved by three separate results according to the magnitude of n.

Before we proceed, let's make clear the graph classes that we talk about.

Definition 2.2. For any odd integer $n \geq 7$, let $\mathcal{G}_n^*$ be the following class of graphs on n vertices such that $G \in \mathcal{G}_n^*$ if its vertices are divided into three parts: a 2-path or 3-clique $G[(x, y, z)]$, and two disjoint cliques A and B, both of order $\frac{n-3}{2}$. Furthermore, there exist two vertices in each of A and B, denoted by a_1, a_2 and b_1, b_2 respectively, any of which is adjacent to both x and z. The only other possible edges in the graph are those linking x and z to the other vertices of A and B, where the following type of pairs a, b are forbidden: $a \in A - \{a_1, a_2\}$ and $b \in B - \{b_1, b_2\}$, and either $N_{G[\{x,z\}]}(a) = N_{G[\{x,z\}]}(b) = \{x\}$ or $N_{G[\{x,z\}]}(a) = N_{G[\{x,z\}]}(b) = \{z\}$.

For any odd integer $n \geq 5$, define $\mathcal{J}_n$ to be the class of graphs $\mathcal{H}_{\frac{n-1}{2}} \vee K^c_{\frac{n+1}{2}} - je - j'e'$, where je means j edges all of which share an end from $K^c_{\frac{n+1}{2}}$ (called an "incomplete end"), with $0 \leq j \leq \frac{n-5}{2}$ and $j'e'$ is similarly defined.

Proposition 2.3. *If G is a 2-connected graph of order $n \leq 8$, and $\max\{d(x), d(y)\} \geq \frac{n-1}{2}$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian, or $G \in \mathcal{G}_7^*$ (certainly when $n = 7$) or $G \in \mathcal{J}_n$.*

Proof. The difference between our condition and Fan's condition allows us to consider only odd-order graphs, i.e. when $n = 3, 5$ or 7. The case $n = 3$ is trivial.

For $n = 5$, by Lemma 2.1, assume G is not Hamiltonian; then there is a 4-cycle $x_1x_2x_3x_4x_1$. To be 2-connected (while avoiding being Hamiltonian), the remaining vertex x_5 should be adjacent to exactly 2 vertices that must be nonconsecutive, say x_2 and x_4. It is easy to see that $x_1 \nsim x_3$. Let $G[x_1, x_3, x_5] = K_3^c$. So in this case $G \in \mathcal{J}_5 = \mathcal{H}_2 \vee K_3^c$, where $G[x_2, x_4] \in \mathcal{H}_2$ is either an edge or a nonedge.

For $n = 7$, by Lemma 2.1, assume G is not Hamiltonian; then there is a 6-cycle $x_1x_2 \cdots x_6x_1$. Let the remaining vertex be x_7. Clearly $2 \leq d(x_7) \leq 3$ (to be both 2-connected and non-Hamiltonian).

(1) If $d(x_7) = 3$, w.l.o.g., let $N(x_7) = \{x_2, x_4, x_6\}$. It's easy to see that x_1, x_3, x_5 must be an independent set, and thus $G[x_1, x_3, x_5, x_7]$ is the empty graph K_4^c. Moreover, at least two of the statements $x_1 \sim x_4$, $x_3 \sim x_6$ and $x_5 \sim x_2$ have to be true. This is because otherwise there will be degree deficiency: if $x_1 \nsim x_4$, $x_3 \nsim x_6$, say, then $d(x_1) + d(x_3) \leq 4$ while $d(x_1, x_3) = 2$, violating the condition. As for $G[x_2, x_4, x_6]$, it may be anything between K_3^c and K_3, as $G[x_1, x_3, x_5, x_7] = K_4^c$ already implies non-Hamiltonian. Therefore in this case $G \in \mathcal{H}_3 \vee K_4^c$ or $G \in \mathcal{H}_3 \vee K_4^c - e$, where "$e$" represents an edge between $\mathcal{H}_3$ and K_4^c (by our notation e is one of x_1x_4, x_3x_6, x_5x_2, isomorphically all the same). This means precisely $G \in J_7$.

