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# SKEW PRODUCT ATTRACTORS AND CONCAVITY

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Abstract. We propose an approach to the attractors of skew products that tries to avoid unnecessary structures on the base space and rejects the assumption on the invariance of an attractor. When nonivertible maps in the base are allowed, one can encounter the mystery of the vanishing attractor. In the second part of the paper, we show that if the fiber maps are concave interval maps, then contraction in the fibers does not depend on the map in the base.

## 1. Introduction

We want to propose a unified approach to many situations where attractors for skew products are considered. This includes random systems, nonautonomous systems, strange nonchaotic attractors, etc. (we allow the reader to continue this list, warning that the terminology may vary). This approach is built on existing ideas, but contains two new ingredients. The first one is a realization that the space can have various structures, so one should consider which results require which structures. The second one is the introduction of the possibility that an attractor is not invariant. Moreover, admitting the possibility that the base map is nonivertible, we encounter "the mystery of the vanishing attractor". An attractor present for an invertible map in the base, vanishes when we forget about the past and replace the base map by a noninvertible one. This happens in spite of the fact that the future dynamics do not depend on the past. From the philosophical point of view this can be interpreted as an application of Mathematics to History: even if our future depends only on our present, making predictions is much more consistent when we take our past into account.[1]

In the second part of the paper we show how changing the point of view allows us to pinpoint the real reasons for existence of a Strange Nonchaotic Attractor if the fibers are one-dimensional and the maps in the fibers are concave.

We concentrate on the discrete case, that is, on iterates of self-maps of some space. Thus, we consider a skew product on a space $X = B \times Y$. The space $B$ is the *base* and $Y$ is the *fiber space*; for each $\vartheta \in B$ the set $\{\vartheta\} \times Y$ is the *fiber over* $\vartheta$. We will denote by $\pi_2 : X \to Y$ the projection $\pi_2(\vartheta, x) = x$. The skew product

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[1]We hope that this remark will allow us to present this paper to the administrators in our universities as a result of the interdisciplinary research.

$F : X \to X$ can be written as

$$F(\vartheta, x) = (R(\vartheta), \psi(\vartheta, x)). \tag{1.1}$$

Here $R$ is a map from the base to itself, and for any $\vartheta \in B$ the map $\psi_\vartheta$, given by $\psi_\vartheta(x) = \psi(\vartheta, x)$ maps the fiber over $\vartheta$ to the fiber over $R(\vartheta)$.

We are interested in the attractors. Attraction of the trajectories to some set occurs in fibers, so we have to assume that the fiber space $Y$ is a metric space. Now, from the point of view of the theory of random maps (see, e.g., [3,7]), what happens in the base is a random process, which influences the maps in fibers. Thus, the system in the base will usually be some measure-preserving transformation. From the point of view of the theory of Strange Nonchaotic Attractors (*SNA*'s; see, e.g., a survey [10]), often the topology (and usually a differentiable structure) is added in the base, while the invariant measure is kept there. Since those systems usually have connections with physical models, this measure is in such cases "natural". The influence of what happens in the base is considered as external forcing. From the point of view of the theory of nonautonomous systems (this name is being used in various meanings; we want to differentiate it from the random systems), no structure in the base is necessary (see, e.g., [12]). Most often, this is just the shift by 1 on (nonnegative) integers or a cyclic permutation on a finite set.

Let us compare a skew product system with a "usual" one, that is, with a continuous map $\Phi : Z \to Z$ of a metric space $Z$ to itself. In both cases, we iterate the map (be it $F$ or $\Phi$) and look for the attractors, that is, loosely speaking, sets to which trajectories converge. However, in the usual case, this convergence is *in the whole space* $Z$, while in the skew product case the convergence is *fiberwise* (i.e. on the sets $\{\vartheta\} \times Y$). This is the basic difference and it has far-reaching consequences.

While, especially in the applications, the phase spaces $Y$ or $Z$ are not necessarily compact, quite often all interesting dynamics happen in some compact subset of those spaces. Moreover, a large part of the "pure" theory of dynamical systems has been built under the assumption that the phase space is compact. Therefore in the sequel we will assume that $Y$ *is a compact metric space*. At a certain moment (in Section 4) we will start investigating closer the special case when $Y$ is a closed interval and the intersection of the attractor with each fiber consists of one point. Then the whole attractor is the graph of a function $\varphi : B \to Y$. We will assume also that the attractor is almost global, that is, the intersection of its basin with every (or almost every) fiber contains all points of the fiber except perhaps the endpoints.

A remarkable paper on SNA's in this setting is the one by Keller [11] where a deep study on the existence and properties of the attractors is conducted for systems on $\mathbb{S}^1 \times [0, +\infty)$, of the form

$$T\begin{pmatrix}\theta\\ x\end{pmatrix} = \begin{pmatrix}R_\omega(\theta)\\ p(x)q(\theta)\end{pmatrix}, \tag{1.2}$$

where $R_\omega(\theta) = \theta + \omega \pmod 1$ is an irrational rotation of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{T}$, $q\colon \mathbb{S}^1 \longrightarrow [0, +\infty)$ is a continuous function and $p\colon [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous, bounded, strictly increasing and strictly concave function. A lot of subsequent analytical studies on the attractors of similar systems rely on the concavity of the function $p$ on the fibers ([1, 6, 8]) and some other studies deal with the relation between the monotonicity of this function and the existence of SNA's ([1, 6]). We

should also mention papers [4, 5, 9], where certain techniques for proving the existence of an SNA when the fiber is one-dimensional, were developed.

