THE DOUBLE RAMIFICATION CYCLE AND THE THETA DIVISOR

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ABSTRACT. We compute the classes of universal theta divisors of degrees zero and $g-1$ over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of curves, with various integer weights on the points, in particular reproving a recent result of Müller.

We also obtain a formula for the class in $CH^g(\mathcal{M}_{g,n}^{ct})$ (moduli of stable curves of compact type) of the double ramification cycle, given by the condition that a fixed linear combination of the marked points is a principal divisor, reproving a recent result of Hain.

Our approach for computing the theta divisor is more direct, via test curves and the geometry of the theta divisor, and works easily over all of $\overline{\mathcal{M}}_{g,n}$. We used our extended result in another paper to study the partial compactification of the double ramification cycle.

1. INTRODUCTION

Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth genus g curves with n labeled distinct marked points, let $\mathcal{M}_{g,n}^{ct}$ denote its partial compactification by stable curves of compact type, and let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification by stable curves. Let $\mathcal{J}_g^d \to \mathcal{M}_g^{ct}$ denote the universal family of Picard varieties (Jacobians) of degree d (recall that the Jacobian of a stable curve of compact type is in fact an abelian variety), and by abuse of notation let $\mathcal{J}_g^d \to \mathcal{M}_{g,n}^{ct}$ also denote its pullback under the forgetful map $\pi : \mathcal{M}_{g,n}^{ct} \to \mathcal{M}_g^{ct}$. For any collection of integers $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ of total degree $\deg \underline{d} := d_1 + \ldots + d_n = d$, define the section $s_{\underline{d}} : \mathcal{M}_{g,n} \to \mathcal{J}_g^d$ by $s_{\underline{d}}(C, p_1, \ldots, p_n) := \sum d_i p_i \in \mathrm{Pic}^d(C)$.

The degree $g-1$ universal Picard variety has the universal theta divisor $\Theta \subset \mathcal{J}_g^{g-1}$. This divisor is defined as the image of the Abel map $\mathcal{C}_g^{g-1} \to \mathcal{J}_g^{g-1}$ of the $(g-1)$-fold symmetric product $\mathcal{C}_g^{g-1}$ given by $(C, p_1, \ldots, p_{g-1}) \mapsto \mathcal{O}_C(p_1 + \cdots + p_{g-1})$. The degree 0 universal Picard variety has the zero section $z_g : \mathcal{M}_g^{ct} \to \mathcal{J}_g^0$. We denote by T the universal symmetric theta divisor $T \subset \mathcal{J}_g^0$ trivialized along the zero section, i.e. analytically the zero divisor of the function $\theta(\tau, z)/\theta(\tau, 0)$.

The main result of this paper is the computations of the pullbacks of Θ and T to $\overline{\mathcal{M}}_{g,n}$, with the answer given in Theorem 6.

Our motivation for computing the pullbacks of the theta divisors is that they can be used to compute the classes of natural geometric loci on the moduli space of curves, which have been studied recently. One example is a result proved (in cohomology) by Hain in [Hai11] (see Section 2 for notation):

Theorem 1. *For* $\deg \underline{d} = 0$, *the class in* $CH^g(\mathcal{M}_{g,n}^{ct}, \mathbb{Q})$ *of the pullback of the zero section of the universal Jacobian variety* $\mathcal{J}_g^0$ *is equal to*

$$[s_{\underline{d}}^* z_g] = \frac{1}{g!} \left[\frac{1}{2} \sum_{i=1}^n d_i^2 K_i - \frac{1}{2} \sum_{P \subseteq I} \left(d_P^2 - \sum_{i \in P} d_i^2 \right) \delta_0^P - \frac{1}{2} \sum_{h>0, P \subseteq I} d_P^2 \delta_h^P \right]^g,$$

where $d_P := \sum_{i \in P} d_i$.

The cycle $s_{\underline{d}}^* z_g$ is known as the *double ramification cycle*, and the question of computing its class is due to Eliashberg. Geometrically, it can be interpreted as the locus of curves admitting a map to $\mathbb{P}^1$ with ramification multiplicities being all the positive d_i over 0, and all the negative d_i over ∞.

The relationship of the double ramification cycle to the pullback of the theta divisor is as follows. Let $p : \mathcal{X}_g \to \mathcal{A}_g$ denote the universal family of principally polarized abelian varieties, let z_g denote the zero section of this family, and let $T \subset \mathcal{X}_g$ denote the universal symmetric theta divisor trivialized along the zero section. Then

$$(1.1) \qquad\qquad [z_g] = \frac{[T]^g}{g!} \in CH^g(\mathcal{X}_g, \mathbb{Q}).$$

Note that the pushforwards $p_*([T]^k)$ were considered by Mumford and studied in detail by van der Geer [vdG99], but (1.1), in the Chow ring, follows from the existence of a multiplicative decomposition for $Rp_*\mathbb{Q}$, proved by Deninger and Murre [DM91] (see eg. [BL04, Cor. 16.5.7], or [Voi12, Prop. 4.3.6, Cor. 4.3.9]). In cohomology (1.1) was proven independently by Hain [Hai11, Prop. 8.1].

