PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 3, March 2015, Pages 1169–1182
S 0002-9939(2014)12359-8
Article electronically published on October 22, 2014

ON THE BLASCHKE CIRCLE DIFFEOMORPHISMS

HAIFENG CHU

(Communicated by Yingfei Yi)

ABSTRACT. We study the analytic linearizability of a special family of analytic circle diffeomorphisms defined by

$$B_{t,a,d}(z) = e^{2\pi i t} z^{d+1} \left(\frac{z+a}{1+az}\right)^d$$

with $t, a \in \mathbb{R},\ d \in \mathbb{N}$, and $a > 2d+1$. Using the quasiconformal surgery procedure we prove that: If $B_{t,a,d}$ is analytically linearizable, then the rational map $B_{t,a,d}$ has a fixed Herman ring with Brjuno type rotation number. Conversely, for any Brjuno number α, we can find a rational map $B_{t,a,d}$ with $t, a \in \mathbb{R},\ d \in \mathbb{N}$, and $a > 2d+1$, such that $B_{t,a,d}|_{S^1}$ has rotation number $\rho(B_{t,a,d}|_{S^1}) = \alpha$ and is analytically linearizable. These present a "bigger family" for the prototype of the local linearization theorem of the analytic circle diffeomorphisms.

1. Introduction

Suppose $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, and $\mathbb{C}$ is the complex plane. We use

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

to denote the open unit disk and $S^1 = \partial\mathbb{D}$ to denote the boundary of $\mathbb{D}$. For any real number $r > 0$, we use

$$\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$$

to denote the open disk of radius r centered at 0.

In this paper we focus our attention on the analytic circle diffeomorphisms

$$B_{t,a,d}(z) = e^{2\pi i t} z^{d+1} \left(\frac{z+a}{1+az}\right)^d \text{ for } t, a \in \mathbb{R},\ d \in \mathbb{N}, \text{ and } a > 2d+1$$

by the techniques of quasiconformal surgery.

Definition 1. We say that $B_{t,a,d}|_{S^1}$ is analytically linearizable if there is an analytic circle homeomorphism $h : S^1 \longrightarrow S^1$, such that

$$B_{t,a,d} \circ h(z) = h\left(e^{2\pi i \rho(B_{t,a,d}|_{S^1})} z\right),$$

where $\rho(B_{t,a,d}|_{S^1})$ is the rotation number of $B_{t,a,d}|_{S^1}$(see section 3). And the map h is called the linearizing map of $B_{t,a,d}$.

Received by the editors January 27, 2013 and, in revised form, June 23, 2013.
2010 *Mathematics Subject Classification.* Primary 37F50; Secondary 37F10.

Since the linearizing map h is analytic, it can be extended to a neighborhood of the unit circle of the form $A(r)$, where we define

$$A(r) = \{z \in \mathbb{C} \mid 1/r < |z| < r\}$$

as the standard ring of radii $1/r$ and r. We denote by $A(R)$ the maximal ring for which h can be analytically continued. Then, it is easy to check that being $B_{t,a,d}|_{S^1}$ analytically linearizable is equivalent to the existence of a Herman ring $\mathcal{A}$ for $B_{t,a,d}$ which is given by $\mathcal{A} = h(A(R))$. In $\mathcal{A}$, every orbit under $B_{t,a,d}$ lies on an invariant closed curve which has rotation number $\rho(B_{t,a,d}|_{S^1})$.

We consider these maps as Blaschke products from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself denoted also by the same notation. It turns out that the polynomial family

$$P_{\lambda,d}(z) = \lambda z(1+z)^d$$

with $\lambda = e^{2i\pi\alpha}, \alpha \in \mathbb{R} - \mathbb{Q}$, corresponds to the Blaschke family

$$B_{t,a,d}(z) = e^{2\pi it} z^{d+1} \left(\frac{z+a}{1+az}\right)^d$$

for $t, a \in \mathbb{R},\ d \in \mathbb{N},$ and $a > 2d+1$. Fortunately the dynamics of $P_{\lambda,d}$ with a fixed Siegel disk is definitely known. So we can transfer these results to this Blaschke family. And the main result in this paper is that:

Theorem 1 (Main Theorem)**.** *If $B_{t,a,d}$ is an analytically linearizable element of the Blaschke family for some $t, a \in \mathbb{R},\ d \in \mathbb{N},$ and $a > 2d+1$, then the rotation number $\rho(B_{t,a,d}|_{S^1})$ is a Brjuno number. Conversely, for every Brjuno number α, there exist $t, a \in \mathbb{R},\ d \in \mathbb{N},$ and $a > 2d+1$, such that $B_{t,a,d}$ is analytically linearizable and $\rho(B_{t,a,d}|_{S^1}) = \alpha$.*

Corollary 1. *If $B_{t,a,d}$ has a fixed Herman ring admitting S^1 as an invariant curve, then the rotation number $\rho(B_{t,a,d}|_{S^1})$ is of Brjuno type.*

Proof. It is easy to check that being $B_{t,a,d}|_{S^1}$ analytically linearizable is equivalent to the existence of a Herman ring for $B_{t,a,d}$ containing S^1 as an invariant curve. And this implies the corollary. □

Remark 1. $\rho(B_{t,a,d}|_{S^1})$ being of Brjuno type does not necessarily imply $B_{t,a,d}$ has a fixed Herman ring containing S^1. Indeed, Herman showed that there exists at least a Brjuno number α not satisfying $\mathcal{H}$(see section 3) and Blaschke product

$$B_{t,a,1}(z) = e^{2\pi it} z^2 \frac{z+a}{1+az}$$

with rotation number $\rho(B_{t,a,1}|_{S^1}) = \alpha$ such that $B_{t,a,1}$ has no Herman ring.

Definition 2. For a holomorphic germ $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we say it is linearizable if there is a conformal (one-to-one holomorphic) map h defined in a neighborhood of 0 and fixing 0 such that

$$h \circ f \circ h^{-1}(z) = e^{2\pi i\alpha} z.$$

We will use the following two theorems to prove the Main Theorem.