When we analyze this system from our new point of view, we notice that the existence of an SNA is caused solely by the concavity properties of the maps in the fibers, and the use of the ergodic theory tools for the basic results is practically unnecessary. Attraction (or contraction) in the fibers is the result of concavity, and not of the averaging.

The paper is organized as follows. In Section 2 we present our general philosophy. Then in Section 3 we study the situation when a skew product has a noninvertible base map. We present an example of a simple system for which a function from the base to the fibers whose graph is an attractor exists, but such function cannot be Borel or measurable for any invariant measure with positive entropy. On the other hand, when you replace the system in the base by its natural extension, there is a very regular (continuous, measurable) function whose graph is an attractor. This attractor vanishes when we forget about the past, although attraction is defined by looking at the future behavior, so we see the mystery of the vanishing attractor, mentioned earlier.

In the rest of the paper we consider systems with interval fibers and concave maps on them. In Section 5 we forget about the structure of the base space and we consider our skew product as a bunch of full orbits. Then, by using the estimates and notions introduced in Section 4, we prove that for nonautonomous dynamical systems concavity always implies contraction on the fibers both when the fiber maps are monotone and nonmonotone.

In Section 6 we use the results from the preceding section in the setting of skew products to study the influence of concavity on the existence of attractors, their basin of attraction and the positiveness almost everywhere of these attractors. We consider skew products similar to the Keller ones, that is, where the fibers are intervals of the form $[0, a]$ with $a > 0$ and the fiber maps $\psi_\theta$ are such that $\psi_\theta(0) = 0$ for every $\theta$ in the base $B$. Then, the set $B \times \{0\}$ is invariant (a set $A$ is invariant if $F(A) \subset A$). As usual, the *pinched set* is defined as the set of all points in the fibers whose image is contained in $B \times \{0\}$ (and, due to the invariance of this set, stay in $B \times \{0\}$ in all subsequent iterates). There are three main conclusions of this section. First, we show that concavity implies the existence of an attractor which is the graph of a function from the base $B$ to the fiber space and the basin of attraction of this attractor contains all points whose forward trajectory is nonpinched. However, these attractors are not necessarily invariant. Finally, we show that when additionally the base map is invertible and preserves an ergodic invariant measure $\mu$, then the function whose graph is the attractor is either 0 $\mu$-almost everywhere or positive $\mu$-almost everywhere. In the latter case, the attractor is invariant and its basin of attraction is of the form $Z \times (0, a]$, where $Z \subset B$ has full $\mu$-measure.

## 2. Skew product attractors

We consider the skew product (1.1), with $Y$ a compact metric space. As we mentioned in the introduction, we consider only attraction in the fibers. That is, the distance of a point $p$ from the attractor is measured as the distance of $p$ from the intersection of the attractor and the fiber containing $p$. Consequently, attraction means that this distance goes to zero.

Normally for an attractor $A$ one defines its *basin of attraction* as the set of points $p$ such that the distance of the $n$-th image of $p$ from $A$ goes to 0 as $n \to \infty$. Here we can repeat this definition, except that, as said earlier, here "distance" means the distance along the fiber (the distance from the empty set is by the definition infinite). In fact, we can speak of the basin of attraction of any set.

**Definition 2.1.** For a set $A \subset X$ its *basin of attraction* (or simply *basin*) is the set of all points $(\vartheta, x) \in X$ such that the distance from $F^n(\vartheta, x)$ to $A \cap (\{R^n(\vartheta)\} \times Y)$ goes to 0 as $n \to \infty$.

From the formal point of view it would be enough to speak of sets and their basins. However, it is customary to speak of *attractors*. In the "normal" case (not a skew product) one usually requires an attractor to be compact, invariant, and to have a large basin. "Large" may mean containing a neighborhood of the attractor or having positive measure, like for the *Milnor Attractor*; see [13]. Here we replace compactness by requiring compact intersection with fibers. However, requiring its invariance is not a natural thing to do, because (with obvious exceptions) the image of a fiber is a different fiber. We will discuss this problem later.

If we want to use measure, we have to consider separately a measure in $B$ and a measure in $Y$. In the base $B$ we can consider any invariant probability (usually ergodic) measure. However, in $Y$ we should consider a natural measure, like the Lebesgue one, and we do not care about its invariance.

Now we can specify what we mean by an attractor for a skew product (1.1).

**Definition 2.2.** A set $A \subset X$ is an *attractor* of $F$ if its intersection with every fiber $\{\vartheta\} \times Y$ is compact and one of the following conditions holds:

(a) for every fiber $\{\vartheta\} \times Y$, its intersection with the basin of $A$ contains a neighborhood of its intersection with $A$ (topological attractor);
(b) for every fiber $\{\vartheta\} \times Y$, its intersection with the basin of $A$ has positive measure (measure attractor).

If there is an invariant probability measure in the base, then we will call $A$ an attractor also if one of the above conditions is satisfied for the fiber over almost every $\vartheta \in B$.

Let us make a comment about invariance. For the usual dynamical systems an attractor is compact, and therefore it contains the $\omega$-limit set of every point from its basin. Thus, we can replace it, if necessary, by its subset defined as the closure of the union of the $\omega$-limit sets of the points from its basin. This subset is automatically invariant. Thus, the requirement that the attractor is invariant is natural and in a sense, is satisfied automatically. In the skew product case it is not even clear how the $\omega$-limit set should be defined if we apply our fiberwise approach. Indeed, (except in the periodic fibers, which often do not exist or their union has measure zero) the distance between the points is measured only in the same fiber, and the trajectory does not visit the same fiber twice. Thus, there is no special reason to require the attractor to be invariant.