A closely related result is the following very recent theorem of Müller [Mül12], which is nearly equivalent to computing the pullback of the degree $g - 1$ theta divisor (our notation is slightly different; see Section 2 for details):

Theorem 2. *For any* $\underline{d}$ *with* $\deg \underline{d} = g-1$, *and such that at least one* d_i *is negative, define the following locus in* $\mathcal{M}_{g,n}$:

$$D_{\underline{d}} := \left\{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \mid h^0(C, d_1 p_1 + \cdots + d_n p_n) \geq 1 \right\},$$

and let $\overline{D}_{\underline{d}}$ *denote its closure in* $\overline{\mathcal{M}}_{g,n}$. *Then in* $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ *the class of* $\overline{D}_{\underline{d}}$ *is equal to*

$$[\overline{D}_{\underline{d}}] = -\lambda_1 + 0 \cdot \delta_{irr} + \frac{1}{2} \sum_{i=1}^n d_i(d_i + 1) K_i - \frac{1}{2} \sum_{P \subseteq I} \left(d_P^2 - \sum_{i \in P} d_i^2 \right) \delta_0^P$$

$$- \frac{1}{2} \sum_{h>0, P \subseteq I} (d_P - h)(d_P - h + 1)\delta_h^P - \sum_{P \subseteq I_+, h > d_P} (h - d_P)\delta_h^P,$$

where $I_+ := \{ i \in I \mid d_i \geq 0 \}$.

Theorem 1 was first proved by Hain in April 2003, using normal functions. His proof became more widely known (and we became aware of it) in February 2011 with the appearance of Hain's preprint [Hai11]. Over the locus of curves with rational tails $\mathcal{M}_{g,n}^{rt}$, this result was reproved by Cavalieri, Marcus, and Wise [CMW11]

in July 2011, using Gromov–Witten theory. We obtained a proof of Theorem 1 in June 2011, and discussed it with Hain and others at PCMI in July 2011, in particular correcting Hain's original formula, see [Hai11]. Theorem 2 was proved by Müller [Mül12] in March 2012. We have decided to make our results available due to the continued interest in the problem.

Our method is more elementary and could also be applied in various similar situations to compute classes related to the theta divisor. In particular in our paper [GZ12] we consider degenerations of abelian varieties and prove an extension of Theorem 1, and primarily of formula (1.1), to the universal family of semiabelic varieties of torus rank one, using our computation of the class $[s_{\underline{d}}^* T]$ on $\overline{\mathcal{M}}_{g,n}$.

We calculate the class of the theta divisor using test curves. In Section 2 we describe a basis of test curves and a basis for $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$, and compute their intersection numbers. This is a standard method, and the calculations are standard, but we briefly summarize them for convenience and for future reference — the result is given in Proposition 3. The more interesting part is computing the intersections of the test curves with the pullback of the theta divisors Θ and T under the map s_d, which is done by using the properties of the theta function and the Abel–Jacobi map. This is the content of Proposition 5. Finally, in Theorem 6 we derive the formulas for the pullbacks of the theta divisors.

We follow the standard notation and conventions for working on the moduli of curves, referring for example to [Mum83], [HM98], [ACG11] for known results, discussion, and further references.

2. Divisors, test curves, and intersection numbers

In this section we describe a basis of divisor classes on $\overline{\mathcal{M}}_{g,n}$ and a collection of test curves, and we compute their intersection numbers. Our computation technique is quite standard, but we include it for the sake of completeness and for possible reference value.

Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ be the forgetful map and let $\pi_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ be the map that forgets all but the ith marked point. Let $I = \{1, 2, \ldots, n\}$ denote the indexing set. For a subset $P \subseteq I$, let P^c denote its complement. We consider the following set of generators for $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$:

- The classes $K_i := c_1(\pi_i^*(\omega_p))$, where ω_p is the relative dualizing sheaf of the universal curve $p : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$.
- The classes δ_h^P of the boundary divisors Δ_h^P, where $P \subseteq I$ and $0 \leq h \leq g$. The generic point of Δ_h^P is a reducible curve consisting of a smooth component of genus h containing the marked points indexed by P and a smooth component of genus $g - h$ with the remaining points, joined at a node. To satisfy the stability condition we assume that $|P| \geq 2$ if $h = 0$ and $|P| \leq g - 2$ if $h = g$. Note that $\delta_h^P = \delta_{g-h}^{P^c}$. In any sum involving δ_h^P, we assume that each divisor class appears only once, so we either explicitly state which one we are adding or we sum with symmetric coefficients. This convention is used by Müller [Mül12] but not by Hain [Hai11]. We use δ to denote the divisor classes on the moduli stack and Δ to denote the divisors on the coarse moduli space.
- The class δ_{irr} of the divisor Δ_{irr}. The generic point of Δ_{irr} is a smooth curve of genus $g - 1$ with two points identified to form a node.
- The first Chern class λ_1 of the Hodge bundle.

The above classes are known to be a basis of $\text{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ for $g \geq 3$. In what follows we assume that $g \geq 3$, but our results also hold for $g = 1$ and $g = 2$ by inspection.

A more common choice of a basis replaces the classes K_i with the classes ψ_i of the cotangent bundles at the marked points p_i. We use the classes K_i in our final result, but we use both K_i and ψ_i in intermediate calculations. These two classes differ by a linear combination of boundary classes (see [AC87, p. 161]):

$$(2.1) \qquad \psi_i = K_i + \sum_{P \ni i, |P| \geq 2} \delta_0^P.$$

Note that the classes K_i are denoted ψ_i in [Hai11], while in [Mül12] the ψ_i have the same meaning as above.

We now define a collection of test curves on $\overline{\mathcal{M}}_{g,n}$.

