

ON THE SEMIPRIMITIVITY OF FINITELY GENERATED ALGEBRAS

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ABSTRACT. Finitely generated associative algebras $A = K\langle a_1, \dots, a_n \rangle$ over a field K defined by homogeneous relations are considered. If there exists an order on the associated free monoid FM_n of rank n such that the set of normal forms of elements of A is a regular language in FM_n , then the algebra A is semiprimitive provided that the associated monomial algebra is semiprime.

Automaton algebras were defined by Ufnarovskii by the condition that the set of normal forms of elements of the algebra is a regular language; see [16]. More precisely, if A is a finitely generated algebra over a field K with a set of generators a_1, \dots, a_n , then let $K\langle x_1, \dots, x_n \rangle$ denote the free K -algebra of rank n and let $\pi : K\langle x_1, \dots, x_n \rangle \rightarrow A$ be the homomorphism such that $\pi(x_i) = a_i$, for $i = 1, \dots, n$. Assume that a well order \prec is given on the free monoid $\text{FM}_n = \langle x_1, \dots, x_n \rangle$ such that the unity of FM_n is the least element and that is compatible with the multiplication in FM_n . The latter means that $v \prec u$ implies that $vw \prec uw$ and $wv \prec wu$ for all $u, v, w \in \text{FM}_n$. Let $I = I_\pi$ be the ideal of FM_n consisting of all leading monomials of elements of $\ker(\pi)$. Then the set $N(A) = \text{FM}_n \setminus I$ is called the set of normal words corresponding to the chosen presentation of the algebra A and the chosen order on FM_n . The minimal set of generators of I is called the set of obstructions. The image $K[\text{FM}_n]/K[I]$ of the free algebra $K[\text{FM}_n]$ is referred to as the monomial algebra associated to A and is an important tool in the combinatorial aspects of finitely generated algebras; see [6, 16]. One says that A is an automaton algebra if $N(A)$ is a regular language; see [16, Section 5.10]. Recall that this means that this set is obtained from a finite subset of FM_n by applying a finite sequence of operations of union, multiplication and operation $*$ defined by $T^* = \bigcup_{i \geq 1} T^i$, for $T \subseteq \text{FM}_n$. It is well known that $N(A)$ is a regular language if and only if the set of obstructions is a regular language [16, Theorem 5.2.2]. Therefore, the class of automaton algebras contains the class of algebras with a finite set of obstructions. In other words, if $K[\text{FM}_n]/K[I]$ is a finitely presented algebra, then A is an automaton algebra. Consequently, due to the definition of Gröbner bases, if an algebra A admits a finite Gröbner basis, then A is an automaton algebra; see [16, §2]. For basic results on regular languages and automata theory we refer the reader to [8].

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There are many results indicating that the class of automaton algebras not only has better computational properties but also several structural properties that behave better than in the class of arbitrary finitely generated algebras. For example, in this context one can refer to results on the Gelfand-Kirillov dimension (see [16, Section 5.10]), results on the radical in the case of monomial automaton algebras ([16, Section 7.6]), results on prime algebras of this type ([1]), and also structural results concerned with the special case of finitely presented monomial algebras ([12, Chapter 24] and [13]).

In general, the notion of an automaton algebra depends on the given presentation and it also depends on the chosen order on the corresponding free monoid FM_n . These and some other aspects of Ufnarovskii's approach were studied in [10, 11]. In particular, a finite presentation of an algebra A does not imply in general that A is an automaton algebra; see for example [10] or [12, Example 24.21]. On the other hand, there are interesting classes of finitely presented algebras that satisfy this property with respect to every degree-lexicographic order on the corresponding FM_n [2].

If L is an ideal in the monoid FM_n , then FM_n/L denotes the Rees factor semigroup defined by $\text{FM}_n/L = (\text{FM}_n \setminus L) \cup \{0\}$ with the operation $a \circ b = ab$ if $ab \notin L$ and $ab = 0$ if $ab \in L$. Clearly, FM_n/L can be identified with the image of FM_n in $K[\text{FM}_n]/K[L]$. By a degree function on the free monoid FM_n we mean a homomorphism d into the additive semigroup of nonnegative integers \mathbb{N} such that $d^{-1}(0) = \{1\}$. This imposes an \mathbb{N} -gradation on the free algebra $K[\text{FM}_n]$ with the degree zero component equal to K . We say that a finitely generated algebra A is defined by a homogeneous presentation if the kernel of the corresponding homomorphism $\pi : K[\text{FM}_n] \rightarrow A$ is homogeneous with respect to the gradation on the free algebra $K[\text{FM}_n]$ determined by a degree function on FM_n . Clearly, in this case, there is an induced gradation on A . By $J(A)$ we denote the Jacobson radical of A . The aim of this note is to prove the following result.