(2) If $d(x_7) = 2$, let x_1 be one of the neighbors of x_7. For the other neighbor, by symmetry, we only need to consider two cases: x_3 or x_4. If x_3 is the other neighbor of x_7, then $x_1x_7x_3x_4x_5x_6x_1$ is a 6-cycle, with x_2 being left out. However, $d(x_7, x_2) = 2$ and $d(x_7) = 2$ implies that $d(x_2) \geq (7-1)/2 = 3$, thus we return to the previous case (by replacing x_7 with x_2). So the only remaining situation is when x_4 is the other neighbor of x_7. Hence x_7 has distance 2 to each of x_2, x_3, x_5, x_6, and therefore $\min\{d(x_2), d(x_3), d(x_5), d(x_6)\} \geq 3$. If any of the relations $x_2 \sim x_5$, $x_2 \sim x_6$, $x_3 \sim x_5$ and $x_3 \sim x_6$ are true, then there is either a 7-cycle or a 6-cycle, leaving a vertex of degree ≥ 3. So the missing degrees are supplemented by $x_2 \sim x_4$, $x_3 \sim x_1$, $x_5 \sim x_1$ and $x_6 \sim x_4$ exactly. Now the structure is clear: we have either a 2-path or a 3-clique ($x_1x_7x_4$), or two 2-cliques (x_2x_3 and x_5x_6 respectively). In each of the two 2-cliques, each of the two vertices is adjacent to both ends of the 2-path. Hence $G \in \mathcal{G}_7^*$. □

At this stage let's point out that while expressed in a slightly different way, our class $\mathcal{G}_n^*$ is a superset of the class $\mathcal{G}_n$ as defined in [2, (i)-(vi), p. 168]. It is natural since we have relatively weaker requirements (without assuming the independence number).

Proposition 2.4. *If G is a 2-connected graph of order $n = 9$, and* $\max\{d(x), d(y)\} \geq 4$ *for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in \mathcal{G}_9^*$ or $G \in \mathcal{J}_9$ or $G = H_9$, where H_9 is the graph prescribed in Theorem* 1.5.

Proof. Suppose G is not Hamiltonian, $G \notin \mathcal{G}_9^*$ and $G \neq H_9$. As we have observed, the only case not covered by Theorem 1.5 is when $\alpha(G) = 5$. Again by Lemma 2.1, assume G is not Hamiltonian; then there is an 8-cycle $x_1x_2 \cdots x_8x_1$. Let the remaining vertex be x_9. Now $2 \leq d(x_9) \leq 4$ is both 2-connected and non-Hamiltonian. The fact $\alpha(G) = 5$ requires a 5-independent set which must include x_9 and four on the 8-cycle, say, x_1, x_3, x_5 and x_7. (1) If $d(x_9) < 4$, then at least three of x_1, x_3, x_5, x_7 have degree at least 4. W.l.o.g., assume $d(x_1), d(x_3), d(x_5) \geq 4$.

This actually means $N(x_1) = N(x_3) = N(x_5) = \{x_2, x_4, x_6, x_8\}$. If $d(x_7) = 4$, then $G \in \mathcal{H}_4 \vee K_5^c - je \subseteq \mathcal{J}_9$, where $0 \leq j \leq 2 = \frac{9-5}{2}$ and the j missing edges share the end x_9. If $d(x_7) < 4$, then $N(x_9) = \{x_2, x_4\}$ and $N(x_7) = \{x_6, x_8\}$, in which case $G \in \mathcal{H}_4 \vee K_5^c - 2e - 2e' \subseteq \mathcal{J}_9$, where x_7 and x_9 are the two incomplete ends of K_5^c.

(2) If $d(x_9) = 4$, w.l.o.g., assume $d(x_1) = d(x_5) = 4$. Either $\max\{d(x_3), d(x_7)\} = 4$, in which case we have $G \in \mathcal{J}_9$, or $N(x_3) = \{x_2, x_4\}$ and $N(x_7) = \{x_6, x_8\}$, in which case we interchange the role of x_3 with that of x_9 and we are back to the situation of (1). □

Now we are prepared to present the big n case.