However, sometimes we are interested in the invariance of an attractor. If an attractor is the graph of a function $\varphi : B \to Y$, then it is invariant if and only if the following *invariance equation* is verified for every $\vartheta \in B$:

$$\varphi(R(\vartheta)) = \psi(\vartheta, \varphi(\vartheta)).$$

In many cases a *pullback attractor* is considered (see, e.g.,[7, 15]). Then, instead of taking a point and its forward orbit, one goes back in time with an iterate of the base map, takes a point there, and returns with the same iterate of the map $F$. While we agree that this notion is very useful in many cases, we cannot agree with the opinion presented in [15] that this notion is better and more natural than the notion of the usual forward attractor. The past may not exist. It may exist but be not unique. And even if it exists and is unique, the aim of considering a dynamical system is to try to predict the future from our knowledge of the present, rather than to predict the present from our knowledge of the past.

We will finish this section by showing why it is a good idea to have a measure structure in $B$. The first reason is the following theorem.

**Theorem 2.3.** *Assume that for a skew product* (1.1) *there is an ergodic invariant measure $\mu$ for $R$ on the base $B$. Then, if the graphs of measurable functions $\varphi_1, \varphi_2 : B \to Y$ are both attractors, it follows that $\varphi_1 = \varphi_2$ $\mu$-almost everywhere.*

*Proof.* Suppose that $\varphi_1$ and $\varphi_2$ are not equal $\mu$-almost everywhere. Then there exists $\varepsilon > 0$ and a measurable set $Z \subset B$ of positive measure, such that $d(\varphi_1(\vartheta), \varphi_2(\vartheta)) > \varepsilon$ for every $\vartheta \in Z$, where $d$ is the metric in $Y$. Since $\mu$ is ergodic, almost every trajectory of $R$ visits $Z$ infinitely often. Therefore, for almost every $\vartheta \in B$ and every $x \in Y$ the maximum of the distances of $\pi_2(F^n(\vartheta, x))$ from $\varphi_1(R^n(\vartheta))$ and from $\varphi_2(R^n(\vartheta))$ is larger than $\varepsilon/2$ for infinitely many $n$'s. This means that it is impossible for the graphs of both $\varphi_1$ and $\varphi_2$ to be attractors.

Now we present an example of what can go wrong if we want to get an attractor everywhere instead of almost everywhere.

**Example 2.4.** Let $B = \{\vartheta_n\}_{n=-\infty}^{\infty} \cup \{-1, 1\}$, where $\vartheta_n = 1 - \frac{1}{n+1}$ if $n \geq 0$ and $\vartheta_n = -1 - \frac{1}{n}$ if $n < 0$. The map $R$ fixes $-1$ and 1, and maps $\vartheta_n$ to $\vartheta_{n+1}$. We take $Y = [0, 1]$ and define $\psi_\vartheta(x)$ to be $x(2 - x)$ if $\vartheta \geq 0$ and $x(2 - x)/4$ if $\vartheta < 0$ (see Figure 1). Assume that the graph of a function $\varphi : B \to [0, 1]$ is an invariant attractor. Since $x(2 - x)/4 \leq x/2$, we have $\varphi(\vartheta_0) \leq \varphi(\vartheta_{-n})/2^n \leq 1/2^n$ for all positive $n$. Thus, $\varphi(\vartheta_0) = 0$, and consequently, $\varphi(\vartheta_n) = 0$ for all $n > 0$. On the other hand, the trajectory of every $x \in (0, 1]$ under the map $x \mapsto x(2 - x)$ goes to 1, so $\varphi(\vartheta_n) \to 1$ as $n \to \infty$. This is a contradiction, and therefore in this case there is no $\varphi$ whose graph is an invariant attractor.

## 3. Noninvertible base map

In this section we consider a model in which the fiber space consists of only two points, and which illustrates very well the problems that we can encounter when considering a noninvertible map in the base. It has a great advantage of being simple, and can be interpreted as flipping a coin. It leads us to the mystery of the vanishing attractor.

In the base $B$ we take the full one-sided shift $R$ on 2 symbols (0 and 1). The fiber space consists of two points (again 0 and 1). We will use the notation $x = (x_0, x_1, x_2, \dots) \in B$ with $x_i \in \{0, 1\}$, so $R(x) = (x_1, x_2, x_3, \dots)$. The map $F : B \times \{0, 1\} \to B \times \{0, 1\}$ is given by

$$F(x, y) = (R(x), x_0).$$

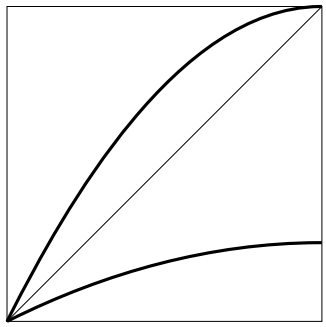

FIGURE 1. Maps $x \mapsto x(2-x)$ and $x \mapsto x(2-x)/4$ on $[0,1]$.

In this setup, the graph of a function $\varphi : B \to \{0,1\}$ is an attractor if and only if for every $(x,y) \in B \times \{0,1\}$ there is $N$ such that for every $n \geq N$

$$\pi_2(F^n(x,y)) = \varphi(R^n(x)). \tag{3.1}$$

We will show first that such a function exists.

**Definition 3.1.** Consider the following equivalence relation in $B$: the points $\vartheta$ and $\sigma$ are equivalent if $R^n(\vartheta) = R^m(\sigma)$ for some nonnegative integers $n, m$. The equivalence classes of this relation will be called the *full orbits* of $R$.