Theorem 0.1. *Assume that $A = K\langle a_1, \dots, a_n \rangle$ is an algebra defined by a homogeneous presentation and such that (with respect to some fixed order on the corresponding free monoid FM_n) the associated monomial algebra $K[\text{FM}_n]/K[I]$ is an automaton algebra. If $K[\text{FM}_n]/K[I]$ is semiprime, then A is semiprimitive.*

We note that Theorem 0.1 extends [9, Theorem 3.2] (which deals with the special case where the ideal I comes from a finite Gröbner basis of A , whence it is a finitely generated ideal of FM_n), at the same time providing a simpler proof that does not depend on a description of the radical of a monomial algebra.

Recall that the monomial algebra $K[\text{FM}_n]/K[I]$ is semiprime if and only if the semigroup FM_n/I has no nonzero nilpotent ideals ([12, Proposition 24.3]).

The following is the key observation needed in the proof of the above theorem.

Lemma 0.2. *Assume that for an ideal L of FM_n the algebra $R = K[\text{FM}_n]/K[L]$ is an automaton algebra. Then the monoid FM_n/L has finitely many right and finitely many left annihilator ideals. Consequently, every nil subsemigroup of FM_n/L is nilpotent.*

Proof. Recall that the set of normal words $N(R)$ is a regular language if and only if its complement $L = \text{FM}_n \setminus N(R)$ is a regular language.

Let $w \in \text{FM}_n \setminus L$. Suppose that w is a zero divisor modulo L and write $r(w) = \{y \in \text{FM}_n \mid wy \in L\}$. So, $r(w)$ corresponds to the right annihilator of w in FM_n/L .

It is well known that L is a regular language if and only if on FM_n there are finitely many classes of the right syntactic congruence ρ_r determined by L ; see [8, Remark 1.9.1 in Chapter 6]. The latter is defined by

$$(u, v) \in \rho_r \text{ if and only if for every } w \in \text{FM}_n \text{ we have: } uw \in L \iff vw \in L.$$

In other words, there are finitely many right ideals of the form $r(w)$, $w \in \text{FM}_n$. Therefore, there exist finitely many right annihilator ideals in the monoid FM_n/L . A symmetric argument shows that there are finitely many left annihilator ideals in FM_n/L .

It is known that if S is a semigroup with a zero element and S satisfies the ascending chain condition on right and on left annihilator ideals, then every nil subsemigroup of S must be nilpotent; see [4, 17.23]. Therefore, the last assertion of the lemma follows. \square

Proof of Theorem 0.1. Suppose that A is not semiprimitive. By the hypothesis, a gradation by the semigroup of nonnegative integers on the free algebra $K[\text{FM}_n]$ is given and we have a homomorphism $\pi : K[\text{FM}_n] \rightarrow A$ such that A can be considered with the induced gradation. By a result of Bergman, the radical of a \mathbb{Z} -graded algebra is homogeneous (see [5, Theorem 30.28]). Hence, there exists a nonzero homogeneous element $\alpha \in \text{J}(A)$. Moreover, every such element must be nilpotent because it has degree > 0 . Choose an element β in $\pi^{-1}(\alpha)$ with the minimal possible leading term with respect to the given order on FM_n . Then $\beta = \sum_{i=1}^m \lambda_i v_i$ with $m \geq 1, 0 \neq \lambda_i \in K$, for some $v_1, \dots, v_m \in \text{FM}_n$, and we may assume that v_1 is the leading term of β . By the definition of I it is then clear that $v_1 \notin I$. Then, for every $w \in \text{FM}_n$ the element $\alpha\pi(w)$ is again a homogeneous element in the radical of A , whence it is nilpotent. Since $v_1 w$ is the leading term of $\beta w = \sum_{i=1}^m \lambda_i v_i w$, it follows easily that $v_1 w$ is nilpotent modulo I . Thus, $v_1 \text{FM}_n$ is a right ideal of FM_n that is nil modulo I . From Lemma 0.2 we know that every nil subsemigroup of FM_n/I is nilpotent. Since $K[\text{FM}_n]/K[I]$ is semiprime, it follows that $v_1 \in I$. This is a contradiction. The result follows. \square

We conclude with a few natural problems.

Problem 1. Can one skip the hypothesis that the defining relations are homogeneous?

The answer is affirmative if the base field K is nondenumerable. Indeed, in this case $\text{J}(A)$ is a nil ideal by a result of Amitsur (see [14, Lemma 7.1.2]), whence the above proof also works. However, in general, this seems to be an open problem even in cases where the associated monomial algebra is finitely presented.

Problem 2. Assume that A is an arbitrary finitely presented algebra defined by homogeneous relations. Is $\text{J}(A)$ (locally) nilpotent? Is this true at least for every semigroup algebra A ?

It is not known whether the Jacobson radical of a finitely presented algebra always is a nil ideal; see [15] and [3, Problem 1.92]. This is a problem attributed to Amitsur and Latyshev. Moreover, there are no examples of finitely presented algebras whose Jacobson radical is not locally nilpotent. The special case where the algebra is nil and one asks whether it must be nilpotent also is a well known open problem ([17]). Notice, however, that there exist finitely presented algebras whose radical is not nilpotent ([7]).

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