Theorem 2.5. *If G is a 2-connected graph of order $n \geq 10$, and $\max\{d(x), d(y)\} \geq 4$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in \mathcal{G}_n^*$ or $G \in \mathcal{J}_n$, where $\mathcal{G}_n^*$, $\mathcal{J}_n$ are two classes of well characterized graphs as described in Definition 2.2.*

Proof. To be precise we shall use "$\geq \lceil \frac{n-1}{2} \rceil$" instead of "$\geq \frac{n-1}{2}$". For convenience assume G satisfies the conditions as described in the statement but it is not Hamiltonian.

By Lemma 2.1, G has an $(n-1)$-cycle (given that G is not Hamiltonian). Let's denote this $(n-1)$-cycle by $C_{n-1} = x_1 x_2 \ldots x_{n-1} x_1$. By "consecutive pair on C_{n-1}", we mean two vertices of the type $x_i x_{i+1}$, and by "pair of distance j on C_{n-1}", we mean two vertices of the type $x_i x_{i+j}$ (the pair is probably of shorter distance in G). Throughout the remaining part of this paper the subscript computation is modulo $n-1$.

Let $v = V(G) - V(C_{n-1})$. Observe that there cannot be consecutive pairs on C_{n-1} within $N(v)$, since by inserting v the graph G would be Hamiltonian.

(1) If $d(v) > \lfloor \frac{n-1}{2} \rfloor$, then there has to be a consecutive pair on C_{n-1} within $N(v)$. So this is impossible.

(2) If $d(v) = \lfloor \frac{n-1}{2} \rfloor$, then $N(v) = \{x_i, x_{i+2}, \ldots\}$. Let $A = \{x_{i+1}, x_{i+3}, \ldots\} \cup \{v\}$ and $|A| = \lceil \frac{n-1}{2} \rceil + 1$. Depending on the parity of n, A is either an independent set (when $n-1$ is even) or an independent set plus exactly one edge formed by one consecutive pair (when $n-1$ is odd). In the latter case $d(v) = \frac{n-1}{2} < \lceil \frac{n-1}{2} \rceil$, so for instance $d(x_{i+1}) \geq \lceil \frac{n-1}{2} \rceil$ as $d(v, x_{i+1}) = 2$. But this is impossible because $d(x_{i+1}) \leq d(v) = \frac{n-1}{2} < \lceil \frac{n-1}{2} \rceil$ (take x_{i+1} out of the consecutive pair). Hence $n-1$ must be even (i.e. n is odd). Now $G[A] = K^c_{\frac{n+1}{2}}$, and at most one vertex of $\{x_{i+1}, x_{i+3}, \ldots\}$ may have degree less than $\frac{n-1}{2} = \lceil \frac{n-1}{2} \rceil$. In conclusion, in this case $G \in \mathcal{H}_{\frac{n-1}{2}} \vee K^c_{\frac{n+1}{2}} - je$, where $0 \leq j \leq \frac{n-5}{2}$. Hence any admissible graph G belongs to the class $\mathcal{J}_n$ by definition.

(3) If $3 \leq d(v) < \lfloor \frac{n-1}{2} \rfloor$, w.l.o.g., let $x_{n-1} \sim v$, and due to no consecutive pairs on C_{n-1} within $N(v)$ let $x_i, x_j \in N(v)$ such that $i < j$ and $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}\} \cap N(v) = \emptyset$. Note that $\{v, x_1, x_{i+1}, x_{j+1}\}$ is an independent set, v has distance 2 to each of the other three and since $d(v) < \lfloor \frac{n-1}{2} \rfloor \leq \lceil \frac{n-1}{2} \rceil$, there must be

$$\min\{d(x_1), d(x_{i+1}), d(x_{j+1})\} \geq \lceil \frac{n-1}{2} \rceil.$$

Now we claim that either $x_1 \sim x_{i+2}$ or $x_{i+1} \sim x_{j+2}$. We shall illustrate the major technique of a "deficiency pair" by the following argument. In general, if $\mathcal{P}$ is a statement, then the notation $\delta(\mathcal{P})$ is to be taken equal to one if $\mathcal{P}$ is true and zero if $\mathcal{P}$ is false. Note that for $1 \leq h \leq i$, either $x_1 \nsim x_{h+1}$ or $x_{i+1} \nsim x_h$ must be true, while for $i+1 \le h \le n-1$, either $x_1 \nsim x_h$ or $x_{i+1} \nsim x_{h+1}$ must be true. This is because, otherwise, a Hamiltonian cycle would be constructed using the two new edges. If the claim is not true, then

