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INTEGRAL REPRESENTATIONS OF CYCLIC GROUPS ACTING ON RELATIVE HOLOMORPHIC DIFFERENTIALS OF DEFORMATIONS OF CURVES WITH AUTOMORPHISMS

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Abstract. We study integral representations of holomorphic differentials on the Oort-Sekiguchi-Suwa component of deformations of curves with cyclic group actions.

1. Introduction

Let X be a nonsingular projective curve defined over an algebraically closed field k of positive characteristic p. If the curve X has genus $g \geq 2$, then it is known that the automorphism group G of X is finite. If p divides $|G|$, then the automorphism group of X behaves in a much more complicated way compared to automorphism group actions in characteristic zero. Wild ramification can appear and the structure of decomposition groups and the different is much more complicated [18].

The representation theory of groups on holomorphic differentials has been a useful tool for studying curves with automorphisms even in characteristic zero [6, chap. V.2]. In the positive characteristic case, extra difficulties arise in the representation theory of the automorphism group which now requires the usage of modular representation theory. The classical problem of determination of the Galois module structure of (poly)differentials, i.e. the study of $k[G]$-module structure of $H^0(X, \Omega_X^{\otimes n})$, remains open in positive characteristic and only some special cases are understood [12], [19], [16], [10].

Also one can consider the deformation problem of curves with automorphisms: Can one find proper, smooth families $\mathcal{X} \to \mathrm{Spec} R$ over a local ring R, with maximal ideal m_R, acted on by G, such that the special fibre $\mathcal{X} \otimes_{\mathrm{Spec} R} R/m_R$ is the original curve? There are certain obstructions [1], [4] to the existence of such families, especially if R is a mixed characteristic ring. As part of this problem one can consider the lifting problem to characteristic zero of a curve with automorphisms. We will use the following lemma.

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Lemma 1. *Let R be a local noetherian integral domain with residue field k and quotient field L. Every finitely generated R-module M with the additional property $\dim_k M \otimes_R k = \dim_L M \otimes_R L = r$ is free of rank r.*

Proof. See [8, Lemma 8.9]. □

By Lemma 1 the modules of relative polydifferentials $M_n = H^0(\mathcal{X}, \Omega_{\mathcal{X}}^{\otimes n})$ are free R-modules that are acted on by G. The aim of this article is to motivate and start the study of the following

Problem. Describe the $R[G]$-module structure of M_n.

This problem is to be studied within the theory of *integral representations*. Traditionally the theory of integral representations considers the $\mathbb{Z}[G]$-module structure, but essentially the theory of $R[G]$-module structure is similar to the $\mathbb{Z}[G]$ theory if R is a principal ideal domain. For cyclic p-groups the possible $\mathbb{Z}[G]$-modules are classified [15], [5]. Here the situation is a little bit easier since we will only consider integral domains that contain a p-th root of unity. In this article we will also consider deformations over affine schemes where R is not a principal ideal domain.

The study of integral representations is even more difficult than the theory of modular representations. Actually one of the main ideas of Brauer in studying modular representation theory is to lift the representation to characteristic zero using complete rings with special fibre k.

The Galois module structure of the special fibre $M_n \otimes_R R/m_R$ remains open, but for Artin-Schreier curves it is known [14], [19], [10]. The relative situation in mixed characteristic rings is described by the Oort-Sekiguchi-Suwa theory [17] and is also well understood, at least for cyclic p-groups of order p.

Using the Oort-Sekiguchi-Suwa theory, Bertin and Mézard [2] gave an explicit such model of equivariant and mixed characteristic deformations of an Artin-Schreier cover with only one ramification point. In this article we will study explicitly the $R[G]$-module structure of M_1 using the Bertin-Mézard deformation as a toy model.

In order to express our main theorem we first introduce some notation:

Proposition 2. *We extend the binomial coefficient $\binom{i}{j}$ by zero for values $i < j$. For every $a \in \mathbb{Z}$, $1 \leq a \leq p$, consider the $a \times a$ matrix $A_a = (a_{ij})$ given by*

$$a_{ij} = \binom{j-1}{i-1}.$$

The matrix (a_{ij}) is lower triangular, and in characteristic p the matrix A_a has order p. Let $G = \langle \sigma \rangle$ be a cyclic group of order p. The matrix A_a defines an indecomposable $\mathbb{Z}[G]$ module of dimension a by sending

$$\rho : \sigma^i \mapsto A_a^i.$$

Proof. Let k be a field of characteristic p. There is a natural representation of a p-group on $k[x]$ by defining $\sigma(x) = x + 1$. Let $k_{a-1}[x]$ be the vector space of polynomials of degree at most $a - 1$. This action with respect to the natural basis $\{1, x, x^2, \ldots, x^{a-1}\}$ has representation matrix A_a. Also, the space of invariants $k_{a-1}[x]^{\langle \sigma \rangle}$ is one dimensional, therefore the G-module $k_{a-1}[x]$ is indecomposable. □

For an algebraically closed field k of positive characteristic we consider the ring $W(k)[\zeta]$ of Witt vectors with one p-root of unity added to it.