Theorem 2 ([23], [4], [25], [26])**.** *An irrational number α is a Brjuno number if and only if every holomorphic germ $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ is linearizable at* 0.

Combining with the theorem above Okuyama gave the following result:

Theorem 3 ([17]). *Let $\lambda = e^{2\pi i\alpha}, \alpha \in \mathbb{R} - \mathbb{Q}$, and $d \geq 2$ be an integer. Then the polynomial $P_{\lambda,d}(z) = \lambda z(1+z)^d$ is linearizable at 0 if and only if $\alpha \in \mathcal{B}$.*

Obviously this polynomial family is another prototype for the linearizability at an irrationally indifferent fixed point, which has generalized Yoccoz's quadratic prototype. And in some sense the Blaschke family $B_{t,a,d}$ presents a prototype for the local linearizaton theorem of the analytic circle diffeomorphisms.

The main idea for the proof of the Main Theorem: first, we suppose the analytic linearizability of $B_{t,a,d}$, so it has a fixed Herman ring containing S^1 as an invariant curve. The general idea of construction is, starting from a rational map with a Herman ring, to obtain a polynomial with a Siegel disk, by means of gluing a rigid rotation to "fill in the hole" of the Herman ring. In our case, we quasiconformally past a rotation inside the unit disk $\mathbb{D}$ and show that the new resulting map is quasiconformally conjugate to $P_{\lambda,d}(z) = \lambda z(1+z)^d$. But the rotation number does not change under the topological conjugation and we obtain $\lambda = e^{2\pi i\rho(B_{t,a,d}|_{S^1})}$. This resulting map has a Siegel disk instead of the original Herman ring and thus by Okuyama's theorem we conclude $\rho(B_{t,a,d}|_{S^1}) \in \mathcal{B}$. Conversely, we discuss the polynomial $P_{\lambda,d}$, and by Brjuno's result $P_{\lambda,d}$ has a Siegel disk. We conjugate $P_{\lambda,d}$ by the reflection with respect to the unit disk and past the dynamics of $P_{\lambda,d}$ with its conjugate in order to obtain a new dynamics having a Herman ring. Finally we show that the new resulting map is quasiconformally conjugate to an element of Blaschke family. The idea of converting Herman rings to Siegel disks and vice versa originated from Shishikura's paper [22].

We need some familiarity with the theory of complex dynamics and quasiconformal mappings. The first topic is covered in the monographs [5], [14] and [15]. For quasiconformal mappings we refer the reader to [1] and [13]. For the circle diffeomorphisms the basic reference would be the monographs [10] and [16]. Some background on the connections between irrationally indifferent fixed points and circle diffeomorphisms and an exposition on the new techniques of Yoccoz and Pérez-Marco can be found in [18], [19] and [20]. Also some number theory will be involved, and the reader may refer to [12] as a reference.

2. Preliminaries

2.1. **Arithmetics.** Let α be an irrational number, denote by $[\alpha]$ the integer part of α i.e. the largest integer not greater than α, by $\{\alpha\} = \alpha - [\alpha]$ the fractional part of α, and define two sequences $\{a_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ by setting $a_0 = [\alpha], \alpha_0 = \{\alpha\}$, and for $n \geq 0$ define inductively by the following relation:

$$a_{n+1} = \left[\frac{1}{\alpha_n}\right] \ , \ \alpha_{n+1} = \left\{\frac{1}{\alpha_n}\right\},$$

so that $1/\alpha_n = a_{n+1} + \alpha_{n+1}$. A sequence $\{\beta_n\}_{n\geq -1}$ is defined associated with $\{\alpha_n\}_{n\geq 0}$ by: $\beta_{-1} = 1$ and for $n > 0$

$$\beta_n = \prod_{i=0}^{n} \alpha_i.$$

We define two sequences $\{p_n\}_{n\geq -2}$ and $\{q_n\}_{n\geq -2}$ recursively by

$$p_{-2} = 0 \ , \ p_{-1} = 1 \ , \ p_n = a_n p_{n-1} + p_{n-2},$$

$$q_{-2} = 1 \ , \ q_{-1} = 0 \ , \ q_n = a_n q_{n-1} + q_{n-2}.$$

It follows easily that

$$q_n \geq (\sqrt{2})^n \quad \text{for } n \geq 2.$$

And so the $q_n \longrightarrow \infty$ at least exponentially fast. The numbers p_n and q_n satisfy $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$, so $(p_n, q_n) = 1$. Moreover,

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \ + \cfrac{1}{a_n}}} = [a_0, a_1, \cdots, a_n].$$

The number p_n/q_n is called the kth convergent of α.

It is well known that

$$\alpha = [a_0, a_1, \cdots, a_n, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \ + \cfrac{1}{a_n + \ddots}}}.$$

Proposition 1. *Let α be an irrational number and define the sequences $\{a_n\}_{n\geq 0}$, $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq -1}$, $\{p_n\}_{n\geq -2}$ and $\{q_n\}_{n\geq -2}$ as above, so that*

$$\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n].$$

Then, for $n \geq 0$, we have the formulas

$$\alpha = \frac{p_n + p_{n-1}\alpha_n}{q_n + q_{n-1}\alpha_n}, \ \beta_n = (-1)^n (q_n \alpha - p_n),$$

$$q_{n+1}\beta_n + q_n \beta_{n+1} = 1, \ \frac{1}{q_{n+1} + q_n} < \beta_n < \frac{1}{q_{n+1}}.$$

The last inequality implies, for $n \geq 0$,

$$\frac{1}{2 q_n q_{n+1}} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Moreover, for all $n \geq 0$,

$$\alpha_n = [0, a_{n+1}, a_{n+2}, \cdots].$$

For more detailed information, the reader may refer to [12]. Now denote by $\mathcal{B}$ the class of all Brjuno numbers. By definition, $\alpha \in \mathcal{B}$ if and only if

$$(\mathcal{B}) \qquad \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty.$$

We can prove that the condition $(\mathcal{B})$ is equivalent to

$$\sum_{n=0}^{\infty} \beta_{n-1} \log \alpha_n^{-1} < +\infty.$$

2.2. **Quasiregular mapping.** Let U be an open subset of $\mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$ be a map.