- **The curves Z_i.** Let $1 \leq i \leq n$; fix a generic smooth curve $(C, p_1, \ldots, \widehat{p}_i, \ldots, p_n) \in \mathcal{M}_{g,n-1}$. We define the family $Z_i \subset \overline{\mathcal{M}}_{g,n}$ by letting the point p_i range over C. The curve Z_i is isomorphic to C.
- **The curves Z_h^P.** Let $P = \{i_1, \ldots, i_m\} \subseteq I$ be a subset, let $P^c = \{j_1, \ldots, j_{n-m}\}$ be the complement, and let $0 \leq h < g$, where we assume that $m \geq 2$ if $h = 0$ and $m < n$ if $h = g - 1$. Fix a generic smooth curve $(C_1, p_{j_1}, \ldots, p_{j_{n-m}})$ in $\mathcal{M}_{g-h,n-m}$, and define the family $Y_h^P \subset \overline{\mathcal{M}}_{g-h,n-m+1}$ by adding a point q_1 and letting it range over C_1. Now fix another generic smooth curve $(C_2, p_{i_1}, \ldots, p_{i_m}, q_2) \in \mathcal{M}_{h,m+1}$, and define the family $Z_h^P \subset \overline{\mathcal{M}}_{g,n}$ by attaching q_1 to q_2 to form a node. The curves Y_h^P and Z_h^P are both isomorphic to C_1.
- **The curve $\mathcal{E}$.** Fix a generic smooth curve $(C_2, p_1, \ldots, p_n, q_2) \in \mathcal{M}_{g-1,n+1}$, and let $\mathcal{E}$ be the family obtained by attaching a varying elliptic curve $(C_1, q_1) \in \overline{\mathcal{M}}_{1,1}$ to the curve C_2 at q_2. We consider $\mathcal{E}$ to be "stacky", i.e. since (C_1, q_1) has an involution, we consider the generic point of $\mathcal{E}$ with coefficient $1/2$.
- **The curve Z_{irr}.** Fix a generic curve $(C_1, p_1, \ldots, p_n, q_1) \in \mathcal{M}_{g-2,n+1}$, and consider, for a fixed generic elliptic curve (E, q_2) in $\mathcal{M}_{1,1}$, the family Z_{irr} obtained by varying a point q_3 over E, and attaching q_1, q_2, q_3 to a rational curve. The curve Z_{irr} is isomorphic to E.

We now compute, in the standard way, the intersection numbers of these test curves with the chosen basis of divisors, which will then imply that these curves form a basis for $N_1(\overline{\mathcal{M}}_{g,n})$.

Proposition 3. *The test curves have the following intersection numbers with the divisors, where we write $(P, h) = (Q, l)$ if $P = Q$ and $h = l$ or if $P^c = Q$ and $g - h = l$.*

$$Z_i \cdot K_j = \begin{cases} 2g - 2, & i = j, \\ 0, & otherwise, \end{cases} \qquad Z_i \cdot \delta_h^P = \begin{cases} 1, & (P, h) = (\{i, j\}, 0), j \neq i \\ 0, & otherwise, \end{cases}$$

$$Z_i \cdot \delta_{irr} = 0, \quad Z_i \cdot \lambda_1 = 0, \quad Z_h^P \cdot K_i = \begin{cases} 2g - 2, & h = 0 \text{ and } i \in P, \\ 1, & h > 0 \text{ and } i \notin P, \\ 0, & otherwise, \end{cases}$$

$$Z_h^P \cdot \delta_l^Q = \begin{cases} 2 - 2(g - h) - |P^c|, & (Q, l) = (P, h), \\ 1, & (Q, l) = (P \sqcup \{j\}, h), j \in P^c, \\ 0, & otherwise, \end{cases}$$

$$Z_h^P \cdot \delta_{irr} = 0, \quad Z_h^P \cdot \lambda_1 = 0, \quad \mathcal{E} \cdot K_i = 0, \quad \mathcal{E} \cdot \delta_h^P = \begin{cases} -1/24, & (P, h) = (\emptyset, 1), \\ 0, & otherwise, \end{cases}$$

$$\mathcal{E} \cdot \delta_{irr} = 1/2, \quad \mathcal{E} \cdot \lambda_1 = 1/24, \quad Z_{irr} \cdot \delta_h^P = \begin{cases} 1, & (P, h) = (\emptyset, 1), \\ 0, & otherwise, \end{cases}$$

$$Z_{irr} \cdot K_i = 0, \quad Z_{irr} \cdot \delta_{irr} = -1, \quad Z_{irr} \cdot \lambda_1 = 0.$$

Proof. We first compute the intersections with the test curves Z_i. If $i \neq j$, then $\pi_j(Z_i) = \{(C, p_j)\}$ is a single point in $\mathcal{M}_{g,1}$, so K_j restricts to a trivial line bundle on Z_i and $Z_i \cdot K_j = 0$. On the other hand, $\pi_i(Z_i)$ is the fiber $\{(C, p_i) \mid p_i \in C\}$ of $\mathcal{M}_{g,1}$ over $C \in \mathcal{M}_g$, so the bundle K_i restricts to the cotangent bundle of C, which has degree $2g - 2$. The curves parametrized by Z_i are smooth except when $p_i = p_j$ for some $j \neq i$, in which case the two marked points lie on a rational tail. This gives the intersection numbers with the boundary divisors. Finally, $\pi(Z_i)$ is a point, so the Hodge bundle is trivial on Z_i and $Z_i \cdot \lambda_1 = 0$.