Proposition 3. *Let S be an integral domain that is a $W(k)[\zeta]$ algebra. Set $\lambda = \zeta - 1$. For $a_0, a_1 \in \mathbb{Z}$, $1 \le a_1 \le p$, we consider the S-module*

$$V'_{a_0,a_1} :=_S \left\langle (\lambda X + 1)^i : a_0 \le i < a_0 + a_1 \right\rangle \subset S(\lambda X + 1).$$

Consider the diagonal integral representation of a cyclic group of order p on V'_{a_0,a_1} by defining

$$\sigma(\lambda X + 1)^i = \zeta^i (\lambda X + 1)^i. \tag{1}$$

Notice that the modules V'_{a_0,a_1} and V'_{a_0+p,a_1} are isomorphic as G-modules. The problem with the module V'_{a_0,a_1} is that there is no good reduction of it modulo the maximal ideal of S. So we define the S-module

$$V_{a_0,a_1} :=_S \left\langle (\lambda X + 1)^{a_0} X^i : 0 \le i < a_1 \right\rangle \subset S(\lambda X + 1, X)$$

instead. The two modules are $\mathrm{GL}_a(\mathrm{Quot}(S))$-equivalent but not $\mathrm{GL}_a(S)$-equivalent. After a $\mathrm{GL}_a(\mathrm{Quot}(S))$ change of basis the representation on V_{a_0,a_1} becomes equivalent to

$$\rho(\sigma) = \mathrm{diag}(\zeta^{a_0}, \zeta^{a_0+1}, \ldots, \zeta^{a_0+a_1-1}) A_{a_1}. \tag{2}$$

The $S[G]$-representation V_{a_0,a_1} is indecomposable.

Proof. We consider the linear change of coordinates from the $(\lambda X+1)^{a_0}(\lambda X+1)^\kappa$, $0 \le \kappa < a_1$, basis to the $(\lambda X + 1)^{a_0} X^\kappa$, $0 \le \kappa < a_1$, basis. This can be done using the binomial theorem as follows:

$$(\lambda X + 1)^{a_0} (\lambda X + 1)^\kappa = \sum_{\nu=0}^{\kappa} \binom{\kappa}{\nu} \lambda^\nu (\lambda X + 1)^{a_0} X^\nu.$$

Observe that this change of basis is invertible if and only if the elements λ are invertible. We can work out the conjugation in terms of the binomial coefficients, but it is easier to observe that the action of σ on polynomials of X is obtained from (1) to be $\sigma(X) = \zeta X + 1$.

Let m be the maximal ideal of $W(k)[\zeta]$. In order to prove that V_{a_0,a_1} is indecomposable we simply take the reduction of S modulo mS and obtain the indecomposable modular representation of Proposition 2. This finishes the proof since a decomposable $S[G]$ integral representation has decomposable reduction as well. □

Remark 4. The isomorphism class of the module V_{a_0,a_1} depends on the equivalence class modulo p of a_0 and of the length a_1. Notice also that the isomorphism class of the reduction of V_{a_0,a_1} modulo the maximal ideal of S depends only on the length a_1. Indeed, the S-module isomorphism (but not the G-module isomorphism unless $a_0 \equiv 0 \mod p$)

$$V_{0,a_1} \to V_{a_0,a_1},$$
$$v \mapsto (\lambda X + 1)^{a_0} v$$

reduces to the identity modulo the maximal ideal of S. This is also compatible with the fact that the isomorphism type of a modular representation for the cyclic group depends only on the rank of the module.

Remark 5. Two modules can be isomorphic over the quotient field of an integral domain S but not be S-isomorphic. This is one of the ingredients of the Bruhat-Tits theory of buildings and also appears in algebraic geometry in the birational versus biregular equivalence.

Definition 6. Define by V_a the indecomposable integral representation $V_{1-p,a}$ of Proposition 3. It has the matrix representation given in (2). The module V_a is free of rank a.

The main theorem of our article is:

Theorem 7. *Let σ be an automorphism of $\mathcal{X}$ of order $p \neq 2$ and conductor m with $m = pq - l$, $1 \leq q$, $1 \leq l \leq p-1$. Let*

$$R = \begin{cases} W(k)[\zeta][[x_1, \ldots, x_q]] & \text{if } l = 1, \\ W(k)[\zeta][[x_1, \ldots, x_{q-1}]] & \text{if } l \neq 1 \end{cases} \tag{3}$$

be the Oort-Sekiguchi-Suwa factor (see also Theorem 9) of the versal deformation ring R_σ. The free R-module $H^0(\mathcal{X}, \Omega_{\mathcal{X}})$ of relative differentials has the following $R[G]$-structure:

$$H^0(\mathcal{X}, \Omega_{\mathcal{X}}) = \bigoplus_{\nu=0}^{p-2} V_\nu^{\delta_\nu}, \tag{4}$$

where

$$\delta_\nu = \begin{cases} q + \left\lceil \frac{(\nu+1)l}{p} \right\rceil - \left\lceil \frac{(2+\nu)l}{p} \right\rceil & \text{if } \nu \leq p-3, \\ q - 1 & \text{if } \nu = p-2. \end{cases}$$

We will use the explicit construction of Bertin and Mézard [2] for constructing models of the family $\mathcal{X} \to \mathrm{Spec} R$. Let π be a local uniformizer of $S = W(k)[\zeta]$. We will first employ the Boseck [3] construction for finding a basis for the space of differentials for Kummer extensions working with base ring $\mathrm{Quot}(S) \otimes R$.