Definition 3. We say that f is $K-$quasiregular ($K \geq 1$) if

- $f \in C(U) \cap W^{1,2}_{loc}(U)$, that is to say: f is continuous and belongs to the Sobolev class $W^{1,2}_{loc}(U)$, i.e. with distributional first derivatives which are locally square-integrable;
- $|\overline{\partial} f| \leq \frac{K-1}{K+1}|\partial f|$ almost everywhere on U.

If f is $K-$quasiregular for some $K \geq 1$, we say that f is quasiregular (For detailed information, the reader may refer to Rickman's book [21].) Quasiregular maps are continuous and almost everywhere differentiable. A map is holomorphic if and only if it is $1-$quasiregular. If f is $K-$quasiregular and injective, then f is called $K-$quasiconformal. Non-constant quasiregular maps are open and $\partial f \neq 0$ a.e. Thus we can define the complex dilatation by $\mu_f = \frac{\overline{\partial} f}{\partial f}$ for f. This function is measurable and $\|\mu_f\|_\infty < 1$. We call each measurable function $\mu : U \longrightarrow \mathbb{C}$ with $\|\mu_f\|_\infty < 1$ a Beltrami coefficient in U. The most important Measurable Riemann Mapping Theorem says that, for any Beltrami coefficient μ, there exists a quasiconformal mapping solving the Beltrami equation $\overline{\partial} f = \mu \partial f$. This implies that a quasiregular map f can always be written as $g \circ h$ where g is holomorphic and h is quasiconformal with $\mu_h = \mu_f$ a.e.

If $f : U \longrightarrow V$ is quasiregular and $\mu : V \longrightarrow \mathbb{C}$ is a Beltrami coefficient, we define the pullback of μ under f by

$$f^*\mu = \frac{\overline{\partial} f + (\mu \circ f)\overline{\partial f}}{\partial f + (\mu \circ f)\partial \overline{f}}.$$

This is the dilatation of $g \circ f$ where g is any quasiregular map with $\mu_g = \mu$. Obviously the pullback is also a Beltrami coefficient.

The important fact about the pullback is the following:

Proposition 2. *Let $f : U \longrightarrow V$ be quasiregular and $\mu : V \longrightarrow \mathbb{C}$ be a Beltrami coefficient. Let $\phi : U \longrightarrow \mathbb{C}$ and $\psi : V \longrightarrow \mathbb{C}$ be two quasiconformal maps with $\mu_\phi = f^*\mu$ and $\mu_\psi = \mu$. Then the mapping $g = \psi \circ f \circ \phi^{-1} : U' \longrightarrow V'$ is holomorphic where $U' = \phi(U),\ V' = \psi(V)$.*

$$\begin{tikzcd}
(U, f^*\mu) \arrow[r, "f"] \arrow[d, "\phi"'] & (V, \mu) \arrow[d, "\psi"'] \\
U' \arrow[r, "g"'] & V'
\end{tikzcd}$$

Proof. As an elementary computation gives that $\mu_{\psi \circ f} = f^*\mu$, the map $\psi \circ f$ can be written as $h \circ \widetilde{\phi}$ where $\widetilde{\phi}$ is quasiconformal with $\mu_{\widetilde{\phi}} = f^*\mu$ and h is holomorphic. Since $\mu_\phi = f^*\mu$, we get that $\widetilde{\phi} \circ \phi^{-1}$ is conformal, hence the map $g = \psi \circ f \circ \phi^{-1} = h \circ \widetilde{\phi} \circ \phi^{-1}$ is holomorphic. □

3. Circle homeomorphisms

Let $f : S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism. Via the universal covering $E : \mathbb{R} \longrightarrow S^1,\ x \longmapsto e^{2\pi i x}$, it can be lifted to an increasing homeomorphism $F : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying: $F(x + 1) = F(x) + 1$ for all $x \in \mathbb{R}$. The lift F is unique up to adding an integer constant. We wish to assign a rotation number to f, which measures the average advance of F over the interval of length 1, that is, the average speed of F. For the lift F, the rotation number is defined as

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

Lemma 1. *The above limit exists and is independent of the choice of x.*

Proof. For a detailed proof we refer the reader to [5]. $\square$

The rotation number of f is defined by: $\rho(f) = \rho(F)\ mod\, \mathbb{Z}$, which is independent of the choice of the lift. Before giving the following theorems we describe the definition of a quasisymmetric map:

Definition 4. A homeomorphism $F : \mathbb{R} \longrightarrow \mathbb{R}$ is $M-$quasisymmetric if it preserves orientation and satisfies $F(\infty) = \infty$ and the inequality

$$\frac{1}{M} \le \frac{F(x+t) - F(x)}{F(x) - F(x-t)} \le M$$

for all $x \in \mathbb{R}$ and $t > 0$.

As an exercise, one can prove that, if $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$, both f and f^{-1} are of class C^1, then f is quasisymmetric.

Poincaré showed that:

Theorem (Poincaré)**.** *The rotation number $\rho(f)$ is a rational number if and only if f has a periodic point. Furthermore, f has a periodic point with period q if and only if its rotation number is rational with denominator q.*

Later Denjoy established that:

Theorem (Denjoy)**.** *A C^2-diffeomorphism f (actually $f \in C^{1+BV}$ is enough i.e.$f \in C^1$ and Df has bounded variation) without periodic points is topologically conjugate to the rigid rotation $z \longmapsto e^{2\pi i\rho(f)}z$, i.e. there exists an orientation-preserving homeomorphism $h : S^1 \longrightarrow S^1$ satisfying $h \circ f(z) = e^{2\pi i\rho(f)}h(z)$.*

Remark 2. This theorem doesn't give us information about the regularity of the conjugacy h. Denjoy proved also that the hypothesis $f \in C^{1+BV}$ cannot be relaxed too much, constructing examples of C^1-diffeomorphisms which are not topologically conjugate to a rotation (see [27] Section 3.4).