**Theorem 3.2.** *For the system defined above, there exists a function $\varphi : B \to \{0,1\}$ whose graph is an attractor with the basin of attraction equal to the whole space.*

*Proof.* We can look at $B$ as the disjoint union of full orbits of $R$. For each such orbit $\mathcal{O}$, we choose one element $x = (x_0, x_1, x_2, \dots) \in \mathcal{O}$. Then for each $n \geq 1$ we set $\varphi(R^n(x)) = x_{n-1}$. Observe that in such a way for every $y \in \{0,1\}$ we have $F(R^{n-1}(x), y) = (R^n(x), \varphi(R^n(x))$. Now, for every $z \in \mathcal{O}$, if $m$ is sufficiently large, there exists $n > 0$ such that $R^{m-1}(z) = R^{n-1}(x)$, so for every $y \in \{0,1\}$ we have $\pi_2(F(R^{m-1}(z), y)) = \varphi(R^m(z))$. Thus, if we define $\varphi$ in an arbitrary way at the remaining points of $\mathcal{O}$ we get $\pi_2(F^m(z,y)) = \pi_2(F(R^{m-1}(z), \widetilde{y})) = \varphi(R^m(z))$ for some $\widetilde{y} \in \{0,1\}$. Hence, if we make this construction for all full orbits of $R$, then the graph of $\varphi$ will be a global attractor.

If $\mu$ is an ergodic invariant measure for $R$, then the question is whether a function $\varphi$, whose graph is an attractor, can be measurable.

**Theorem 3.3.** *For the system described above and an ergodic invariant probability measure $\mu$ on $B$, if there exists a $\mu$-measurable function $\varphi : B \to \{0,1\}$ whose graph is an attractor with the basin of attraction $Z \times \{0,1\}$ for some set $Z \subset B$ of $\mu$ measure 1, then the entropy of $\mu$ is zero.*

*Proof.* Set $A = \{x \in B : x_0 = \varphi(R(x))\}$ and $C = \bigcap_{n=0}^{\infty} R^{-n}(A)$. With this notation, if $n \geq 1$, then

$$\pi_2(F^n(x,y)) = (R^{n-1}(x))_0,$$

so (3.1) is equivalent to $R^{n-1}(x) \in A$. Thus, the graph of $\varphi$ is an attractor on a set of full measure if and only if for almost every $x \in B$ there is $N \geq 1$ such that

$R^N(x) \in C$, that is, if

$$\mu\left(\bigcup_{N=1}^{\infty} R^{-N}(C)\right) = 1.$$

Note also that $\sigma(C) \subset C$.

Clearly, if $\varphi$ is measurable, then $A$ is measurable, so $C$ is also measurable. Since $C$ is invariant and $\mu$ is ergodic, $C$ has measure 0 or 1. If it has measure 0, the union of its preimages has measure 0, a contradiction. This proves that $\mu(C) = 1$. Since $C \subset A$, we get $\mu(A) = 1$.

Let $\xi : B \to B$ be the map that replaces $x_0$ by $1 - x_0$. By the definition, at most one of the points $x, \xi(x)$ can belong to $A$. Therefore, $A \cap \xi(A) = \emptyset$. This means that the shift $R$ is one-to-one $\mu$-almost everywhere. Since $R$ has a one-sided generator, this implies that $h_\mu(R) = 0$ (see, e.g., [14]).

Since there exist ergodic invariant probability measures for $R$ with positive entropy, we get the following corollary.

**Corollary 3.4.** *For the system above, there is no Borel function $\varphi : B \to \{0, 1\}$ whose graph is an attractor.*

Consider now what happens if instead of one-sided shift in the base we consider its natural extension, the two-sided shift. In this case the graph of the map $\psi$, given by

$$\psi(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) = x_{-1}$$

is clearly an attractor with the whole space as the basin of attraction (because this graph is the image of the whole space). This attractor is invariant, the function $\psi$ is continuous, and therefore measurable for all invariant measures. To summarize – this is the best attractor one can dream of. Yet when we return to our original system (which can be interpreted as forgetting about the past), this attractor vanishes. This is a paradox, because attractors are defined by looking forward in time, so why should forgetting of the past have any influence on them? We call this strange phenomenon "the mystery of the vanishing attractor". Another, less degenerate but more complicated, instance of this "mystery" can be found in the forthcoming paper [2] for a system where the base is also the 2-shift, the fiber space is the interval $[0, 1]$ and the fiber maps are two homeomorphisms, piecewise linear with two pieces, that fix 0 and 1.

## 4. $\alpha$-concavity

In this section we introduce the notion of $\alpha$-concavity and we obtain estimates that relate the $\alpha$-concavity with the contraction of an interval map, both in the monotone and nonmonotone case. These estimates will be useful later.

**Definition 4.1.** Let $f$ be a continuous real-valued function on a closed interval $I$ of the real line and let $\alpha \geq 0$. The function $f$ will be called *$\alpha$-concave* if the function $f_\alpha$, given by

$$f_\alpha(x) = f(x) + \alpha x^2$$

is concave.

The following properties of an $\alpha$-concave function $f$ follow immediately from the definition:

(a) $f$ is concave;
(b) if $\alpha > 0$, then $f$ is strictly concave;
(c) if $0 \le \beta \le \alpha$, then $f$ is $\beta$-concave.

Now we will prove two inequalities satisfied by $\alpha$-concave functions that we will need later.