Then we will select a basis B so that the R-lattice generated by it has full rank and gives a well defined reduction modulo the ideal πR. A detailed analysis for the family of Artin-Schreier extensions given by $\mathcal{X} \times_R R/\pi R$ proves that the basis B chosen before is indeed a basis for $H^0(\mathcal{X}, \Omega_{\mathcal{X}})$. The proof of Theorem 7 is given by detailed computations with the basis elements chosen. As an application we easily obtain a classical theorem due to Hurwitz on the Galois module structure of Kummer extensions. We also obtain the known modular representation structure of the special fiber by reduction of the integral representation.

2. Explicit deformation theory

2.1. **Holomorphic differentials of Kummer extensions.** The following theorem will be used for constructing a basis on the characteristic zero fibers.

Theorem 8 (Boseck)**.** *Let L be a not necessarily algebraically closed field of characteristic $p \geq 0$, $(p, n) = 1$. Consider the Kummer extension F of $L(x) := F_0$ given by the equation $y^n = f(x)$ where $f(x) = a \prod_{i=1}^r p_i(x)^{l_i}$, the $p_i(x)$ are monic irreducible polynomials of degree d_i, and the exponents l_i satisfy $0 < l_i < n$. The place at infinity of $L(x)$ is assumed to be nonramified in F/F_0, and this is equivalent to $\deg(f) = \sum d_i l_i \equiv 0 \mod n$. The ramified places in F/F_0 correspond*

to the irreducible polynomials p_i and are ramified in extension F/F_0 with index $e_i = n/(n, l_i)$. We will denote by g_i the number of places of F extending the place p_i. Set $t := \deg(f)/n$ and $\lambda_i = e_i l_i/n$. Let f_i be the residual degree of the places p_i in extension F/F_0. For every $\mu = 1, \ldots, n-1$ we define $m_i^{(\mu)}$, $\rho_i^{(\mu)}$ by the division

$$\mu\lambda_i = m_i^{(\mu)} e_i + \rho_i^{(\mu)}, \text{ where } 0 \le \rho_i^{(\mu)} \le e_i - 1.$$

We set

$$t^{(\mu)} = \frac{1}{n} \sum_{i=1}^{r} d_i f_i g_i \rho_i^{(\mu)}$$

and

$$g_\mu(x) = \prod_{i=1}^{r} p_i(x)^{m_i^{(\mu)}}.$$

A basis of holomorphic differentials is given by

$$x^\nu g_\mu(x) y^{-\mu} dx,$$

for $\mu = 1, \ldots, n-1$, $0 \le \nu \le t^{(\mu)} - 2$.

Proof. See [3, II, pp. 48-50]. □

2.2. **The Bertin-Mézard model.** Let k be an algebraically closed field of positive characteristic $p > 0$. Consider the Witt ring $W(k)[\zeta]$ extended by the p-th root of unity, and let $L = \mathrm{Quot}(W(k)[\zeta])$. Let $\lambda = \zeta - 1$.

We consider the Kummer extension of a rational function field $L(x)$ defined by the extension

$$(X + \lambda^{-1})^p = x^{-m} + \lambda^{-p}.$$

Write $m = pq - l$ where $0 < l \le p - 1$.

Set $\lambda X + 1 = y/x^q$. We have then the model

$$y^p = (\lambda^p + x^m)x^l = \lambda^p x^l + (x^q)^p. \tag{5}$$

Notice that the polynomial $f(x) = (\lambda^p + x^m)x^l$ on the right hand side has degree $m + l = qp$ divisible by p so that the place at infinity is not ramified.

More generally we replace x^q on the right hand side of (5) by

$$a(x) = x^q + x_1 x^{q-1} + \cdots + x_q, \tag{6}$$

where $x_q = 0$ if $l \ne 1$. This gives the Kummer extension

$$\big(\lambda\xi + a(x)\big)^p = \lambda^p x^l + a(x)^p, \tag{7}$$

where $\xi = Xa(x)$, and if we set

$$y = \lambda\xi + a(x) = a(x)(\lambda X + 1), \tag{8}$$

we have

$$y^p = \lambda^p x^l + a(x)^p. \tag{9}$$

Observe that (9) becomes (5) if we set $x_1 = \cdots = x_q = 0$.

It is known [2] that the global deformation functor is prorepresentable by a ring R_σ. Bertin and Mézard for the case we are studying proved the following theorem.

Theorem 9. *Let σ be an automorphism of the special fibre X, of order $p \neq 2$ and conductor m, with $m = pq - l$, $1 \leq q$, $1 \leq l \leq p-1$. The versal ring R_σ has a formally smooth quotient $R_\sigma \twoheadrightarrow R$ called the Oort-Sekiguchi-Suwa (OSS) factor; the ring R is given in* (3)*. One model of a family over the OSS factor R is given by setting $\xi = Xa(x)$, where $a(x)$ is defined in* (6)*, and by taking the normalization of the fibers given by* (7)*.*

We will use the Boseck construction for bases for holomorphic differentials on the generic fibre. Set $\bar{x} = (x_1, \ldots, x_q)$ and consider the polynomial

$$f_{\bar{x}}(x) = \lambda^p x^l + a(x)^p.$$

We consider the decomposition of the polynomial $f_{\bar{x}}(x)$ as a product of irreducible polynomials $p_i(x)$ of degree d_i:

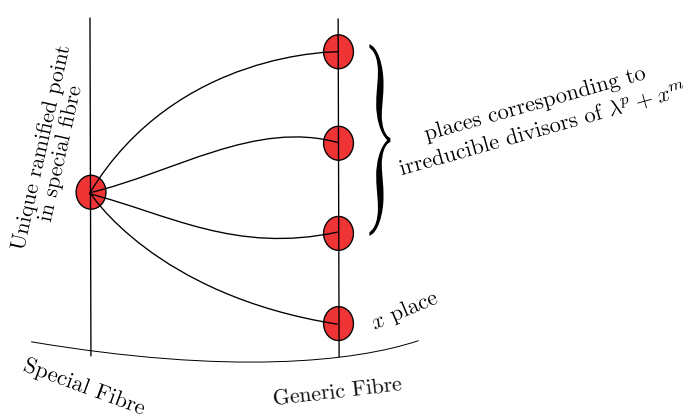
$$f_{\bar{x}}(x) = \prod p_i(x)^{l_i}.$$

Consider the $\bar{x} = (0, \ldots, 0)$ case, i.e. $f_{\bar{x}}(x) = \lambda^p x^l + x^{pq} = x^l(\lambda^p + x^m)$. In this case it is clear (compute the discriminant of the polynomial $x^m - a$) that $l_i = 1$ for irreducible factors of $\lambda^p + x^m$ and the only possible factor where $l_1 := l > 1$ is x. Since $(l_i, p) = 1$ we have that every place P_i corresponding to an irreducible factor p_i of $f_{\bar{x}}(x)$ ramifies completely in the cover $X \to \mathbb{P}^1_L$. Therefore, all inertia degrees are $f_i = 1$ and the number of places g_i above P_i is $g_i = 1$. Notice that the number of places that are ramified is $m + 1$.

We can arrive at the same conclusion for the general $f_{\bar{x}}(x)$. Indeed, if $l = 1$, then the polynomial $f_{\bar{x}}(x)$ is of degree $m + 1$ and should have $m + 1$ distinct roots in an algebraic closure (otherwise the genus of the curve given in (5) differs from the genus of the curve given in (9)). We can apply the same method for the $l > 1$ case. We notice that $a(x)^p x^{-l}$ is a polynomial and we have

$$f_{\bar{x}}(x) = x^l(\lambda^p + a(x)^p x^{-l}).$$

That is to say, we have that the factor x of multiplicity l and the remaining factor $\lambda^p + a(x)^p x^{-l}$ is a polynomial of degree $m = pq - l$ that should have distinct roots; otherwise the genera of the curves defined in (5) and (9) are different.



Following the notation of Boseck we have $\lambda_i = e_i l_i/p = l_i$ and we also define for every $\mu = 1, 2, \ldots, p-1$ the integers

$$\mu\lambda_i = m_i^{(\mu)} e_i + \rho_i^{(\mu)},$$

i.e.

$$m_i^{(\mu)} = \left\lfloor \frac{\mu l_i}{p} \right\rfloor \text{ and } \rho_i^{(\mu)} = p \left\langle \frac{\mu l_i}{p} \right\rangle = \mu l_i - \left\lfloor \frac{\mu l_i}{p} \right\rfloor p.$$

This means that for all $2 \leq i \leq r$ such that $l_i = 1$ we have

$$m_i^{(\mu)} = 0 \text{ and } \rho_i^{(\mu)} = \mu \text{ for all } 1 \leq i \leq r,$$

while the same holds for m_1, ρ_1, only if $l = 1$, i.e. only if $m = pq - 1$. Therefore, the polynomials

$$g_\mu(x) = \prod_{i=1}^{r} p_i(x)^{m_i^{(\mu)}}$$

that are defined in the work of Boseck have only $x^{m_1^{(\mu)}}$ as a factor.

We set

$$t^{(\mu)} = \frac{1}{p} \sum_{i=1}^{r} d_i f_i g_i \rho_i^{(\mu)} = \frac{m\mu + \mu l - \left\lfloor \frac{\mu l}{p} \right\rfloor p}{p} = \mu q - \left\lfloor \frac{\mu l}{p} \right\rfloor.$$

The set of holomorphic differentials is given by

$$x^{\nu + \lfloor \frac{l\mu}{p} \rfloor} y^{-\mu} dx, \text{ where } 1 \leq \mu \leq p-1, 0 \leq \nu \leq t^{(\mu)} - 2 = \mu q - \left\lfloor \frac{\mu l}{p} \right\rfloor - 2. \tag{10}$$

Let us write $y = a(x)(\lambda X + 1)$; set $N := \nu + \left\lfloor \frac{l\mu}{p} \right\rfloor$ and $a := p - 1 - \mu$. We have proved the following proposition:

Proposition 10. *The set of differentials of the form*

$$x^N a(x)^a \frac{(\lambda X + 1)^a}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx, \tag{11}$$

where

$$0 \leq a < p - 1 \text{ and } l - \left\lceil \frac{(1+a)l}{p} \right\rceil \leq N \leq (p - 1 - a)q - 2, \tag{12}$$

forms a basis of holomorphic differentials.

This basis is not suitable for taking the reduction modulo the maximal ideal of the ring $S = W(k)[\zeta]$. We will select a different basis so that it has good reduction.

Lemma 11. *For every holomorphic differential in the basis given in Proposition* 10 *and for* $0 \leq k \leq a$ *the differential*

$$x^N a(x)^a \frac{(\lambda X + 1)^k}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx \tag{13}$$

is also holomorphic.