If we normalize h by $h(1) = 1$, then h is unique. The equivalent statement for the lift F is the following: If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing C^2-diffeomorphism(or $F \in C^{1+BV}$) satisfying $F(x+1) = F(x) + 1$ and $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, then there exists an increasing diffeomorphism $H : \mathbb{R} \longrightarrow \mathbb{R}$ with $H(x+1) = H(x) + 1$ and

$$H \circ F(x) = H(x) + \rho(F)$$

for all $x \in \mathbb{R}$. In this situation, we call the map $x \longmapsto x + \rho$ a rigid rotation. Again the normalization $H(0) = 0$ makes H unique. In the rest of the article we restrict our consideration on the analytic circle diffeomorphism, to which Denjoy's theorem certainly applies. In [2] it was proved that if the rotation number verified a Diophantine condition and if the analytic diffeomorphism f is close enough to a rigid rotation, then the conjugation is analytic. At the same time examples of analytic diffeomorphisms, with irrational rotation number, for which the conjugation is not even absolutely continuous were given. In [10] Arnold's result was improved obtaining a global result: there exists $\mathscr{A} \subset [0,1]$, with full Lebesgue measure, such that if the rotation number of the C^∞-diffeomorphism belongs to $\mathscr{A}$, then it is C^∞ conjugate to a rigid rotation. A similar result holds for finitely differentiable diffeomorphisms, but in this case the conjugacy is less regular. This phenomenon of loss differentiability is typical of small divisor problems. Later Yoccoz found the sharp conditions on the rotation number for both local (restriction on being close to a rotation) and global statements [27].

Global Conjugacy Theorem (Yoccoz)**.** *If $\rho \in \mathcal{H}$, then every analytic circle diffeomorphism with rotation number ρ is analytically linearizable. If $\rho \notin \mathcal{H}$, then there exists an analytic circle diffeomorphism with rotation number ρ which is not analytically linearizable.*

Remark 3. The exact definition of $\mathcal{H}$ can be found in [18] or [27], and the set $\mathcal{H} \subsetneq \mathcal{B}$ has full Lebesgue measure and is invariant under the action of the group $SL(2,\mathbb{Z})$. Also $\mathcal{H}$ is an $F_{\sigma\delta}$-set, i.e. a countable intersection of F_σ-sets(for detailed information, refer to [27]).

Local Conjugacy Theorem. *If $\rho \in \mathcal{B}$, then there exists $R = R(\rho)$ such that any analytic circle diffeomorphism with rotation number ρ which extends univalently to the annulus $A(R) = \{z \in \mathbb{C} \mid 1/R < |z| < R\}$ is analytically linearizable. If $\rho \notin \mathcal{B}$, then the Blachke product $B_{t,a,1}(z) = e^{2\pi i t} z^2 \dfrac{z+a}{1+az}$, with $a > 3, \rho(B_{t,a,1}) = \rho$, is not analytically linearizable.*

Remark 4. For any R one can choose $a = a(R) > 3$ large enough such that the Blachke product $B_{t,a,1}$ is univalent in $A(R)$. Note that the condition of univalency in a large annulus implies proximity to the rigid rotation by the classical distortion theorem.

Although the Local Conjugacy Theorem implies the second part of the Main Theorem, we will give a constructive proof to realize the existence of the Local Conjugacy Theorem.

It is still an open problem to determine a prototype for the global version.

4. Quasiconformal surgery

Quasiconformal surgery is based on the quasiconformal mapping theory. Herman made intensive use of quasiconformal surgery. Actually, he implicitly used this method which had earlier been introduced to polynomial-like by Douady and Hubbard [8], and which has been further developed by Shishikura [22]. One of the basic ideas behind the use of quasiconformal mappings is to consider two dynamical systems acting in different parts of the plane and to construct a new system that combines the dynamics of both.

Now we will give a key quasiconformal extension lemma which is very important in constructing new dynamics.

Quasiconformal Extension Lemma. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two doubly connected domains bounded by analytic Jordan curves, and $h = (h_1, h_2) : \partial\mathcal{A}_1 \longrightarrow \partial\mathcal{A}_2$ be an orientation-preserving real-analytic diffeomorphism where h_1 maps the interior boundary curve of $\mathcal{A}_1$ to the interior boundary of $\mathcal{A}_2$ and h_2 maps the exterior* *boundary curve of* $\mathcal{A}_1$ *to the exterior boundary of* $\mathcal{A}_2$*. Then* h *can be extended to a quasiconformal mapping* $\widehat{h} : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ *with boundary value* $h = (h_1, h_2)$*.*

Proof. First unwind the two annuli $\mathcal{A}_1$ and $\mathcal{A}_2$ by using some suitable universal coverings to the horizontal strips $S_{\mathcal{A}_j} = \{z = x + iy \mid 0 < y < R_j\}$, for $j = 1, 2$ respectively. The map h_1 lifts to $x \longmapsto H_1(x)$ where H_1 is a quasisymmetric map. Also, $H_1(x) = x + P_1(x)$ where P_1 is $1-$periodic and can be chosen so that $P(0) \in [0, 1)$. Similarly, h_2 lifts to a mapping

$$x + iR_1 \longmapsto H_2(x) + iR_2$$

where $H_2(x) = x + P_2(x)$. The properties of H_2 and P_2 are the same as observed above for H_1 and P_1. We need to find a quasiconformal interpolation gluing H_1 on the lower boundary of the strip $S_{\mathcal{A}_1}$ and H_2 on the upper boundary of $S_{\mathcal{A}_1}$. Observe that we can assume without a loss of generality that $R_1 = R_2 = R$. Indeed, we can build an interpolation map in this setup and then compose it with a quasiconformal mapping $x + iy \longmapsto x + i\frac{R_2}{R_1}y$. Similarly, if we further reduce the situation to $R = 2$, the added dilatation will be $\max\{R^2/4, 4/R^2\}$, whichever turns out to be equal to or greater than 1. To extend H_1, let us apply Beurling-Ahlfors' extension: $x + iy \longmapsto u + iv$ where

$$u(x, y) = \frac{1}{2}\int_0^1 (H_1(x + t) + H_1(x - t))\, dt$$

and

$$v(x, y) = \int_0^1 (H_1(x + t) - H_1(x - t))\, dt.$$

This extension is quasiconformal with a bound depending only on the dilatation of H_1. Due to an elementary computation, we see that $u(x, 1) = x + \int_0^1 P_1(t)dt$ while $v(x, 1) = 1$. So the extension preserves the line $y = 1$ and merely translates it by $\int_0^1 P_1(t)dt$. This is a constant between -1 and 2. So by composing the extension with a quasiconformal translation we get an extension fixing the line $y = 1$ pointwise. A mirror argument shows that H_2 can be extended from the line $y = 2$ to fix the line $y = 1$ pointwise. The union H of the two extensions gives as the desired interpolation. Let $\widehat{h}$ be the projection of H. This completes the proof. □

We end the brief account with a powerful tool which we will use later. The following "Generalized QC-lemma" comes from the original QC-lemma due to Shishikura [22] when he estimated the number of non-repelling periodic cycles for a non-linear rational map with fixed degree.