Assume that the left endpoint of $I$ is 0. Given two points $u, v > 0$ we define

$$\kappa(u,v) := \frac{|v-u|}{\min\{u,v\}}.$$

**Lemma 4.2.** *Assume that $f$ is $\alpha$-concave in the interval $[0,y]$ and $f(0) = 0 < f(y)$. Let $x \in (0,y)$ be such that $0 < f(x) < f(y)$. Then,*

$$\frac{\kappa(f(x), f(y))}{\kappa(x,y)} \le \frac{f(y)}{f(y) + \alpha y^2}.$$

*Proof.* By concavity of $f_\alpha$ we have $f_\alpha(x)/x \ge f_\alpha(y)/y$. Therefore

$$\frac{f(x)}{x} \ge \frac{f(y)}{y} + \alpha(y-x), \tag{4.1}$$

so

$$f(x) \ge \frac{xf(y)}{y} + \alpha x(y-x).$$

Thus,

$$f(y) - f(x) \le \frac{yf(y)}{y} - \frac{xf(y)}{y} - \alpha x(y-x) = (y-x)\left(\frac{f(y)}{y} - \alpha x\right).$$

From this and (4.1) we get

$$\begin{aligned}\frac{\kappa(f(x), f(y))}{\kappa(x,y)} &= \frac{f(y)-f(x)}{f(x)} \cdot \frac{x}{y-x} \le \frac{\frac{f(y)}{y} - \alpha x}{\frac{f(y)}{y} + \alpha(y-x)} \\ &= 1 - \frac{\alpha y}{\frac{f(y)}{y} + \alpha(y-x)} \le 1 - \frac{\alpha y}{\frac{f(y)}{y} + \alpha y} = \frac{\frac{f(y)}{y}}{\frac{f(y)}{y} + \alpha y} = \frac{f(y)}{f(y) + \alpha y^2}.\end{aligned}$$

Recall that for a concave map the one-sided derivatives are well defined. We will denote the left one-sided derivative of $f$ by $f'_-$.

Let $f$ be a strictly concave nonnegative function on the interval $[0,a]$, with $f(0) = 0$. Observe that there exists a unique point $c \in [0,a]$ such that $f(c) = \max\{f(x) \colon x \in [0,a]\}$, that is, $f$ is strictly increasing on $[0,c]$ and strictly decreasing on $[c,a]$ (but note that it may happen that $c = a$).

**Definition 4.3.** By strict concavity, for every $x \in (0,c]$ we have $f'_-(x) < f(x)/x$. Usually this inequality, with the absolute value of the derivative, can be extended further to the right of $c$. Set

$$b = \sup\left\{x \in [0,a] \colon |f'_-(x)| < \frac{f(x)}{x}\right\}. \tag{4.2}$$

Since often the absolute values of the slopes of the tangent line to the graph of $f$ at $b$ and the line joining $(0,0)$ with $(b, f(b))$ are equal, we will call $b$ the *isoclinic point* of $f$.

Note that $c \le b \le a$ and $f(b) > 0$. However, we can prove more.

**Lemma 4.4.** *Let $f$ be a strictly concave nonnegative map of an interval $[0, a]$ to itself, with $f(0) = 0$, whose isoclinic point is $b$. Then $b \ge a/2$.*

*Proof.* Suppose that $b < a/2$. Then $c < a/2 < a$, so by strict concavity,

$$|f'_-(a/2)| < \frac{f(a/2) - f(a)}{a - a/2} \le \frac{f(a/2)}{a/2}.$$

Thus, by the definition, $a/2 \le b$.

**Lemma 4.5.** *Assume that $f$ is $\alpha$-concave in the interval $[0, a]$ with $f(0) = 0$ and $\alpha > 0$. Let $x, y \in (0, b)$ be such that $x < y$ and $0 < f(y) < f(x)$. Then*

$$\frac{\kappa(f(x), f(y))}{\kappa(x, y)} < 1 - \frac{\alpha b(b - x)}{f(b)}. \tag{4.3}$$

*Proof.* With our assumptions we have

$$\frac{\kappa(f(x), f(y))}{\kappa(x, y)} = \frac{f(x) - f(y)}{f(y)} \cdot \frac{x}{y - x}. \tag{4.4}$$

The assumptions $x < y$ and $f(y) < f(x)$ imply that $y > c$. Hence, $f(b) < f(y)$ because $y < b$, so

$$\frac{x}{f(y)} < \frac{b}{f(b)}. \tag{4.5}$$

Moreover, by the $\alpha$-concavity of $f$ we get

$$\begin{aligned} \frac{f(x) - f(y)}{y - x} &\le \frac{f(x) - f(b)}{b - x} \\ &= \frac{f_\alpha(x) - \alpha x^2 - f_\alpha(b) + \alpha b^2}{b - x} = \frac{f_\alpha(x) - f_\alpha(b)}{b - x} + \alpha(b + x). \end{aligned} \tag{4.6}$$

The left one-sided derivative of $f$ is continuous from the left, and therefore

$$-f'_-(b) = |f'_-(b)| \le \frac{f(b)}{b}.$$

By this and concavity of $f_\alpha$ we get

$$\frac{f_\alpha(x) - f_\alpha(b)}{b - x} \le -(f_\alpha)'_-(b) = -f'_-(b) - 2\alpha b \le \frac{f(b)}{b} - 2\alpha b.$$

Together with (4.6), this gives

$$\frac{f(x) - f(y)}{y - x} \le \frac{f(b)}{b} - \alpha(b - x).$$

This inequality together with (4.4) and (4.5) implies (4.3).

## 5. Nonautonomous systems

When we forget about the structure of the base space, our skew product becomes a bunch of full orbits. With this in mind, in this section we study the relation between concavity and contraction on the fibers for nonautonomous dynamical systems by using the estimates and notions introduced in Section 4. To this end we introduce the notion of *equiconcavity* which makes the notion of $\alpha$-concavity independent of the "scale" of each map in the system (that is, independent on the supremum of each of these maps).