Proof. For $k = a$ we have nothing to show. Let us assume that $0 \leq k < a$. The differentials of the form

$$x^{N'} a(x)^k \frac{(\lambda X + 1)^k}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx \tag{14}$$

for $l - \left\lceil \frac{(1+k)l}{p} \right\rceil \leq N' \leq (p-1-k)q - 2$ are holomorphic. We will show that every differential given in (13) is written as a linear combination of holomorphic differentials given in (14).

Indeed we have to show that we can select $\lambda_{N'}$ such that

$$\sum_{N'=l-\left\lceil \frac{(1+k)l}{p} \right\rceil}^{(p-1-k)q-2} \lambda_{N'} x^{N'} a(x)^k = x^N a(x)^a$$

or equivalently that

$$\sum_{N'=l-\left\lceil \frac{(1+k)l}{p} \right\rceil}^{(p-1-k)q-2} \lambda_{N'} x^{N'} = x^N a(x)^{a-k}. \tag{15}$$

We now observe that $a(x)^{a-k}$ is a polynomial of degree $q(a-k)$ and that the possible values of N are in the range given by (12). This means that on the right hand side of (15) appear monomials of degree at most $N_1 = N + q(a-k)$, where N_1 satisfies

$$N_1 \leq (p-1-a)q - 2 + q(a-k) = (p-1-k)q - 2$$

and at least

$$N_2 := \begin{cases} N & \text{if } l = 1, \\ N + a - k & \text{if } l > 1. \end{cases}$$

The distinction in the two cases appears since in the $l > 1$ case the polynomial $a(x)$ has zero constant term.

For $l = 1$ it is enough to prove that

$$l - \left\lceil \frac{(1+k)}{p} \right\rceil \leq l - \left\lceil \frac{(1+a)}{p} \right\rceil,$$

which is immediate since $1 \leq 1 + k \leq 1 + a \leq p - 1$.

For the $l > 1$ we have to prove that

$$\left\lceil \frac{(1+a)l}{p} \right\rceil - \left\lceil \frac{(1+k)l}{p} \right\rceil \leq (a-k). \tag{16}$$

Write $a = k + t$ for some $t > 0$. Then we have

$$(1+a)l = \pi_a p + v_a, \quad (1+k)l = \pi_k p + v_k \text{ where } 0 \leq v_a, v_k < p,$$

so

$$(1+a)l = \pi_k p + v_k + tl$$

and

$$v_k + tl = p \left\lfloor \frac{v_k + tl}{p} \right\rfloor + v \text{ for some } 0 \leq v < p.$$

This way we see that

$$\pi_a = \pi_k + \left\lfloor \frac{v_k + tl}{p} \right\rfloor,$$

but

$$\left\lfloor \frac{v_k + tl}{p} \right\rfloor \leq \left\lfloor \frac{v_k + tp}{p} \right\rfloor = t + \left\lfloor \frac{v_k}{p} \right\rfloor = t;$$

therefore (16) holds. □

Corollary 12. *If for every $p(x) \in R[x]$ the differential*

$$p(x)a(x)^a \frac{(\lambda X + 1)^a}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx$$

is holomorphic, then the differentials

$$p(x)a(x)^a \frac{(\lambda X + 1)^k}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx$$

are also holomorphic.

For $0 \leq a \leq p-2$ we consider the set of admissible N, i.e. N that satisfy the inequalities $\underline{N_a} \leq N \leq \overline{N_a}$, where $\underline{N_a} = l - \left\lceil \frac{(1+a)l}{p} \right\rceil$ and $\overline{N_a} = (p-1-a)q - 2$. For example $\underline{N_{p-2}} = 0$ and $\overline{N_{p-2}} = q - 2$.

We consider the space of holomorphic differentials as a graded space with grading given by the exponent of $(\lambda X + 1)^a$, i.e.

$$\Omega_X = \bigoplus_{0 \leq a \leq p-2} \Omega_X^a.$$

Notice that every space Ω_X^a has dimension:

$$\dim \Omega_X^a = \overline{N_a} - \underline{N_a} + 1 = (p-1-a)q - 2 - l + \left\lceil \frac{(1+a)l}{p} \right\rceil + 1. \tag{17}$$

Corollary 12 implies that for every $0 \leq k \leq a \leq p-2$ there are linear injections

$$L_{a,k} : \Omega_X^a \to \Omega_X^k$$

so that

$$L_{a,k}\left(p(x)a(x)^a \frac{(\lambda X + 1)^a}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx \right) = p(x)a(x)^a \frac{(\lambda X + 1)^k}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx.$$

Define

$$C_{p-2} := \Omega_X^{p-2}.$$

We define C_{p-3} so that Ω_X^{p-3} is a direct sum in the category of vector spaces:

$$\Omega_X^{p-3} = L_{p-2,p-3}(C_{p-2}) \oplus C_{p-3}.$$

We proceed inductively. We define C_{p-4} so that

$$\Omega_X^{p-4} = L_{p-2,p-4}(C_{p-2}) \oplus L_{p-3,p-4}(C_{p-3}) \oplus C_{p-4}.$$

More generally,

$$\Omega_X^{p-k} = \bigoplus_{\nu=p-k+1}^{p-2} L_{\nu,p-k}(C_\nu) \oplus C_{p-k} = \bigoplus_{\nu=p-k}^{p-2} L_{\nu,p-k}(C_\nu). \tag{18}$$