Generalized QC-lemma. *Let* $U \subset \widehat{\mathbb{C}}$ *be a domain and* $f : U \longrightarrow U$ *a quasiregular map. Suppose that* $\{f^n\}_{n\geq 1}$ *is a family of uniform* $K-$*quasiregular mappings with fixed* K*. Then there exists a quasiconformal mapping* $\phi : U \longrightarrow V$ *such that* $\phi \circ f^n \circ \phi^{-1} : V \longrightarrow V$ *is a holomorphic map for every* $n \geq 1$*. In particular, when* $U = \widehat{\mathbb{C}}$*,* $\phi \circ f^n \circ \phi^{-1}$ *is a rational map for every* $n \geq 1$*.*

For the sake of completeness, we will give the main steps of the proof here, and one can find a more detailed proof by referring to [11], [24].

Proof. We will divide the proof the Main Proposition into two steps.

Step 1. We first assume that $G = \{f^n\}$ is a countable $K-$quasiconformal group generated by f. For any element $g : U \longrightarrow U$ of group G, let μ be a Beltrami coefficient on U. As before we define the pull-back of μ under g by

$$g^*\mu = \frac{\overline{\partial} g + (\mu \circ g)\overline{\partial g}}{\partial g + (\mu \circ g)\partial \overline{g}}.$$

Our goal is to find the conditions for Beltrami coefficient μ defined on S^2 guaranteeing that if a quasiconformal mapping h of S^2 has the complex dilatation μ, then h is a solution of our problem. Clearly, we must set

$$\mu|_{S^2-U} \equiv 0.$$

The conformity of $h \circ g \circ h^{-1}|_V$, where $V = h(U)$, for $g \in G$ is equivalent to

$$g^*\mu = \mu \text{ a.e.}$$

Let

$$T_{g,z}(w) = \frac{\overline{\partial} g(z) + w\overline{\partial g}(z)}{\partial g(z) + w\partial \overline{g}(z)} = \frac{a + \overline{b}w}{b + \overline{a}w}$$

with $a = \overline{\partial} g(z), b = \partial g(z)$. Then $T_{g,z} \in M\ddot{o}b(\mathbb{D})$, so

$$g^*\mu(z) = T_{g,z}(\mu(g(z))) = \mu_{f\circ g}(z) \tag{4.1}$$

with $\mu_f = \mu$ a.e. For $z \in U$, we consider the sets $M_z = \{\mu_g(z) \mid g \in G\}$. Then we have $M_z \subset B(k)$ (Euclidean disk) with $k = \frac{K-1}{K+1}$ and

$$T_{g,z}(M_{g(z)}) = M_z \tag{4.2}$$

because of $\{T_{g,z}(\mu_f(g(z))) \mid f \in G\} = \{\mu_{f\circ g}(z) \mid f \in G\} = \{\mu_f(z) \mid f \in G\}$ for almost all $z \in U$ and every $g \in G$ if G is countable. Let us assume that there is a map $X \longmapsto P(X)$ that assigns a point for every non-empty subset $X \subset \mathbb{D}$ which is bounded in the hyperbolic metric in such a way that

$$P(\beta(X)) = \beta(P(X)) \tag{4.3}$$

for every $\beta \in M\ddot{o}b(\mathbb{D})$. Then, for countable we construct the map P defined on $\mathscr{B}$ as follows:

A Technique Lemma. *For any $X \in \mathscr{B}$, there is a unique closed hyperbolic disk $D(z, r)$ with center z and radius $r \geq 0$ with the following properties:*
(i) $D(z, r) \supset X$, and
(ii) if $D(w, r') \supset X,\ w \neq z$, then $r' > r$.

We define $P(X) = z$, i.e. the center of the hyperbolic disk, and this definition is well defined. To see the existence of $D(z, r)$ we can reason as follows. Setting

$$r = \mathrm{Inf}\{s \mid \exists z \in \mathbb{D} \text{ such that } D(z, s) \supset X\}.$$

It is easy to see there is a unique point taking this infimum, and for the uniqueness we prove as follows: In any case there is a smallest $r \geq 0$ such that if $r' > r$ there is $w \in \mathbb{D}$ with $D(w, r') \supset X$. Next it is easy to see that there is at least one $z \in \mathbb{D}$ such that $D(z, r) \supset X$. Assume there is another point $w \in \mathbb{D}$ with $D(w, r) \supset X$. Let x be one of the two points of $\partial D(z, r) \cap \partial D(w, r)$ and let y be the orthogonal projection (in hyperbolic geometry) of x onto the hyperbolic line through z and w. Consider the hyperbolic triangle with vertices $z,\ y$, and x. It has a right angle at y and therefore it is geometrically evident that $d(z, x) = r > d(y, x)$. This follows also from the relation $\cosh r = \cosh d(z, y) \cosh d(y, x)$. But then, if we take $r' = d(y, x)$,

$r' < r$, and $D(y, r') \supset D(z, r) \cap D(w, r) \supset X$. This proves the uniqueness of z. Therefore, if we let $P(X)$ be the center of the smallest closed hyperbolic disk containing X, then we have a well-defined map P. Clearly, $P(\beta(X)) = \beta(P(X))$ for any $\beta \in M\ddot{o}b(\mathbb{D})$. It also has the following property: *(iii) If $X \subset D(w, s)$, $X \neq \emptyset$, then $P(X) \subset D(w, s)$.* To see the validity of (iii), note first that $r \leq s$ if r is the radius of the smallest disk containing X. Then, if $d(w, P(X)) > s$, we can reason as above and find $D(y, r') \supset X$ with $r' < r$. Therefore $d(w, P(X)) \leq s$. Now we define a map μ by setting

$$\mu(z) = \begin{cases} 0, & z \in S^2 - U, \\ P(M_z), & z \in U. \end{cases}$$

Since $M_z \subset D(0, r)$, where $r = d(0, k)$, for almost all $z \in U$, $\|\mu\|_\infty \leq k$ by (iii). And a simple argument gives that μ is measurable. This completes the first step of the proof.