The remaining test curves are all supported on the boundary. To compute the intersection numbers with these curves we use the technique of §2 of [Fab99]. The boundary divisor Δ_h^P is the image of the map $\eta : \overline{\mathcal{M}}_{g-h,P^c \sqcup \{r_1\}} \times \overline{\mathcal{M}}_{h,P \sqcup \{r_2\}} \to \overline{\mathcal{M}}_{g,n}$ identifying the marked points r_1 and r_2. We compute the intersection numbers with the test curves by pulling back to this product. We denote by pr_1 and pr_2 the projection maps to the two components. The pullback $\eta^* \psi_i$ is equal to either $pr_1^* \psi_i$ if $i \in P^c$ or $pr_2^* \psi_i$ if $i \in P$. According to [Fab99], the pullback $\eta^* \delta_h^P$ is either $-pr_1^* \psi_{r_1} - pr_2^* \psi_{r_2} + pr_2^* \delta_{g-2h}^{I \sqcup \{r_2\}}$ if $g - 2h \geq 0$ and $P = I$ or $-pr_1^* \psi_{r_1} - pr_2^* \psi_{r_2}$ otherwise.

To compute $Z_h^P \cdot K_i$ for $h > 0$, we note that the family Z_h^P does not parametrize any curves with rational tails, so by (2.1) we have $Z_h^P \cdot K_i = Z_h^P \cdot \psi_i$. The preimage $\eta^{-1}(Z_h^P)$ is equal to $Y_h^P \times pt$, therefore $Z_h^P \cdot \psi_i = 0$ if $i \in P$. If $i \in P^c$, then passing to the first factor in the product we have

$$Z_h^P \cdot \psi_i = Y_h^P \cdot \psi_i = Y_h^P \cdot \left(K_i + \sum_{i \in Q \subseteq P^c \sqcup \{r_1\}} \delta_0^Q \right) = 1,$$

since $Y_h^P \cdot K_i = 0$ as above, and the only boundary divisor in the sum that intersects Y_h^P is $\delta_0^{\{p_i, r_1\}}$.

For the intersection numbers $Z_0^P \cdot K_i$, we note that the projection π_i collapses rational tails, so the image $\pi_i(Z_0^P)$ is the point $(C_2, p_i) \in \overline{\mathcal{M}}_{g,1}$ if $i \in P^c$, or the curve $\{(C_2, q_1) \mid q_1 \in C_2\}$ if $i \in P$. Therefore, $Z_0^P \cdot K_i$ is zero in the first case and $2g - 2$ in the second.

The intersections of the test curves Z_h^P with the boundary divisors correspond to the possible degenerations of the parameterized curves. All of the non-empty intersections are transverse and equal to one, except that the curve Z_h^P lies on the divisor Δ_h^P. Therefore, we again restrict to the first factor and obtain

$$Z_h^P \cdot \delta_h^P = -Y_h^P \cdot \psi_{r_1} = -Y_h^P \cdot \left(K_{r_1} + \sum_{Q \subseteq P^c, Q \neq \emptyset} \delta_0^{Q \sqcup \{r_1\}} \right)$$

$$= -Y_h^P \cdot K_{r_1} - \sum_{k \in P^c} Y_h^P \cdot \delta_0^{\{p_k, r_1\}} = -(2(g - h) - 2) - |P^c|.$$

Finally, the Hodge bundle on Z_h^P is trivial, so $Z_h^P \cdot \lambda_1 = 0$.

The curve $\mathcal{E}$ lies in the divisor $\Delta_1^\emptyset$ and intersects Δ_{irr}. The corresponding intersection numbers, as well as $\mathcal{E} \cdot \lambda_1$, were computed by Wolpert (see 4.15 and proof of Lemma 5.4 in [Wol83]). Note, however, that for us $\delta_1^\emptyset$ and $\mathcal{E}$ are stacky with generic automorphism group of degree two, so the intersection numbers differ by factors of $1/2$ (see also [Mül12, Lemma 4.2]). Finally, for any i the curve $\pi_i(\mathcal{E})$ is a fixed curve with a fixed marked point and an attached varying elliptic tail, so $\mathcal{E} \cdot K_i = 0$.

The curve Z_{irr} lies in the boundary divisors $\Delta_2^\emptyset$ and Δ_{irr}, intersects $\Delta_1^\emptyset$ at one point, and does not intersect the other boundary divisors. To compute $Z_{irr} \cdot \delta_2^\emptyset$, we pull back to $\overline{\mathcal{M}}_{g-2,n+1} \times \overline{\mathcal{M}}_{2,1}$ as above. The divisor class $\delta_2^\emptyset$ pulls back to $-pr_1^*\psi_{r_1} - pr_2^*\psi_{r_2}$, and on the pullback of Z_{irr} both r_1 and r_2 are fixed points on fixed components, so $Z \cdot \delta_2^\emptyset = 0$.