**Definition 5.1.** Let $(f_n)_{n=1}^{\infty}$ be a sequence of maps from the interval $[0,a]$ to itself such that $f_n(0)=0$ for every $n$. Such a sequence will be called *pinched* when there exists an $n$ such that $f_n$ is identically zero. Also, it will be called *equiconcave* if there exists a positive constant $\beta$ such that each $f_n$ is $\beta\gamma_n$-concave, where $\gamma_n$ is the supremum of $f_n$.

We will consider a nonautonomous dynamical system given by a sequence $(f_n)_{n=1}^{\infty}$. That is, we apply first $f_1$, then $f_2$, etc. We will use the standard notation for the trajectories, that is, if the starting point with index 0 belongs to $[0,a]$, then we define by induction the point with index $n$ as the image under $f_n$ of the point with index $n-1$. For instance, $x_1=f_1(x_0)$, $x_2=f_2(x_1)$, etc.

Let us consider first the case when all maps $f_n$ are nondecreasing.

**Theorem 5.2.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of monotone maps from the interval $[0,a]$ to itself such that $f_n(0)=0$ for every $n$. Assume also that this sequence is either pinched or equiconcave. Then for every $x_0,y_0\in(0,a]$ we have*

$$\lim_{n\to\infty}|x_n-y_n|=0. \tag{5.1}$$

*Proof.* Assume first that $(f_n)_{n=1}^{\infty}$ is pinched and let $N$ be the smallest positive integer such that $f_N$ is identically zero. Then $x_N=y_N=0$, and therefore $x_n=y_n=0$ for every $n\geq N$. This completes the proof of the theorem in this case.

Suppose now that $(f_n)_{n=1}^{\infty}$ is equiconcave. Without loss of generality we may assume that $0<x_0<y_0$, and therefore $0<x_n<y_n$ for every $n$. Then we have

$$|x_n-y_n|=x_n\kappa(x_n,y_n)<a\kappa(x_n,y_n)=a\kappa(x_0,y_0)\prod_{k=0}^{n-1}\frac{\kappa(x_{k+1},y_{k+1})}{\kappa(x_k,y_k)}. \tag{5.2}$$

By Lemma 4.2 and since $\gamma_{k+1}\geq y_{k+1}$, we have

$$\frac{\kappa(x_{k+1},y_{k+1})}{\kappa(x_k,y_k)}\leq\frac{y_{k+1}}{y_{k+1}+\beta\gamma_{k+1}y_k^2}\leq\frac{1}{1+\beta y_k^2}.$$

If there is $\varepsilon>0$ such that $y_k\geq\varepsilon$ for infinitely many indices $k$, then by the above estimate infinitely many terms $\kappa(x_{k+1},y_{k+1})/\kappa(x_k,y_k)$ are bounded from above by $1/(1+\beta\varepsilon^2)$, which is smaller than 1. Taking into account that by Lemma 4.2 $\kappa(x_{k+1},y_{k+1})/\kappa(x_k,y_k)<1$ for all $k$, we see that in this case the product in (5.2) goes to 0 as $n\to\infty$. Thus, by (5.2), (5.1) holds.

If there is no such $\varepsilon$, then by the definition of the limit, $\lim_{n\to\infty}y_n=0$, and since $0<x_n<y_n$, (5.1) also holds.

Now we discard the assumption of monotonicity of the maps $f_n$. However, we need some bound on the points $x_n$ and $y_n$. In a general case we cannot expect any contraction, as the simple example of $f_n(x)=4x(1-x)$ and $a=1$ shows. Let $b_n$ be the isoclinic point of $f_n$.

The following theorem is a generalization of Theorem 5.2.

**Theorem 5.3.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of maps from the interval $[0,a]$ to itself such that $f_n(0)=0$ for every $n$. Assume also that this sequence is either pinched or equiconcave. Then for every $x_0,y_0\in(0,a]$ such that $x_n,y_n<b_n$ for all $n$, we have*

$$\lim_{n\to\infty}|x_n-y_n|=0. \tag{5.3}$$

*Proof.* In the pinched case the proof is exactly the same as for Theorem 5.2. In the equiconcave case, it is very similar, so we only point out the differences.

The first difference is that it may happen that $x_N = y_N$ for some $n$, but then $x_n = y_n$ for all $n > N$, so (5.3) holds.

The second difference is that it may happen that $x_k > y_k$; then we just switch the roles of $x$ and $y$.

The third, and most important difference is that it may happen that $x_k < y_k$ but $x_{k+1} > y_{k+1}$ (or $x_k > y_k$ but $x_{k+1} < y_{k+1}$, then again we switch the roles of $x$ and $y$). In this case, instead of Lemma 4.2 we use Lemma 4.5. We get, taking into account that $f(b_k) \leq \gamma_k$ and Lemma 4.4,

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} \leq 1 - \frac{\beta\gamma_k b_k(b_k - x_k)}{f(b_k)} \leq 1 - \frac{\beta a(b_k - x_k)}{2}. \tag{5.4}$$

Now, instead of checking whether there are infinitely many indices $k$ for which $y_k \geq \varepsilon$, we check whether there are infinitely many indices $k$ for which either $x_{k+1} < y_{k+1}$ and $y_k \geq \varepsilon$ or $x_{k+1} > y_{k+1}$ and $b_k - x_k \leq \varepsilon$ (we assume that $x_k < y_k$). If yes, we get (5.3) in the same way as in the proof of Theorem 5.2, but taking additionally into account inequality (5.4). If there is no $\varepsilon > 0$ for which this is true, then the sequence $(x_k - y_k)$ splits into two subsequences. On one of them both $x_k$ and $y_k$ go to 0, so $x_k - y_k \to 0$; on the other one both $b_k - x_k$ and $b_k - y_k$ go to 0, so also $x_k - y_k \to 0$.