Step 2. Let us assume that G is a $K-$quasiregular semigroup generated by γ. Thus $G = \{f^n \mid n \geq 1\}$ or $G = \{f^n \mid n \geq 0\}$. Then the above arguments in Step 1 are still valid until the equality $T_{g,z}(M_{g(z)}) = M_z$ and we replace the conformity of $h \circ g \circ h^{-1}|_V$ by holomorphy. In this situation, we only have

$$T_{g,z}(M_{g(z)}) \subseteq M_z.$$

Therefore the set M_z needs to be replaced by another set with the required invariance property. It is hard to see how this could be done except by replacing M_z by a larger set. This then means that the larger set might no longer be a subset of $B(k)$. We can only hope that the larger set will be contained in $B(k_1)$ for some fixed $k_1 < 1$. In this case we add to M_z all complex dilatations that would be there if G were a group generated by f. Thus we let $\widetilde{M}_z$ consist of the complex dilatations at z of all mappings of the form f^n where $n \geq 0$, or a branch of f^{-n} for some $n \geq 1$ whenever such a branch is defined in a neighborhood of z, or of the form $f^{-m} f^n$ where $m, n \geq 0$. Clearly $M_z \subseteq \widetilde{M}_z$. Pick $z \in U$ and $g \in G$. As f goes through all maps considered in definition of $\widetilde{M}_{g(z)}$, the maps $f \circ g$ go through exactly all maps considered in the definition of $\widetilde{M}_z$. As the branches their inverses are locally quasiconformal, the formulas and equalities in Step 1 still apply, and we see that $T_{g,z}(\widetilde{M}_{g(z)}) = \widetilde{M}_z$ holds with equality. We now complete the proof of the Main Proposition following the process of Step 1. □

5. Proof of the Main Theorem

Proof. Let us assume that $B_{t,a,d}$ is an analytically linearizable element of the Blaschke family for some $t, a \in \mathbb{R}$, $d \in \mathbb{N}$, and $a > 2d+1$. For the rest of the proof, t, a, and d are kept fixed and the dependence on them will be suppressed in the notation. Denote $\rho \in \mathbb{R}/\mathbb{Z}$ by the rotation number of $B|_{S^1}$ and define $\lambda = E(\rho) \in S^1$. Reflection with respect to the unit circle will be denoted by $J(z) = 1/\bar{z}$.

First, analytic linearizability implies that B has a Herman ring $\mathcal{A} \supset S^1$, i.e. a maximal doubly connected rotation domain. Equivalently, to say that $B|_{S^1}$ is analytically linearizable means there exists an analytic map $h : S^1 \longrightarrow S^1$ such that

$$B \circ h = h \circ R_\rho,$$

where $R_\rho(w) = e^{2\pi i\rho}w$. Since the linearization h is real analytic, it can extend to a complex diffeomorphism on some small neighborhood of the unit circle of the form $A(R)$ where $A(R)$ is a standard annulus defined as before. We denote by $A(R)$ the maximal ring for which h can be analytically continued. Then it is easy to check that being $B|_{S^1}$ analytically linearizable is equivalent to the existence of a Herman ring $\mathcal{A}$ for B containing S^1 as an invariant curve. Then there exists a conformal map $\varphi : \mathcal{A} \longrightarrow A(R)$. We normalize φ such that the interior boundary of $\mathcal{A}$ is mapped onto the interior boundary of $A(R)$ and satisfies $\varphi(1) = 1$. An elementary computation gives that B commutes with J. So $J(\mathcal{A})$ is also a Herman ring containing S^1 and this implies $J(\mathcal{A}) = \mathcal{A}$. Then $\widetilde{\varphi} = J\circ\varphi\circ J$ is also a conformal map from $\mathcal{A}$ onto $A(R)$ subject to the same normalization, and thus $\varphi \circ J = J \circ \varphi$. Also φ conjugates B to the rigid rotation $w \longmapsto e^{2\pi i\rho}w$ where ρ is the rotation number of B.

Now we begin to construct a new map $\psi : \mathcal{A} \cup \overline{Int\mathcal{A}} \longrightarrow D_R(0)$ which is quasiconformal with $\psi(0) = 0$ and coincides with φ in $\mathcal{A} \setminus \mathbb{D}$ (here $Int\mathcal{A}$ is the interior of the annulus $\mathcal{A}$). For this purpose, we use our Quasiconformal Extension Lemma in the following way:

Let $h_1 = Id$ on the $\partial\mathbb{D}\epsilon$ with ϵ small enough and $h_2 = \varphi$ on $\partial\mathbb{D}$, so the Quasiconformal Extension Lemma gives a quasiconformal extension denoted by ψ satisfying our conditions.

We define a new map $g : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ as follows:

$$g(z) = \begin{cases} B(z) & \text{if } |z| \geq 1, \\ \psi^{-1} \circ R_\rho \circ \psi(z) & \text{if } |z| < 1, \end{cases}$$

where $R_\rho(w) = e^{2\pi i\rho}w$. Then g is well defined and g is quasiconformal in $\mathbb{D}$, holomorphic in $\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. Thus g is quasiregular on $\widehat{\mathbb{C}}$, holomorphic outside the unit disk $\mathbb{D}$ and has a rotation domain about the fixed point 0. The rotation number of $g\,|_{S^1}$ is ρ. Let K denote the dilatation of ψ. Then g^n is K^2-quasiregular for all $n \geq 0$ and the Generalized QC-lemma gives an unique quasiconformal map $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ such that: $G = \phi \circ g \circ \phi^{-1}$ is holomorphic, with $\phi(0) = 0, \phi(-a) = -1$ and $\phi(\infty) = \infty$.