After setting $a = p - k$ we have

$$\Omega_X^a = \bigoplus_{\nu=a}^{p-2} L_{\nu,a}(C_\nu).$$

We now write

$$\Omega_X = \bigoplus_{a=0}^{p-2} \bigoplus_{\nu=a}^{p-2} L_{\nu,a}(C_\nu) = \bigoplus_{\nu=0}^{p-2} \bigoplus_{a=0}^{\nu} L_{\nu,a}(C_\nu).$$

Recall that by Proposition 10 the differentials in Ω_X^ν are of the form

$$f(x)a(x)^\nu \frac{(\lambda X+1)^\nu}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx,$$

where $f(x)$ is a polynomial in $R[x]$ with degree bounded by eq. (12):

$$l - \left\lceil \frac{(1+\nu)l}{p} \right\rceil \leq \deg f(x) \leq (p-1-\nu)q - 2. \tag{19}$$

Fix a basis

$$\left\{ f_1^{(\nu)} a(x)^\nu \frac{(\lambda X+1)^\nu}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx, \ldots, f_{\dim C_\nu}^{(\nu)} a(x)^\nu \frac{(\lambda X+1)^\nu}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx \right\}$$

of C_ν, where $f_i^{(\nu)}$ are polynomials in $R[x]$ with degree bounded by (19). Then the set

$$\left\{ f_\kappa^{(\nu)} a(x)^\nu \frac{(\lambda X+1)^a}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx : 1 \leq \kappa \leq \dim C_\nu, 0 \leq a \leq \nu \right\}$$

is a basis of the space $\bigoplus_{a=0}^{\nu} L_{\nu,a}(C_\nu)$, by the injectivity of the $L_{\nu,a}$'s. Furthermore, Proposition 3 implies that

$$\bigoplus_{a=0}^{\nu} L_{\nu,a}(C_\nu) = V_\nu^{\dim C_\nu},$$

where V_ν is the indecomposable factor defined in Definition 6.

Corollary 13. *A natural basis for $\bigoplus_{a=0}^{\nu} L_{\nu,a}(C_\nu)$ with respect to the reduction is*

$$\left\{ f_\kappa^{(\nu)} a(x)^\nu \frac{X^a}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx : 1 \leq \kappa \leq \dim C_\nu, 0 \leq a \leq \nu \right\}. \tag{20}$$

The degrees of the polynomials $f_\kappa^{(\nu)}$ are bounded by (19)*. Notice that Lemma* 11 *and its proof implies that we can select as a basis*

$$\left\{ {f'}_\kappa^{(\nu)} a(x)^a \frac{X^a}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx : 1 \leq \kappa \leq \dim C_\nu, 0 \leq a \leq \nu \right\}, \tag{21}$$

where the degrees of the polynomials ${f'}_\kappa^{(\nu)}$ now satisfy the bound

$$l - \left\lceil \frac{(1+a)l}{p} \right\rceil \leq \deg {f'}_\kappa^{(\nu)}(x) \leq (p-1-a)q - 2. \tag{22}$$

Using the decomposition in (18) and (17) we obtain that

$$\dim C_{p-2} = \dim \Omega_X^{p-2} = q - 1,$$

$$\dim C_{p-k} \quad = \quad \dim \Omega_X^{p-k} - \dim \Omega_X^{p-k+1} \text{ for } k \geq 3.$$

So

$$\dim C_a = \begin{cases} \dim \Omega_X^a - \Omega_X^{a+1} = q + \left\lceil \frac{(a+1)l}{p} \right\rceil - \left\lceil \frac{(2+a)l}{p} \right\rceil & \text{if } a \leq p-3, \\ q-1 & \text{if } a = p-2. \end{cases}$$

The proof of our main Theorem 7 will be complete if we show that the differentials chosen in Corollary 13 are $\mathcal{X}$ holomorphic. For this we will study the characteristic p-fibers of our family $\mathcal{X} \to \mathrm{Spec} R$.

3. On the finite characteristic fibres

Recall that $S = W(k)[\zeta]$ and R is defined in Theorem 7 to be the Oort-Sekiguchi-Suwa factor. These are rings of characteristic zero, and here we focus on their reduction modulo the ideal generated by the uniformizer π of S. The quotient is a polynomial ring over the algebraically closed field k of characteristic $p > 0$.

In this section we consider the reduction of the model given in (9):

$$\big(\lambda\xi + a(x)\big)^p = \lambda^p x^l + a(x)^p,$$

modulo the ideal generated by π. Using the fact that for $\lambda = \zeta - 1$, $p\lambda^{-j} \equiv 0 \mod \pi$ for $0 \leq j < p-1$ and that $p\lambda^{-(p-1)} \equiv -1 \mod \pi$, Bertin and Mézard in [2, sec. 4.3] arrived at the equation

$$X^p - X = \frac{x^l}{a(x)^p}, \text{ where } X = \frac{\xi}{a(x)}. \tag{23}$$

This equation, that lives in positive characteristic, is Artin-Schreier but not in normal form since the right hand side has poles of orders divisible by p. Write (keep in mind that $l_1 = l$)