To calculate the intersection number $Z_{irr} \cdot \delta_{irr}$, we consider Δ_{irr} as the image of the map $\xi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ which glues together the last two marked points r_1 and r_2, and pull back to the moduli space $\overline{\mathcal{M}}_{g-1,n+2}$. According to [Fab99], the pullback of δ_{irr} to itself is

$$\xi^*\delta_{irr} = -\psi_{r_1} - \psi_{r_2} + \delta_{irr} + \sum_{h=0;\, 1 \in Q \subseteq I; (h,Q) \neq (g-1,I)}^{g-1} \left(\delta_h^{Q \cup \{r_1\}} + \delta_h^{Q \cup \{r_2\}} \right).$$

Geometrically, the preimage of a point under ξ is the normalization of the corresponding marked curve at a non-separating node. Therefore, the preimage $Y_{irr} = \xi^{-1}(Z_{irr})$ is a family that parametrizes reducible curves having three components: the fixed genus $g-2$ curve $(C_1, p_1, \ldots, p_n, q_1)$ containing the first n marked points, an elliptic curve (E, q_2, r_2) with q_2 fixed and r_2 varying, and a rational tail connecting q_1 and q_2, and containing r_1. To compute $Z_{irr} \cdot \Delta_{irr}$, we need to compute the intersections of Y_{irr} with the divisor classes on $\overline{\mathcal{M}}_{g-1,n+2}$ occurring in the above formula for $\xi^*\delta_{irr}$.

The curve Y_{irr} lies in the boundary divisors $\delta_1^{\{r_2\}} = \delta_{g-2}^{I \cup \{r_1\}}$ and $\delta_1^{\{r_1,r_2\}} = \delta_{g-2}^I$, intersects at one point the boundary divisor $\delta_1^\emptyset = \delta_{g-2}^{I \cup \{r_1,r_2\}}$ (when r_2 hits q_2 it moves off E onto a second rational bridge), and does not intersect the other boundary divisors. Also, it is clear that $Y_{irr} \cdot \psi_{r_1} = 0$.

Looking at the formula for $\xi^*\delta_{irr}$, we only need to compute $Y_{irr} \cdot \psi_{r_2}$ and $Y_{irr} \cdot \delta_1^{\{r_2\}}$. The computation of $Y_{irr} \cdot \delta_1^{\{r_2\}}$ is similar to the ones we did previously. We pull back a second time via the map $\eta : \overline{\mathcal{M}}_{g-2,n+2} \times \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g-1,n+2}$ whose image is $\Delta_1^{\{r_2\}}$. Let t_1 and t_2 denote the points of attachment, then $\delta_1^{\{r_2\}}$ pulls back to $-pr_1^*\psi_{t_1} - pr_2^*\psi_{t_2}$. The preimage $\eta^{-1}(Y_{irr})$ is $pt \times X_{irr}$, where pt is a fixed point and X_{irr} is a family parametrizing the fixed elliptic curve (E, r_2, t_2), with the moving point r_2. Therefore

$$Y_{irr} \cdot \delta_1^{r_2} = -X_{irr} \cdot \psi_{t_2} = -X_{irr} \cdot \left(K_{t_2} + \delta_0^{\{r_2,t_2\}} \right) = -1.$$

Similarly, we see that $Y_{irr} \cdot \psi_{r_2} = 0$. Indeed,

$$\psi_{r_2} = K_{r_2} + \sum_{Q \subset I \cup \{r_1\}, Q \neq \emptyset} \delta_0^{Q \cup \{r_2\}},$$

none of these boundary divisors intersect Y_{irr}, and K_{r_2} integrates to zero on Y_{irr} because it is an elliptic curve. Putting all this together, we get that $Z_{irr} \cdot \delta_{irr} = -1$.

Finally, the Hodge bundle on Z_{irr} is trivial, so $Z_{irr} \cdot \lambda_1 = 0$, and $Z_{irr} \cdot K_i = 0$ because the marked points are all fixed on a fixed component. $\qquad\square$

A straightforward computation (noting that the matrix of intersections above is close to being diagonal) shows that the matrix of intersections of our test curves with the chosen basis of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ is non-degenerate, and we thus get

Corollary 4. *The curve classes Z_j, Z_l^Q, $\mathcal{E}$, and Z_{irr} generate over $\mathbb{Q}$ the group $N_1(\overline{\mathcal{M}}_{g,n})$ of numerical equivalence classes of curves on $\overline{\mathcal{M}}_{g,n}$.*

3. The class of the theta divisor

In this section, we compute the intersection numbers of the test curves defined in the previous section with the pullbacks $[s_{\underline{d}}^*\Theta]$ and $[s_{\underline{d}}^*T]$ of the theta divisors, and then prove the main theorems.

Proposition 5. *For $\deg \underline{d} = 0$, we have*

$$Z_i \cdot [s_{\underline{d}}^*T] = d_i^2 g, \ Z_h^P \cdot [s_{\underline{d}}^*T] = d_P^2(g-h).$$

For $\deg \underline{d} = g - 1$, we have

$$Z_i \cdot [s_{\underline{d}}^*\Theta] = d_i^2 g, \ Z_h^P \cdot [s_{\underline{d}}^*\Theta] = (d_P - h)^2(g-h)$$

(we compute the intersections with $\mathcal{E}$ and Z_{irr} separately).

Proof. We compute these intersection numbers by noting that the restrictions of $s_{\underline{d}}$ to the test curves can be understood as Abel–Jacobi embeddings. We first compute, for $\deg \underline{d} = 0$, the intersections with $[s_{\underline{d}}^*T]$.