## 6. Equiconcave attractors

Let us now analyze how the results of Section 5 fit into the scheme presented in Section 2.

We use the notation from Section 2.

**Definition 6.1.** If $Y = [0, a]$, the family $\{\psi_\vartheta\}_{\vartheta \in B}$ will be called *equiconcave* if $\psi_\vartheta(0) = 0$ for every $\vartheta \in B$ and there exists a positive constant $\beta$ such that each $\psi_\vartheta$ is $\beta\gamma_\vartheta$-concave, where $\gamma_\vartheta$ is the supremum of $\psi_\vartheta$. Note that now we included the pinched case in the definition of equiconcavity. Indeed, if $\psi_\vartheta$ is identically 0, then $\gamma_\vartheta = 0$ and $\psi_\vartheta$ is 0-concave.

**Definition 6.2.** If $Y = [0, a]$ and the family $\{\psi_\vartheta\}_{\vartheta \in B}$ satisfies $\psi_\vartheta(0) = 0$ for each $\vartheta \in B$ and is equiconcave, then we will call the system $(X, F)$ an *equiconcave skew product*. If additionally all functions $\psi_\vartheta$ are monotone, the system will be called a *monotone equiconcave skew product*. If we replace the assumption of monotonicity by the assumption that

$$\psi_\vartheta([0, a]) \subset [0, b_{R(\vartheta)})$$

for all $\vartheta \in B$, where $b_{R(\vartheta)}$ is the isoclinic point of $\psi_{R(\vartheta)}$, then the system will be called an *isoclinic equiconcave skew product*. Remember that the isoclinic point of a monotone function is $a$. Hence, every monotone equiconcave skew product is an isoclinic equiconcave skew product.

In many standard examples of systems with strange nonchaotic attractors one defines $\psi$ as a product: $\psi(\vartheta, x) = f(x)g(\vartheta)$. In those examples, if $f$ is $\alpha$-concave for some $\alpha > 0$, then $\{\psi_\vartheta\}_{\vartheta \in B}$ is equiconcave and the isoclinic point for all $\psi_\vartheta$ is the same as for $f$. In particular, if additionally $f$ is monotone, then we get a monotone equiconcave skew product, and if the product of the maxima of $f$ and $g$ is smaller than the isoclinic point for $f$, then we get an isoclinic equiconcave skew product.

To state the results in a short way, we need more definitions.

**Definition 6.3.** A graph of a function $\varphi : B \to [0, a]$ will be called *preinvariant* if for every $\vartheta \in B$ there exists $N$ such that for every $n \geq N$ we have

$$F(R^n(\vartheta), \varphi(R^n(\vartheta))) = (R^{n+1}(\vartheta), \varphi(R^{n+1}(\vartheta))). \tag{6.1}$$

A point $\vartheta \in B$ will be called *pinching* if there are infinitely many positive integers $n$ such that $\psi_{R^n(\vartheta)}$ is identically equal to 0.

First we do not endow $B$ with any extra structure.

**Theorem 6.4.** *Let the system $(X, F)$ with base $B$ and fiber space $[0, a]$ be an isoclinic equiconcave skew product and let $\varphi : B \to [0, a]$ be a preinvariant function, positive at any point that is not pinching. Then the basin of the graph of $\varphi$ contains all points whose forward trajectory does not pass through $B \times \{0\}$.*

*Proof.* Let $(\vartheta, x)$ be a point whose forward trajectory does not pass through $B \times \{0\}$. By the definition of preinvariance, there is $N$ such that (6.1) holds for every $n \geq N$. None of the points $R^n(\vartheta)$, $n = 0, 1, 2, \ldots$, is pinched, so $\pi_2(F^N(\vartheta, x))$ is positive. By the assumptions on $(\vartheta, x)$, $\varphi(R^N(\vartheta))$ is also positive. Therefore, by Theorem 5.3, the distance between $\pi_2(F^n(\vartheta, x))$ and $\varphi(R^n(\vartheta))$ goes to 0 as $n \to \infty$.

In the proof of the next theorem, we will use a similar method as in the proof of Theorem 3.2.

**Theorem 6.5.** *Let the system $(X, F)$ with base $B$ and fiber space $[0, a]$ be an isoclinic equiconcave skew product. Then there exists a preinvariant function $\varphi : B \to [0, a]$, positive at any point that is not pinched.*

*Proof.* Since we do not require any special behavior of $\varphi$ across the fibers, it is enough to define $\varphi$ on each full orbit $\mathcal{O}$ of $R$. We will consider three possible cases.

The first case is when there is $\vartheta \in \mathcal{O}$ which is pinched. Then every element of $\mathcal{O}$ is pinched. In this case we set $\varphi$ identically 0 on $\mathcal{O}$.

The second case is when no element of $\mathcal{O}$ is pinched and $\mathcal{O}$ is neither periodic nor preperiodic. Then we fix $\vartheta_0 \in \mathcal{O}$ such that for $n = 0, 1, 2, \ldots$ the map $\psi_{R^n(\vartheta_0)}$ is not identically 0, choose some value $a_0 \in (0, a]$ and set

$$\varphi(R^n(\vartheta_0)) = \pi_2(F^n(\vartheta_0, a_0)) \tag{6.2}$$

for $n = 0, 1, 2, \ldots$. At all other points of $\mathcal{O}$ we set arbitrary positive values of $\varphi$.