$$\begin{aligned} n-1 \le 2\lceil \frac{n-1}{2} \rceil &\le d(x_1) + d(x_{i+1}) \\ &\le n-1+\delta(x_1 \sim x_1) + \delta(x_{i+1} \sim x_{i+1}) - |\{x_{j+1}\}| = n-2, \end{aligned}$$

a contradiction. Note that the two judgement functions are to take care of certain neglected points occurred from subscript manipulations. The technique above will be used frequently in our proof. The number of points neglected, which is equivalent to the number of δ supplements, is decided by the nature of a specific configuration.

Hence the claim $x_{i+1} \sim x_{j+2}$ is true. We get an $(n-1)$-cycle

$$x_i v x_j x_{j-1} \cdots x_{i+1} x_{j+2} x_{j+3} \cdots x_i.$$

The only vertex being left out, x_{j+1}, has degree at least $\lceil \frac{n-1}{2} \rceil \ge \lfloor \frac{n-1}{2} \rfloor$, so we are in the previous cases again.

(4) If $d(v) = 2$. W.l.o.g., assume $N(v) = \{x_{n-1}, x_i\}$. Clearly, $2 \le i \le n-2$. The degree condition gives that

$$\min\{d(x_1), d(x_{i-1}), d(x_{i+1}), d(x_{n-2})\} \ge \lceil \frac{n-1}{2} \rceil,$$

as each of the four vertices above has distance 2 to v. Note if x_{n-1} and x_i are taken away, then C_{n-1} is turned into two parts (where it is possible that there are edges between them). For convenience, denote $L = \{x_1, x_2, \dots, x_{i-1}\}$ and $R = \{x_{i+1}, x_{i+2}, \dots, x_{n-2}\}$, whereas $|L| = i-1$ and $|R| = n-2-i$. We discuss it according to the following cases.

Case 1. Neither x_1 nor x_{i+1} has any neighbor in the part that it does not belong to. That is, $N(x_1) \cap R = \emptyset$, and $N(x_{i+1}) \cap L = \emptyset$. We shall show that $G[L \cup R]$ is disconnected with the two components L and R.

Now, $d(x_1) \ge \lceil \frac{n-1}{2} \rceil$ gives that $|\{x_{n-1}, x_i\} \cup L - \{x_1\}| = i \ge \lceil \frac{n-1}{2} \rceil$ and similarly $d(x_{i+1}) \ge \lceil \frac{n-1}{2} \rceil$ gives that $n-1-i \ge \lceil \frac{n-1}{2} \rceil$. Therefore n must be odd and $d(x_1) = d(x_{i+1}) = \frac{n-1}{2}$ with $i = \frac{n-1}{2}$. Specifically $N(x_1) = \{x_{n-1}, x_i\} \cup L - \{x_1\}$ and $N(x_{i+1}) = \{x_{n-1}, x_i\} \cup R - \{x_{i+1}\}$.

Then there must be no edge between $L - x_1$ and $R - x_{i+1}$. Otherwise, suppose $x_l \sim x_r$, where $2 \le l \le i-1$ and $i+2 \le r \le n-2$. But we get a Hamiltonian cycle:

$$v x_i x_{i-1} \cdots x_{l+1} x_1 x_2 \cdots x_l x_r x_{r-1} \cdots x_{i+1} x_{r+1} x_{r+2} \cdots x_{n-1} v.$$

This means that $G[L]$ and $G[R]$ are both cliques. Actually, if $x_{l_1} \nsim x_{l_2}$, where $2 \le l_1, l_2 \le i-1$, say, then $d(x_{l_1}, x_{l_2}) = 2$ as they are both neighbors of x_1, but the degree condition will be unsatisfied.