Since $\pi(Z_i) = C$ is a single point in $\mathcal{M}_g$, the image $s_{\underline{d}}(Z_i)$ lies inside $\mathrm{Pic}^0(C)$. The restriction of $s_{\underline{d}}$ to Z_i is therefore the composition of an Abel–Jacobi embedding $Z_i \to \mathrm{Pic}^0(C)$ and a multiplication by d_i map on $\mathrm{Pic}^0(C)$. The theta function restricted to the Abel–Jacobi image of the curve has degree g, while the pullback under the multiplication has degree d_i^2 on divisors. Therefore we get $Z_i \cdot [s_{\underline{d}}^*T] = d_i^2 g$.

Similarly, the Jacobian variety of any curve parameterized by Z_h^P is $\mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2)$. The limit of the Abel–Jacobi embedding is a more delicate issue. Indeed, note that the Abel-Jacobi mapping is naturally an embedding $C \to \mathrm{Pic}^1(C)$. Thus to have a map $C \to \mathrm{Pic}^0(C)$, we need to choose a basepoint for the embedding. In a family of curves degenerating to some $C_1 \cup C_2$ with one node, the limit of the chosen base point must lie on both C_1 and C_2, for the limit of the Abel–Jacobi embedding to be well-defined. Therefore it must be the point $q_1 \in C_1$, which is identified with $q_2 \in C_2$ to form the node. For the case of $\deg \underline{d} = 0$, we thus have in the limit

$$(3.1) \qquad s_{\underline{d}}((C_1, p_{j_1}, \ldots, p_{j_{n-m}}, q_1), (C_2, p_{i_1}, \ldots, p_{i_m}, q_2))$$

$$= \left(\sum_{j \in P^c} d_j p_j + d_P q_1, \sum_{i \in P} d_i p_i - d_P q_2 \right) \in \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2).$$

We recall that the theta function on a decomposable abelian variety is the product of the theta functions on the two factors. The point q_1 varies along the curve C_1 while q_2 is fixed, hence the second term is a constant map. The first term is the composition of the Abel–Jacobi embedding of C_1 with a multiplication by d_P, and thus has degree $d_P^2(g-h)$.

The computations for the pullback $[s_{\underline{d}}^*\Theta]$, for the case $\deg \underline{d} = g - 1$, are similar. First, the intersections with Z_i are the same as for $[s_{\underline{d}}^*T]$, as the degree 0 and $g - 1$ Picard varieties of a curve with rational tails are isomorphic.

The intersection numbers of Z_h^P with $[s_{\underline{d}}^*\Theta]$ are not the same, because the limit of the Abel–Jacobi map in degree $g - 1$ is different. As a smooth curve C degenerates to a nodal curve $C_1 \cup C_2$, the Jacobian variety $\mathrm{Pic}^{g-1}(C)$ becomes $\mathrm{Pic}^{g-h-1}(C_1) \times \mathrm{Pic}^{h-1}(C_2)$. Therefore,

$$(3.2) \qquad s_{\underline{d}}((C_1, p_{j_1}, \ldots, p_{j_{n-m}}, q_1), (C_2, p_{i_1}, \ldots, p_{i_m}, q_2))$$

$$= \left(\sum_{j \in P^c} d_j p_j + (d_P - h)q_1, \sum_{i \in P} d_i p_i - (d_P - h + 1)q_2 \right)$$

$$\in \mathrm{Pic}^{g-h-1}(C_1) \times \mathrm{Pic}^{h-1}(C_2),$$

and by the same reasoning we obtain $Z_h^P \cdot [s_{\underline{d}}^*\Theta] = (d_P - h)^2(g - h)$. $\qquad \square$

We are now ready to prove our main result.

Theorem 6. *For* $\deg \underline{d} = 0$*, the class* $[s_{\underline{d}}^*T] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ *of the pullback of the universal symmetric theta divisor trivialized along the zero section is equal to*

$$(3.3) \quad [s_{\underline{d}}^*T] = 0 \cdot \lambda_1 + 0 \cdot \delta_{irr} + \frac{1}{2}\sum_{i=1}^n d_i^2 K_i - \frac{1}{2}\sum_{P \subseteq I}\left(d_P^2 - \sum_{i \in P} d_i^2 \right)\delta_0^P - \frac{1}{2}\sum_{h>0, P \subseteq I} d_P^2 \delta_h^P.$$

We denote $d_P := \sum_{i \in P} d_i$*, and the last sum includes every boundary divisor class exactly once.*

For $\deg \underline{d} = g - 1$*, the class* $[s_{\underline{d}}^*\Theta] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ *of the pullback of the universal theta divisor is equal to*

$$(3.4) \qquad [s_{\underline{d}}^*\Theta] = -\lambda_1 + \frac{1}{8}\delta_{irr} + \frac{1}{2}\sum_{i=1}^n d_i(d_i + 1)K_i - \frac{1}{2}\sum_{P \subseteq I}\left(d_P^2 - \sum_{i \in P} d_i^2 \right)\delta_0^P$$

$$- \frac{1}{2}\sum_{h>0, P \subseteq I}(d_P - h)(d_P - h + 1)\delta_h^P.$$

Moreover, the class of the divisor $\overline{D}_{\underline{d}}$ *considered by Müller* [Mül12] *is expressed in terms of* $[s_{\underline{d}}^*\Theta]$ *as*

$$(3.5) \qquad [\overline{D}_{\underline{d}}] = [s_{\underline{d}}^*\Theta] - \sum_{P \subset I_+, d_P < h}(h - d_P)\delta_h^P - \delta_{irr}/8,$$

where $I_+ := \{i \in I \mid d_i > 0\}$*, thus reproving the formula in Theorem 2.*

Proof. According to Corollary 4, the curve classes Z_j, Z_h^P, $\mathcal{E}$, and Z_{irr} generate $N_1(\overline{\mathcal{M}}_{g,n})$. Moreover, the test curves Z_j and Z_h^P have zero intersection with the divisors λ_1 and δ_{irr}. Therefore to calculate the coefficients of the divisors K_i and δ_h^P in the theta divisors it suffices to intersect both sides of (3.3) and (3.4) with the test curves Z_j and Z_h^P, using Propositions 3 and 5, and to verify that they are equal. This is a tedious but straightforward calculation.