By the construction, we know G has the following properties:

- G is an entire function on $\mathbb{C}$;
- ∞ is a pole of G with order $(d + 1)$, and $G^{-1}(\infty) = \infty$;
- G has a Siegel disk $\phi(A \cup \overline{\mathbb{D}})$ around $z = 0$, with rotation number ρ, and hence $G'(0) = e^{2\pi i\rho}$;
- G has two zeros: one simple zero at 0, the other at (-1) with multiplicity d.

The only such map is the polynomial $P_{\lambda,d}(z) = \lambda z(1 + z)^d$ with $\lambda = e^{2\pi i\rho}$. So $G = P_{\lambda,d}$ and then Okuyama's theorem implies $\rho \in \mathcal{B}$. This completes the first part of the theorem.

Now let us take any $\rho \in \mathcal{B}$ and define $\lambda = e^{2\pi i\rho}$. Our goal is to find an element of the Blaschke family with rotation number ρ which is analytically linearizable. By Brjuno's result the family $P_{\lambda,d}(z) = \lambda z(1 + z)^d$ has a Siegel disk centered at 0. Let γ be an invariant analytic curve in the Siegel disk and choose $m > 0$ satisfying

$2/m < dist(\gamma, 0)$. Define $\widetilde{P}_{\lambda,d}(z) = mP_{\lambda,d}(z/m)$. Then $\widetilde{P}_{\lambda,d}$ is polynomial with the Siegel disk denoted by $S \supset \mathbb{D}_2$ about 0. Moreover, $\widetilde{\gamma} = m\gamma \subset \mathbb{C} \setminus \mathbb{D}_2$. Let $\phi : S \to \mathbb{D}_R$ be the conformal mapping with $\phi(0) = 0$, and $\phi(\widetilde{\gamma}) = \{w \in \mathbb{C} \mid |w| = 2\}$. So $\phi \circ \widetilde{P}_{\lambda,d}(z) = \lambda\phi(z)$ for $z \in S$. Define $\widetilde{\phi}(z) = \phi(z)$ for $z \in Ext\widetilde{\gamma}$, $\widetilde{\phi}(z) = z$ for $z \in S^1$. By the QC-Extension lemma we can extend $\widetilde{\phi}$ to a quasiconformal map defined in the domain bounded by $\widetilde{\gamma}$ and S^1. Let us also denote $\widetilde{\phi}$ by the extension. Define

$$g(z) = \begin{cases} \widetilde{P}_{\lambda,d}(z) & \text{if } z \in Ext\widetilde{\gamma}, \\ \widetilde{\phi}^{-1}(\lambda\widetilde{\phi}(z)) & \text{if } z \in Int\widetilde{\gamma} \text{ and } |z| \geq 1. \end{cases}$$

(Here $Ext\widetilde{\gamma}$ and $Int\widetilde{\gamma}$ is the exterior and interior of the simple curve $\widetilde{\gamma}$.)

Analytic curves are removable sets for quasiregular maps, so $g(z)$ is quasiregular in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $g(S^1) = S^1$. By reflection, it may be extended to a quasiregular self-map of $\widehat{\mathbb{C}}$ with $g(0) = 0,\ g(\infty) = \infty$. The domain $\mathcal{A}$ where g is not holomorphic is bounded by $\widetilde{\gamma}$ and $J(\widetilde{\gamma})$. There the map is quasiconformally conjugate to the rotation via $\widetilde{\phi}$, which also extends by the reflection. Thus again g satisfies the assumptions of the Generalized QC-lemma, and we can define an invariant Beltrami coefficient μ in the following way:

$$\mu(z) = \begin{cases} \mu_{\widetilde{\phi}}(z), & z \in \mathcal{A}, \\ (g^n)^*\mu_{\widetilde{\phi}}(z), & z \in g^{-n}(\mathcal{A}) \setminus g^{-n+1}(\mathcal{A}), \\ 0, & \text{otherwise.} \end{cases}$$

The map g commutes with J and this implies $J^*\mu = \mu$. Let $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal map with $\varphi(0) = 0,\ \varphi(1) = 1,\ \varphi(\infty) = \infty$ and $\mu_\varphi = \mu$ a.e. Then $J \circ \varphi \circ J$ is a map satisfying the same conditions and by the uniqueness part of the Measurable Riemann Mapping Theorem $J \circ \varphi \circ J = \varphi$; thus φ commutes with J, too. In particular, $\varphi(S^1) = S^1$.

We assume $\varphi(-m) = -a$, and define $f = \varphi \circ g \circ \varphi^{-1}$. Then f is a holomorphic map of $\widehat{\mathbb{C}}$ commuting with J and satisfying $f(0) = 0,\ f(-a) = 0$. By construction $f(S^1) = S^1$ and f is quasiconformal conjugate to a rotation on $\varphi(\mathcal{A}) \supset S^1$. This implies that $f|_{S^1}$ is analytically linearizable. The rotation number of $f|_{S^1}$ is ρ as it does not change under the topological conjugation.

By the construction, we know f has the following properties:

- f is a rational function on $\widehat{\mathbb{C}}$;
- f has two poles: one pole at ∞ with multiplicity $(d+1)$, the other at $-1/\bar{a}$ with multiplicity d;
- f has a rotation domain around S^1, with rotation number ρ;
- f has two zeros: one zero at 0 with multiplicity d, the other at $(-a)$ with multiplicity $(d-1)$.

Now we will determine the precise formula of f which will satisfy our demand.

Lemma 2. *A rational map f of degree d carries the unit circle into itself if and only if it can be written as a Blaschke product*

$$f(z) = e^{2\pi i t_0}\beta_{a_1}\beta_{a_2}, \cdots, \beta_{a_d} \quad \textit{with} \quad \beta_{a_i} = \frac{z + a_i}{1 + \bar{a}_i z}$$

for some constants $e^{2\pi i t_0} \in \partial\mathbb{D}$ and $a_1, a_2,\ \cdots, a_d \in \mathbb{C} \setminus \partial\mathbb{D}$.