The third case is when no element of $\mathcal{O}$ is pinched and $\mathcal{O}$ is periodic or preperiodic. Then we fix $\vartheta_0 \in \mathcal{O}$ which is periodic for $R$. Let $k$ be its period. Then $F^k$ restricted to the fiber over $\vartheta_0$ has a fixed point $(\vartheta_0, 0)$. If it has another fixed point $(\vartheta_0, x)$, we choose this $x$ as $a_0$. Otherwise, we set $a_0 = 0$. Then we set (6.2) for $n = 0, 1, 2, \ldots, k-1$. At all other points of $\mathcal{O}$ we set arbitrary positive values of $\varphi$.

It is clear that the function $\varphi$ constructed in the way described above is preinvariant and is positive at any point that is not pinched.

From the above two theorems we obtain immediately the following corollary.

**Corollary 6.6.** *Let the system $(X, F)$ with base $B$ and fiber space $[0, a]$ be an isoclinic equiconcave skew product. Then there exists a function $\varphi : B \to [0, a]$, whose graph has the basin of attraction containing all points whose forward trajectory does not pass through $B \times \{0\}$.*

According to our terminology, we cannot call the above graph an attractor if we do not have any information about how large the set of pinching points is. However, informally we think about it as an attractor.

Observe that using this method we cannot always make an attractor invariant. If $\varphi$ is defined at $R(\vartheta)$, we can try to define $\varphi(\vartheta)$ as $\psi_\vartheta^{-1}(\varphi(R(\vartheta)))$, but it may happen that the image of $[0,a]$ under $\psi_\vartheta$ does not contain $\varphi(R(\vartheta))$.

Note that in general at this stage we do not have any uniqueness of the attractor. Indeed, our choices in the construction were to a great degree arbitrary.

Now we introduce topology. We assume that $B$ is a compact metric space and that $F$ is continuous. We can ask whether the function $\varphi$, whose graph is an attractor, can be chosen continuous, semicontinuous, Borel, etc. As we have seen in Section 3, if $R$ is not a homeomorphism, then the situation may be very complicated. Similarly, the general isoclinic equiconcave case is more complicated than the monotone equiconcave one (see [1]). Thus, for simplicity let us restrict our attention to a monotone equiconcave skew product with a homeomorphism in the base. This makes sense, since we are presenting mainly counterexamples.

Example 2.4 shows that even with those strong assumptions we cannot count on getting an invariant attractor. However, in this example a preinvariant attracting graph of a continuous function of course exists; just take $\varphi$ identically equal to 1.

On the other hand, if $(B,R)$ is the circle with an irrational rotation and one of the maps $\psi_\vartheta$ is identically equal to 0, then a continuous function $\varphi$ whose graph is an attractor has to be equal to 0 on a dense subset of a circle, so it has to be identically 0. However, one can easily produce examples where such $\varphi$ is not an attractor (see [11]). This means that requiring $\varphi$ to be continuous will not work.

**Question 6.7.** Let $(X,F)$ with base $B$ and fiber space $[0,a]$ be a monotone equiconcave skew product, for which the base map is a homeomorphism. Does it follow that there exists a Borel function $\varphi : B \to [0,a]$, whose graph has the basin of attraction containing all points whose forward trajectory does not pass through $B \times \{0\}$?

Now we consider the case when for a monotone equiconcave skew product the base map $R$ is invertible and preserves an ergodic invariant probability measure $\mu$ on $B$. This setup gives us more space for maneuvers, because now the function $\varphi$ has to be defined only almost everywhere.

We construct first a pullback attractor. This was the method of the getting of the invariant graph described for instance in [11]. Namely, we set $\varphi_n(\vartheta)$ to be equal to the second component of $F^n(\vartheta,a)$. Since $F$ is monotone in the fibers, the sequence $(\varphi_n)_{n=0}^\infty$ is decreasing, and therefore convergent pointwise on the whole $B$. Denote its limit by $\varphi_K$. From the definition it follows immediately that the graph of $\varphi_K$ is invariant. Therefore the set of points at which $\varphi_K$ is 0 is also invariant. By ergodicity of $\mu$, we see that either $\varphi_K$ is 0 almost everywhere, or it is positive almost everywhere. However, note that Example 2.4 shows that $\varphi_K$ (which is 0 at all points $\vartheta_n$) may be not an attractor everywhere.

In this situation we get the following theorem which is a generalization of the known results proving the existence of attractors under concavity (see, for instance, [6, 8, 11]).

**Theorem 6.8.** *Let the system $(X,F)$ with base $B$ and fiber space $[0,a]$ be a monotone equiconcave skew product, and let the base map $R$ be invertible and preserve an ergodic invariant probability measure $\mu$ on $B$. If the function $\varphi_K$ is positive almost*

*everywhere, then its graph is an attractor with the basin of attraction containing the set $Z \times (0, a]$ for some set $Z \subset B$ of full measure $\mu$.*

*Proof.* Assume that $\varphi_K$ is positive almost everywhere. Let $Z$ be the set of those points of $B$ at which $\varphi_K$ is positive. This set is invariant for $R$, so we can consider the restriction of $F$ to the set $Z \times [0, a]$. This restriction satisfies the assumptions of Corollary 6.6, so the graph of $\varphi_K$ restricted to $Z$ is an attractor whose basin contains $Z \times (0, a]$. Since $Z$ is of full measure, this completes the proof.

**Question 6.9.** Does Theorem 6.8 hold if we drop the assumption that $\varphi_K$ is positive almost everywhere?

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