Computing the remaining coefficients is somewhat more complicated, since the test curves $\mathcal{E}$ and Z_{irr} parametrize curves of non-compact type, and we need to understand the degeneration of the Abel–Jacobi map to such curves.

We first compute the remaining coefficients in $[s_{\underline{d}}^* T]$. To calculate the intersection number $\mathcal{E} \cdot [s_{\underline{d}}^* T]$, note that a generic point of $\mathcal{E}$ is a smooth elliptic curve C_1 attached to a fixed genus $g-1$ curve C_2 with all the marked points on it. According to (3.1), the limit of the Abel–Jacobi map at that point is $(0, \sum d_i p_i) \in \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2)$, where the second term is constant. The value of the theta function trivialized along the zero section at all such points is the constant number $\theta(\tau, \sum d_i p_i)/\theta(\tau, 0)$, where τ is the period matrix of C_2, and we can choose the marked points p_i so that it is non-zero. This number does not change as the elliptic curve degenerates, therefore, $\mathcal{E} \cdot [s_{\underline{d}}^* T] = 0$.

The calculation of $Z_{irr} \cdot [s_{\underline{d}}^* T]$ is similar. Consider a reducible curve consisting of a smooth genus $g-2$ component $(C_1, p_1, \ldots, p_n, q_1)$ attached to a smooth genus two component (C_2, q_2). The value of the theta function trivialized along the zero section at such a curve depends only on the position of the marked points on the first component. Any point in Z_{irr} lies in a one-parameter family consisting of such curves, hence the theta function is constant on Z_{irr} and $Z_{irr} \cdot [s_{\underline{d}}^* T] = 0$. Intersecting both sides of (3.3) with $\mathcal{E}$ and Z_{irr} and using Proposition 3 shows that the coefficient of δ_{irr} in $[s_{\underline{d}}^* T]$ is also zero.

Before finding the remaining coefficients of $[s_{\underline{d}}^* \Theta]$, we first prove formula (3.5). Over the smooth locus $\mathcal{M}_{g,n}$ the classes $D_{\underline{d}}$ and $s_{\underline{d}}^* \Theta$ coincide by definition. However, the theta function vanishes identically on certain boundary components, so the divisor $s_{\underline{d}}^* \Theta$ is reducible and contains some boundary components with multiplicities that we now compute.

Consider a boundary divisor $\Delta_h^P = \overline{\mathcal{M}}_{g-h, n-m+1} \times \overline{\mathcal{M}}_{h, m+1}$. The restriction of the theta function to a generic point of this divisor is given by formula (3.2). It may happen that on one of the two components all marked points have non-negative weights. In this case the image of the divisor on that component lies entirely inside the theta divisor of the corresponding Jacobian. Assume without loss of generality that this happens on the second component. The order of vanishing of the theta function on such a boundary divisor is equal to $h - d_P$ by the Riemann theta singularity theorem (indeed, as per formula (3.2) for the Abel-Jacobi map on reducible stable curves we see that we are computing the multiplicity of the Riemann theta function at the specified effective divisor) to relate $[\overline{D}_{\underline{d}}]$ and $[s_{\underline{d}}^* \Theta]$ we need to subtract the corresponding multiple of δ_h^P.

Along Δ_{irr}, the generic vanishing order of the theta function is equal to $1/8$. This can be seen by looking at the Fourier–Jacobi expansion of the theta function (see e.g., the lowest order term in [Don87, Lemma 3.1.1]). We note that we are dealing with the actual universal theta function and not with the polarization on semiabelic varieties. Subtracting all the generic vanishing we get

$$[\overline{D}_{\underline{d}}] = [s_{\underline{d}}^* \Theta] - \sum_{P \subset I_+, d_P < h} (h - d_P) \delta_h^P - \delta_{irr}/8,$$

proving formula (3.5).

It remains to compute the coefficients of λ_1 and δ_{irr} in $[s_{\underline{d}}^* \Theta]$. First, we intersect with the curve $\mathcal{E}$. Once all the generic vanishing is taken out, the restrictions of $\overline{D}_{\underline{d}}$ and of $s_{\underline{d}}^* T$ to the curve $\mathcal{E}$ are the same, as indeed on a stable curve whose

normalization consists of two components of genera 1 and $g-1$, the degree $g-1$ and 0 Abel maps have the same behavior on the elliptic curve (for which the genus minus one is equal to zero). Thus we deduce from $s_{\underline{d}}^* T \cdot \mathcal{E} = 0$ also that $[\overline{D}_{\underline{d}}] \cdot \mathcal{E} = 0$ (compare also to [Mül12, Lemma 4.2]). Denoting the coefficients of λ_1 and δ_{irr} in $[s_{\underline{d}}^*\Theta]$ by a and b, respectively, and using (3.5) and the intersection numbers from Proposition 3, we see that $a + 12b = 1/2$.