$$a(x) = x^{l_1} \prod_{i=2}^{r} p_i(x)^{l_i} \mod \pi S,$$

where $p_i(x)$ are irreducible polynomials in $S[[x_1, \ldots, x_q]][x]$. The valuations of the denominator in (23) are not prime to p. Bertin and Mézard proved that the normalization of $R[[x]]$ in the Galois extension of the generic fibre is the ring $R[[\eta]]$, where $\eta^l = \xi$. The group action of the generator σ of G, on the characteristic p fiber, is then given by

$$\sigma(\xi) = \xi + a(x) \Rightarrow \sigma(\eta) = \eta \left(1 + \frac{a(x)}{\eta^l}\right)^{1/l},$$

so

$$\begin{aligned} \sigma(\eta) - \eta &= \eta \sum_{\nu=1}^{\infty} \binom{\frac{1}{l}}{\nu} \left(\frac{a(x)}{\eta^l}\right)^{\nu} \\ &= \left(\frac{a(x)}{\eta^{l-1}}\right) \left(\sum_{\nu=1}^{\infty} \binom{\frac{1}{l}}{\nu} \left(\frac{a(x)}{\eta^l}\right)^{\nu-1}\right). \end{aligned} \tag{24}$$

Notice that for $l > 1$ the polynomial $a(x)$ has at least one root, and since $x = \eta^p u(\eta)$, where $u(\eta)$ is a unit in $R[[\eta]]$, we have that $a(x)/\eta^{l-1}$ has reduced order $pq - l + 1 = m + 1$ and by Weierstrass preparation theorem can be written as

$$\left(\frac{a(x)}{\eta^{l-1}}\right) = \left(\eta^{pq-l+1} + a_{pq-l}(x_1, \ldots, x_q)\eta^{pq-l} + \cdots + a_0(x_1, \ldots, x_q)\right) U(\eta),$$

where $a_i(0,\dots,0)=0$. According to [7, sec. 2.1] the ramification locus corresponds to the irreducible factors of the distinguished Weierstrass polynomial that differs from $a(x)$ only by a unit. Therefore only the places $p_i(x)$ corresponding to factors of $a(x)$ are ramified. If l_i is the multiplicity of the polynomial $p_i(x)$ in the decomposition of $a(x)$, then the conductor is given by

$$m_i = \begin{cases} pl_i - 1 & \text{if } i \neq 1, \\ pl_1 - l_1 & \text{if } i = 1. \end{cases}$$

Remark 14. Notice that since $\sum l_i = q$, the contributions to the different

$$\sum (p-1)(m_i+1) = (p-1)(pq-l+1) = (p-1)(m+1)$$

are as expected [7, sec. 3.4], [11, sec. 5].

Let P_i be the unique places above the polynomials $p_i(x)$ and P_1 above x. We compute the divisors

$$\operatorname{div}(X) = \operatorname{div}_0(X) - \sum_{i=2}^{r} l_i p P_i - (l_1 p - l) P_1.$$

Remark 15. Notice that in the case of normalized Artin-Schreier curves the coefficients in front of the poles of the generating functions is just the conductor of the corresponding place; see [3, eq. 26]. This cannot be true in our case since the conductors are not divisible by p.

The differential dx is a differential on $X/G \cong \mathbb{P}^1$. We will denote by P_∞ the place at infinity of the function field of X/G. We would like to compute its divisor, seen as a differential on X. For this we use proposition IV.2.3 in [8, p. 301]. By computation,

$$\operatorname{div}(dx) = \sum_{i=2}^{r} (p-1) p l_i P_i + (p-1)(l_1 p - l + 1) P_1 - 2\operatorname{Con}(P_\infty),$$

where $\operatorname{Con}(P_\infty)$ is the sum of places extending P_∞ in the Artin-Schreier extension, and

$$\begin{aligned} \operatorname{div}(X^\mu dx) = &\sum_{i=2}^{r} (p-1-\mu) p l_i P_i + \big((p-1-\mu)(l_1 p - l) + (p-1)\big) P_1 \\ &- 2\operatorname{Con}(P_\infty) + \mu \operatorname{div}_0(X). \end{aligned}$$

Following Boseck we now set

$$m_i^{(\mu)} = (p-1-\mu) l_i \text{ and } m_1^{(\mu)} = \left\lfloor \frac{(p-1-\mu)(l_1 p - l) + (p-1)}{p} \right\rfloor.$$

Then we form the polynomials

$$\begin{aligned} g_\mu(x) &= \prod_{i=1}^{r} p_i(x)^{m_i^{(\mu)}} = \left(\frac{a(x)}{x^{l_1}}\right)^{p-1-\mu} x^{\left\lfloor \frac{(p-1-\mu)(l_1 p - l)+(p-1)}{p} \right\rfloor} \\ &= a(x)^{p-1-\mu} x^{\left\lfloor \frac{-l(p-1-\mu)+(p-1)}{p} \right\rfloor}. \end{aligned}$$

Therefore

$$g_\mu(x)^{-1} X^\mu dx$$

is a holomorphic differential if

$$t^{(\mu)} = \sum_{i=1}^{r} d_i m_i^{(\mu)} = q(p-1-\mu) - l + \left\lfloor \frac{l(1+\mu)+p-1}{p} \right\rfloor \geq 2.$$