Proof. Given f, we can simply choose any solution to the equation $f(a) = 0$; then divide $f(z)$ by $\beta_a(z)$ to obtain a rational map of lower degree and continues inductively. □

By the above lemma and the properties of the map f, we can write f in the following form:

$$f(z) = e^{2\pi i t_0} z^{d+1} \left(\frac{z+a}{1+\bar{a}z} \right)^d \quad t_0 \in \mathbb{R}.$$

Without loss of generality, we can assume $a \in \mathbb{R}$; otherwise we can make a suitable rotation congujation. So we have the standard form:

$$f(z) = e^{2\pi i t} z^{d+1} \left(\frac{z+a}{1+az} \right)^d \quad t, a \in \mathbb{R}.$$

An elementary computation gives that:

$$f'(z) = e^{2\pi i t} z^{d+1} \left(\frac{z+a}{1+az} \right)^{d-1} \frac{(2d+a^2+1) + a(d+1)(z+1/z)}{(1+az)^2}.$$

Due to the fact that f has no critical point on S^1, $a > 2d+1$. In conclusion, the map f, denoted by $B_{t,a,d}$, is the exact element of the Blaschke product we need, and this completes the proof of the Main Theorem. □

Acknowledgments

The author would like to thank Professor Yunping Jiang who has contributed with useful discussions and Yuefei Wang for his help and constant encouragement. This research was partially supported by grants from the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences and from the Morningside Center of Mathematics.

References

[1] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, 2nd ed., University Lecture Series, vol. 38, American Mathematical Society, Providence, RI, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR2241787 (2009d:30001)

[2] V. I. Arnol′d, *Small denominators. I. Mapping the circle onto itself* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 21–86. MR0140699 (25 #4113)

[3] A. D. Brjuno, *On convergence of transforms of differential equations to the normal form* (Russian), Dokl. Akad. Nauk SSSR **165** (1965), 987–989. MR0192098 (33 #325)

[4] A. D. Brojuno, *Analytic form of differential equations*, Trans. Moscow Math. Soc., **25** (1971) 131-288 and **26**(1972) 199-239.

[5] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR1230383 (94h:30033)

[6] Adrien Douady, *Disques de Siegel et anneaux de Herman* (French), Astérisque **152-153** (1987), 4, 151–172 (1988). Séminaire Bourbaki, Vol. 1986/87. MR936853 (89g:30049)

[7] A. Denjoy, *Sur les courbes définies par les équations differentilles à la surface du tore*, J. Math. Pour et Appl. **11**, série 9, 333-375,(1932).

[8] Adrien Douady and John Hamal Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 2, 287–343. MR816367 (87f:58083)

[9] Lukas Geyer, *Siegel discs, Herman rings and the Arnold family*, Trans. Amer. Math. Soc. **353** (2001), no. 9, 3661–3683, DOI 10.1090/S0002-9947-01-02662-9. MR1837254 (2002d:37071)

[10] M. Herman, *Sur les conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHES, **49** (1979), 5-234.

[11] A. Hinkkanen, *Uniformly quasiregular semigroups in two dimensions*, Ann. Acad. Sci. Fenn. Math. **21** (1996), no. 1, 205–222. MR1375517 (96m:30029)

[12] G. H, Hardy and E. M. Wright, *The Theory of Numbers*, Oxford Univ. Press, London, 1938.

[13] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, 2nd ed., Springer-Verlag, New York, 1973. Translated from the German by K. W. Lucas; Die Grundlehren der mathematischen Wissenschaften, Band 126. MR0344463 (49 #9202)

[14] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR1312365 (96b:58097)

[15] John Milnor, *Dynamics in one complex variable*, 3rd ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR2193309 (2006g:37070)

[16] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR1239171 (95a:58035)

[17] Yûsuke Okuyama, *Non-linearizability of n-subhyperbolic polynomials at irrationally indifferent fixed points*, J. Math. Soc. Japan **53** (2001), no. 4, 847–874, DOI 10.2969/jmsj/05340847. MR1852886 (2002j:37061)

[18] R. Pérez-Marco, *Solution complète au problème de Siegel delinéarisation d'u application holomorphe au voisinage d'un point fixe (d'apres J.-C. Yoccoz)*, Sém. Bourbaki, n° (753)(1992): Astérisque (206), 273-310.

[19] R. Pérez-Marco, *Sur les dynamiques holomorphes non linéarisables et une conjecture de V.I. Arnold*, Ann. Sci. École Norm. Sup., **26** (1993), 193-217.

[20] R. Pérez-Marco, *Total convergence or general divergence in small divisors*, Comm. Math. Phys. **223** (2001), no. 3, 451–464, DOI 10.1007/s002200100457. MR1866162 (2003d:37063)

[21] Seppo Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 26, Springer-Verlag, Berlin, 1993. MR1238941 (95g:30026)

[22] Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 1–29. MR892140 (88i:58099)

[23] Carl Ludwig Siegel, *Iteration of analytic functions*, Ann. of Math. (2) **43** (1942), 607–612. MR0007044 (4,76c)

[24] Pekka Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), no. 1, 73–78. MR595178 (82c:30031)

[25] Jean-Christophe Yoccoz, *Linéarisation des germes de difféomorphismes holomorphes de* $(\mathbf{C},0)$ (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 1, 55–58. MR929279 (89i:58123)

[26] J. C. Yoccoz, *Petits diviseurs en dimension* 1, Astéisque No. **231** (1995).

[27] Jean-Christophe Yoccoz, *Analytic linearization of circle diffeomorphisms*, Dynamical systems and small divisors (Cetraro, 1998), Lecture Notes in Math., vol. 1784, Springer, Berlin, 2002, pp. 125–173, DOI 10.1007/978-3-540-47928-4_3. MR1924912 (2004c:37073)

School of Mathematics, Northwest University, Xi'an Shaanxi 710100, People's Republic of China

E-mail address: chuhaifeng@amss.ac.cn