To finish the computation of the class $[s_{\underline{d}}^*\Theta]$, we thus need to find one more relation. Intersecting with the test curve Z_{irr} is tricky, as it requires having an explicit description of the behavior of the map $s_{\underline{d}}$ on irreducible stable curves. This can be accomplished by a careful study of the Fourier-Jacobi expansion, but we take another approach. Consider the restriction of the class $[s_{\underline{d}}^*\Theta]$ to the image of the gluing map $i : \mathcal{M}_{2,1}^{ct} \times \overline{\mathcal{M}}_{g-2,n+1} \to \overline{\mathcal{M}}_{g,n}$. We note that on $\mathcal{M}_2^{ct}$ (and thus also by pullback on $\mathcal{M}_{2,1}^{ct}$) the class λ_1 is a boundary, namely $\lambda_1 = \delta_1^{\emptyset}/5$ (see [HM98], p. 171). Therefore, computing the pullback $i^*[s_{\underline{d}}^*\Theta]$ gives us the one extra condition that we need. Using the pullback formulas described in [Fab99], we see that

$$i^*[s_{\underline{d}}^*\Theta] = pr_1^*(a\lambda_1 + \psi) + pr_2^*(\cdots),$$

where the second term is not relevant for us.

To compute this class geometrically, let $\mathcal{P}\mathrm{ic}^1 = \mathcal{X}_2$ be the universal Picard variety over $\mathcal{M}_2^{ct} = \mathcal{A}_2$, and consider the map $s_1 : \mathcal{M}_{2,1}^{ct} \to \mathcal{P}\mathrm{ic}^1$ defined by $s_1(C,p) = p \in \mathrm{Pic}^1(C)$. We need to compute $s_1^*\theta$, where $\theta \subset \mathcal{P}\mathrm{ic}^1$ is the universal theta divisor. The image of s_1 is exactly θ, so we compute its pullback using the adjunction formula. As a map of course moduli spaces s_1 is injective, but as a map of stacks s_1 may have a non-trivial ramification divisor due to the presence of automorphisms.

The generic points of $\mathcal{M}_{2,1}^{ct}$ and of $\mathcal{P}\mathrm{ic}^1$ have trivial automorphism groups. The moduli space $\mathcal{M}_{2,1}^{ct}$ has two divisors whose generic points have non-trivial automorphisms, namely the locus of Weierstrass points and the boundary divisor $\delta_1^{\emptyset}$. On $\mathcal{P}\mathrm{ic}^1$, the images of these two divisors under s_1 are the locus of 2-torsion points and the subfamily of the universal family on $\mathcal{A}_{1,1}$, the moduli space of products of elliptic curves, that is trivial on one of the factors. The generic automorphism group both of the source and the target divisor is $\mathbb{Z}/2\mathbb{Z}$ in each of these two cases, so the map s_1 is unramified in codimension one, and therefore by the adjunction formula

$$K_{\mathcal{M}_{2,1}^{ct}} = (K_{\mathcal{P}\mathrm{ic}^1} + \theta)|_{\theta}.$$

It is well-known (see [ACG11, Thm. 7.15]) that $K_{\mathcal{M}_2^{ct}} = 13\lambda_1 - 2\delta_1^{\emptyset}$. Denoting by $p : \mathcal{M}_{2,1}^{ct} \to \mathcal{M}_2^{ct}$ the projection map, we see that the canonical class of $\mathcal{M}_{2,1}^{ct}$ is

$$K_{\mathcal{M}_{2,1}^{ct}} = p^*(K_{\mathcal{M}_2^{ct}}) + c_1\left(\omega_{\mathcal{M}_{2,1}^{ct}/\mathcal{M}_2^{ct}}\right) = 13\lambda_1 - 2\delta_1^{\emptyset} + \psi.$$

Similarly, let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ denote the universal family of principally polarized abelian varieties. It is well-known that $K_{\mathcal{A}_g} = (g+1)L$, where $L = \det \mathbb{E}$ is the determinant of the Hodge bundle $\mathbb{E} = \pi_*(\omega_{\mathcal{X}_g/\mathcal{A}_g})$. Therefore,

$$K_{\mathcal{P}\mathrm{ic}^1} = K_{\mathcal{X}_2} = \pi^* K_{\mathcal{A}_2} + c_1(\omega_{\mathcal{X}_g/\mathcal{A}_g}) = \pi^*(3L) + \pi^*L = 4\pi^*L.$$

By definition L pulls back to λ_1, so we finally obtain

$$\theta|_\theta = K_{\mathcal{M}_{2,1}^{ct}} - K_{\mathcal{P}\mathrm{ic}^1}|_\theta = \psi + 13\lambda_1 - 2\delta_1^\emptyset - 4\lambda_1 = \psi + 9\lambda_1 - 2\delta_1^\emptyset = -\lambda_1 + \psi,$$

using that $5\lambda_1 = \delta_1^\emptyset$ on $\mathcal{M}_{2,1}^{ct}$ in the last step. It follows that the coefficient a of the class $[s_{\underline{d}}^*\Theta]$ is equal to -1, completing the proof. $\qquad\square$

Proof of Theorem 1. The theorem directly follows from formula (3.3) for $[s_{\underline{d}}^*T]$, restricted to the moduli space $\mathcal{M}_{g,n}^{ct}$ of curves of compact type over which the universal family $\mathcal{X}_g \to \mathcal{A}_g$ extends, and from formula (1.1) for the zero section. $\quad\square$

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