Notice that for $\mu = p-1$ the above formula gives $t^{(p-1)} = 0$, so $\mu = p-1$ is not permitted and $0 \leq \mu \leq p-2$. By computation we see

$$\left\{ x^{\nu} g_{\mu}(x)^{-1} X^{\mu} dx, \;\; \mu = 0, \ldots, p-1, t^{(\mu)} \geq 2, \nu = 0, \ldots, t^{(\mu)} - 2 \right\}$$

is a basis of holomorphic differentials. We set $N := \nu - \left\lfloor \frac{-l(p-1-\mu)+(p-1)}{p} \right\rfloor$ and we observe that

$$l - \left\lfloor \frac{l(1+\mu)+(p-1)}{p} \right\rfloor \leq N \leq q(p-1-\mu) - 2. \tag{25}$$

Since for every integer a we have $\left\lfloor \frac{a+p-1}{p} \right\rfloor = \left\lceil \frac{a}{p} \right\rceil$, we see that inequality (25) is equivalent to inequality (22) of Corollary 13 and every basis element in the basis given in (21) of Corollary 13 has a reduction that is a linear combination of elements of the set

$$\left\{ x^N a(x)^{\mu} \frac{X^{\mu}}{a(x)^{p-1}} dx \right\}$$

with N satisfying inequality (25) and $0 \leq \mu \leq p-2$. This set forms a basis of holomorphic differentials of the special fibre. The reader should always keep in mind that $\lambda \equiv 0$ modulo the maximal ideal of the Witt ring.

Therefore, the union of the bases given in Corollary 13 is not just a basis of holomorphic differentials on the characteristic zero fibers but it forms a free basis of the module of relative differentials $H^0(\Omega_{\mathcal{X}}, \mathcal{X})$.

3.1. **On a theorem of Hurwitz.** The following is a classical result due to Hurwitz [9] (see also [13, Theorem 3.5, p. 600]) that characterizes the dimension of the ζ^i eigenvalues of the generator of a p-cyclic group on the space of holomorphic differentials. Let K be an algebraically closed field of characteristic $\mathrm{char}(K)$.

Theorem 16 (Hurwitz)**.** *Let F/E be a cyclic Galois extension of function fields with Galois group the cyclic group G, of order n, $(\mathrm{char}(K), n) = 1$ or $\mathrm{char}(K) = 0$. This is given in Kummer form $y^n = u$, where $u \in E$ and $u^l \notin E$ for every $l \mid n$. For every place P_i of E that is ramified in F/E with ramification index e_i, we set $\Phi(i) = \frac{e_i v_{P_i}(u)}{n}$. Set*

$$\Gamma_{\kappa} := \sum_{i=1}^{r} \left\langle \frac{\kappa \Phi(i)}{e_i} \right\rangle.$$

For $\kappa = 0, \ldots, n-1$, we have n distinct irreducible representations of degree 1. Every representation of the cyclic group G is decomposable as a direct sum of one dimensional representations. The one dimensional representation where a generator of G acts on the eigenvector by multiplication by a n-th primitive root of 1 raised to the κ-power is called the κ-th representation. Let g_E denote the genus of E. The κ-th representation occurs $d_{\kappa} = \Gamma_{n-\kappa} - 1 + g_E$ times in the representation of G in the holomorphic differentials Ω_F of F when $\kappa \neq 0$, and g_E times when $\kappa = 0$.

We will apply this theorem for the Kummer extension defined in (5). So in our case $n = p$, the group G is a cyclic group and the characteristic is zero on the generic fibre. We have $\Phi(i) = 1$ for all places that correspond to divisors of $\lambda^p + x^m$ and $\Phi(r) = l$ for the last place. We compute

$$\begin{aligned}\Gamma_\kappa &= \sum_{i=1}^{r} \left\langle \frac{\kappa\Phi(i)}{p} \right\rangle \\ &= m\left(\frac{\kappa}{p} - \left\lfloor \frac{\kappa}{p} \right\rfloor\right) + \frac{\kappa l}{p} - \left\lfloor \frac{\kappa l}{p} \right\rfloor \\ &= \kappa\frac{m+l}{p} - \left\lfloor \frac{\kappa l}{p} \right\rfloor \\ &= \kappa q - \left\lfloor \frac{\kappa l}{p} \right\rfloor .\end{aligned}$$

The dimension of the eigenspace of the eigenvalue ζ^μ is $\Gamma_{p-\mu} - 1$. We compute

$$\Gamma_{p-\mu} - 1 = (p-\mu)q - \left\lfloor \frac{(p-\mu)l}{p} \right\rfloor - 1. \tag{26}$$

We would like to obtain the result of Hurwitz using our result on integral representation. Observe that the action of σ on V_n has eigenvalues $\zeta, \zeta^2, \ldots, \zeta^{n+1}$; thus there is a contribution from V_n to the eigenspace of the eigenvalue ζ^μ only if $n + 1 \geq \mu$. Therefore only the summands for which $n \geq \mu - 1$ contribute $\dim C_n$ to the eigenspace. The dimension of the desired eigenspace is

$$\begin{aligned}\sum_{n=\mu-1}^{p-2} \dim C_n &= \sum_{n=\mu-1}^{p-3} \dim C_n + \dim C_{p-2} \\ &= q(p-\mu-1) + \left\lceil \frac{\mu l}{p} \right\rceil - \left\lceil \frac{(p-1)l}{p} \right\rceil + q - 1 \\ &= q(p-\mu) + \left\lceil \frac{\mu l}{p} \right\rceil - l - 1.\end{aligned}$$

Of course, this result coincides with the result given by (26).

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