Let G_1, G_2 be the following graphs of order n with a neat structure: for $i = 1, 2$, $G_i[\{x_{n-1}, v, x_i\}]$ is a 2-path, and $G_i[L]$ and $G_i[R]$ are two disjoint cliques, both of order $\frac{n-3}{2}$ (again n must be odd); as for G_1, each of the ends of L and R, i.e. $x_1, x_{i-1}, x_{i+1}, x_{n-2}$, are adjacent to both ends of the 2-path, i.e. x_{n-1} and x_i; as for $\mathcal{H}_2$, every vertex in the two cliques is adjacent to both ends of the 2-path. In conclusion $G_1 \subseteq G \subseteq G_2$, except that the following type of pairs x_l, x_r are forbidden: $2 \le l \le i-1$ and $i+2 \le r \le n-2$, either $N_{\{x_{n-1}, x_i\}}(x_l) = N_{\{x_{n-1}, x_i\}}(x_r) = \{x_{n-1}\}$ or $N_{\{x_{n-1}, x_i\}}(x_l) = N_{\{x_{n-1}, x_i\}}(x_r) = \{x_i\}$. This is because such a forbidden pair is of distance 2 while having insufficient degrees. So any admissible graph G belongs to the class $\mathcal{G}_n^*$ by definition.

Case 2. W.l.o.g., assume x_1 is adjacent to some vertex of $R = \{x_{i+1}, x_{i+2}, \ldots, x_{n-2}\}$. Apparently $x_1 \nsim x_{i+1}$ and $x_{n-2} \nsim x_{i-1}$, for otherwise there would be an n-cycle. We may further assume that $x_1 \nsim x_{i+2}$ because otherwise we obtain an $(n-1)$-cycle covering v while excluding x_{i+1}, thus returning us to the first 2 cases discussed.

Now let $x_k \in N_R(x_1)$ be chosen with the smallest subscript so that $i+3 \leq k \leq n-2$. Then obviously $x_{k+1} \nsim x_{i-1}$ and $x_{k-1} \nsim x_{i-1}$.

We claim that $x_{i+1} \sim x_k$: for $x_r \in \{x_{i+1}, x_{i+2}, \ldots, x_{n-1}\}$, $x_1 \sim x_r$ and $x_{i+1} \sim x_{r+1}$ cannot hold simultaneously or there will be an n-cycle; similarly for $x_l \in \{x_1, x_2, \ldots, x_i\}$, $x_1 \sim x_{l+1}$ and $x_{i+1} \sim x_l$ do not both hold. If $x_{i+1} \nsim x_k$, then on one hand $d(x_1) + d(x_{i+1}) \leq n-1+\delta(x_1 \sim x_1) + \delta(x_{i+1} \sim x_{i+1}) - |\{x_{k-1}\}| = n-2$, while on the other hand $d(x_1) + d(x_{i+1}) \geq 2\lceil \frac{n-1}{2} \rceil > n-2$, a contradiction. Thus indeed $x_{i+1} \sim x_k$.

This means $x_{k+1} \nsim x_{k-1}$, for otherwise, there would be the Hamiltonian cycle

$$vx_ix_{i-1}\cdots x_1x_kx_{i+1}x_{i+2}\cdots x_{k-1}x_{k+1}x_{k+2}\cdots x_{n-1}v.$$

So now $d(x_{k+1}, x_{k-1}) = 2$ and hence $\max\{d(x_{k+1}), d(x_{k-1})\} \geq \lceil \frac{n-1}{2} \rceil$.

Subcase 2.1 $x_{i-1} \nsim x_k$.

2.1.1 $d(x_{k+1}) \geq \lceil \frac{n-1}{2} \rceil$.

Note that for $h \in \{1, 2, \ldots, n-2\}$, $x_{i-1} \sim x_h$ and $x_{k+1} \sim x_{h+1}$ cannot both be true. Let's assume $x_{k-2} \nsim x_{i-1}$. Because (i) $x_{i-1} \nsim x_k$ and $x_{k+1} \nsim x_{k+1}$ and (ii) $x_{i-1} \nsim x_{k-2}$ and $x_{k+1} \nsim x_{k-1}$, we have $d(x_{i-1}) + d(x_{k+1}) \leq n-2+\delta(x_{i-1}, x_{n-1}) + \delta(x_{k+1}, x_1) - |\{x_k, x_{k-2}\}| = n-2$, which is contradictory to the degree requirement.

Hence, it has to be $x_{k-2} \sim x_{i-1}$. Now because of the existence of an $(n-1)$-cycle,

$$vx_ix_{i+1}\cdots x_{k-2}x_{i-1}x_{i-2}\cdots x_1x_kx_{k+1}\cdots x_{n-1}v,$$

we may assume that the only missing vertex x_{k-1} has degree 2 and in particular $x_{k-1} \nsim x_{n-2}$.

Next for $x_l \in \{x_1, x_2, \ldots, x_{i-1}\}$, $x_{n-2} \sim x_l$ and $x_{i-1} \sim x_{l-1}$ do not both hold; meanwhile for $x_r \in \{x_i, x_{i+1}, \ldots, x_{n-3}\}$, $x_{n-2} \sim x_r$ and $x_{i-1} \sim x_{r+1}$ do not both hold either. By pairing up x_{n-2} and x_{i-1}, we again derive a contradiction on the degree sum: $d(x_{n-2}) + d(x_{i-1}) \leq n-3+\delta(x_{n-2}, x_{n-2}) + \delta(x_{n-2}, x_{n-1}) + \delta(x_{i-1}, x_{i-1}) + \delta(x_{i-1}, x_i) - |\{x_{k-1}\}| = n-2$.

2.1.2 $d(x_{k-1}) \geq \lceil \frac{n-1}{2} \rceil$.

This is relatively easy due to the configuration with x_{k-1} in between. Again for $x_l \in \{x_2, x_3, \ldots, x_{i-1}\}$, $x_{k-1} \sim x_l$ and $x_{i-1} \sim x_{l-1}$ do not both hold; meanwhile for $x_r \in \{x_i, x_{i+1}, \ldots, x_{n-2}\}$, $x_{k-1} \sim x_r$ and $x_{i-1} \sim x_{r+1}$ do not both hold either. Furthermore $x_{k-1} \nsim x_{k-1}$ and $x_{i-1} \nsim x_k$, so that $d(x_{k-1}) + d(x_{i-1}) \leq n-3+\delta(x_{k-1}, x_1) + \delta(x_{k-1}, x_{n-1}) + \delta(x_{i-1}, x_{i-1}) + \delta(x_{i-1}, x_i) - |\{x_{k-1}\}| \leq n-2$ (recall that $\delta(x_{k-1}, x_1) = 0$ by the selection of x_k).

To summarize what we have so far, the entire Subcase 2.1 is impossible. Hence, we proceed to the case $x_{i-1} \sim x_k$.

Subcase 2.2 $x_{i-1} \sim x_k$.

Now that $x_1 \sim x_k$ and $x_{i-1} \sim x_k$, symmetrically, we have $N(x_{k-1}) \cap \{x_{n-2}, x_1, x_{i-1}\} = \emptyset$ and $N(x_{k+1}) \cap \{x_1, x_{i-1}, x_{i+1}\} = \emptyset$ to avoid generating any n-cycle.

2.2.1 $d(x_{k+1}) \geq \lceil \frac{n-1}{2} \rceil$.

For $x_r \in \{x_{i+1}, x_{i+2}, \ldots, x_{n-2}\}$, $x_{k+1} \sim x_{r+1}$ and $x_1 \sim x_r$ do not both hold, and for $x_l \in \{x_1, x_2, \ldots, x_{i-1}\}$, $x_{k+1} \sim x_l$ and $x_1 \sim x_{l+1}$ do not both hold either.

For $r = k-2$, however, note that $i+3 \leq k$ and thus $k-2 \geq i+1$, and both $x_{k+1} \nsim x_{k-1}$ and $x_1 \nsim x_{k-2}$.

Hence $d(x_{k+1}) + d(x_1) \leq n-3+\delta(x_{k+1} \sim x_{i+1}) + \delta(x_{k+1} \sim x_i) + \delta(x_1 \sim x_1) + \delta(x_1 \sim x_{n-1}) - |\{x_{k-2}\}| \leq n-2$, contradiction.

2.2.2 $d(x_{k-1}) \geq \lceil \frac{n-1}{2} \rceil$. If $x_{i-1} \nsim x_{k+2}$, then similar to the argument of 2.1.2, by pairing up x_{k-1} and x_{i-1}, there is a contradiction (the only difference is replacing "$-|\{x_{k-1}\}|$" by "$-|\{x_{k+1}\}|$"). So it must be $x_{i-1} \sim x_{k+2}$.

Similar to the argument in 2.1.1, by pairing up x_{k-1} and x_1, it implies $x_1 \sim x_{k+2}$ by the fact that $x_{k-1} \nsim x_{k+1}$ (otherwise $d(x_{k-1}) + d(x_1) \leq n-2+\delta(x_{k-1}, x_{n-1}) + \delta(x_1, x_1) - |\{x_{k+1}\}| \leq n-2$). Recall that $x_{i-1} \nsim x_{k+1}$ and $x_1 \nsim x_{k+1}$. Inductively, we claim that x_{i-1} and x_1 are both nonadjacent to each vertex in $\{x_{k+1}, x_{k+3}, \ldots\}$ and are both adjacent to each vertex in $\{x_k, x_{k+2}, \ldots\}$. This is because if the claimed pattern first breaks at the place of "k plus odd", then it incurs an n-cycle, while if the pattern first breaks at "k plus even", then there is degree sum deficiency. Now since it is known that $x_{i-1} \nsim x_{n-2}$, there must be $x_{i-1} \nsim x_{n-3}$ (thus $n-k-3$ is an even number).

Thus we get an $(n-1)$-cycle

$$vx_{n-1}x_1x_2\cdots x_{i-1}x_{n-3}x_{n-4}\cdots x_iv,$$

leaving only x_{n-2}, which is a vertex of degree $\geq \lceil \frac{n-1}{2} \rceil \geq 3$. But then we get back to what we have discussed before.

The proof is now completed. □

The main result (Theorem 1.7) of this work is therefore a consequence of Propositions 2.3, 2.4 and Theorem 2.5.

References

[1] Ahmed Ainouche and Nicos Christofides, *Conditions for the existence of Hamiltonian circuits in graphs based on vertex degrees*, J. London Math. Soc. (2) **32** (1985), no. 3, 385–391, DOI 10.1112/jlms/s2-32.3.385. MR825912 (87f:05106)

[2] A. Benhocine and A. P. Wojda, *The Geng-Hua Fan conditions for pancyclic or Hamilton-connected graphs*, J. Combin. Theory Ser. B **42** (1987), no. 2, 167–180, DOI 10.1016/0095-8956(87)90038-4. MR884252 (88f:05071)

[3] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976. MR0411988 (54 #117)

[4] Reinhard Diestel, *Graph theory*, 3rd ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005. MR2159259 (2006e:05001)

[5] G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), 69–81. MR0047308 (13,856e)

[6] Genghua Fan, *New sufficient conditions for cycles in graphs*, J. Combin. Theory Ser. B **37** (1984), no. 3, 221–227, DOI 10.1016/0095-8956(84)90054-6. MR769364 (86c:05083)

[7] Michael R. Garey and David S. Johnson, *Computers and intractability*, A guide to the theory of NP-completeness. A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979. MR519066 (80g:68056)

[8] Ronald J. Gould, *Advances on the Hamiltonian problem—a survey*, Graphs Combin. **19** (2003), no. 1, 7–52, DOI 10.1007/s00373-002-0492-x. MR1974368 (2004a:05092)

[9] Shengjia Li, Ruijuan Li, and Jinfeng Feng, *An efficient condition for a graph to be Hamiltonian*, Discrete Appl. Math. **155** (2007), no. 14, 1842–1845, DOI 10.1016/j.dam.2007.03.013. MR2354331 (2008g:05122)

[10] Oystein Ore, *Note on Hamilton circuits*, Amer. Math. Monthly **67** (1960), 55. MR0118683 (22 #9454)

[11] Michael Sipser, *Introduction to the theory of computation*, second ed., Course Technology, Cambridge, 2005.
[12] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996. MR1367739 (96i:05001)

Department of Mathematics, Qiongzhou University, Sanya, Hainan 572022, People's Republic of China
E-mail address: kewen@bxemail.com

School of Mathematical Sciences, and LMAM, Peking University, Beijing 100871, People's Republic of China
E-mail address: csong@math.pku.edu.cn

Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008-5152
E-mail address: ping.zhang@wmich.edu