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THE LOCALIZED SINGLE-VALUED EXTENSION PROPERTY AND RIESZ OPERATORS

PIETRO AIENA AND VLADIMIR MULLER

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ABSTRACT. The localized single-valued extension property is stable under commuting Riesz perturbations.

The single-valued extension property (SVEP) dates back to the early days of local spectral theory and appeared first in the work of Dunford ([5], [6]). The localized version of SVEP, considered in this article, was introduced by Finch [7], and has now developed into one of the major tools in the connection of local spectral theory and Fredholm theory for operators on Banach spaces; see the recent books [9] and [1].

To fix notation, throughout this article, let X be a non-zero complex infinite dimensional Banach space, and denote by $L(X)$ the Banach algebra of all bounded linear operators on X. As usual, given $T \in L(X)$, let $\ker T$ and $T(X)$ stand for the kernel and range of T, the spectrum of T is denoted by $\sigma(T)$, while the spectral radius of T is denoted by $r(T)$.

Definition 0.1. An operator $T \in L(X)$ is said to have the *single-valued extension property at a point* $\lambda \in \mathbb{C}$ (for brevity, SVEP at λ) provided that, for every open disc $D \subseteq \mathbb{C}$ centered at λ, the only analytic function $f : D \to X$ that satisfies

$$(\mu I - T)f(\mu) = 0 \qquad \text{for all } \mu \in D$$

is the function $f \equiv 0$ on D. Moreover, T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

The *quasi-nilpotent part* of $T \in L(X)$ is defined as the set

$$H_0(T) := \{x \in X : \lim_{n\to 0} \|T^n x\|^{\frac{1}{n}} = 0\},$$

while the *analytic core* of T is defined as $K(T) := \{x \in X :$ there exist $c > 0$ and a sequence $(x_n)_{n\geq 1} \subseteq X$ such that $Tx_1 = x, Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$, and $||x_n|| \leq c^n ||x||$ for all $n \in \mathbb{N}\}$; see [1] for details on these subspaces.

Lemma 0.2 ([3], or [1, Theorem 2.22])**.** *Suppose that* $T \in L(X)$. *Then* T *has SVEP at* λ *if and only if* $\ker(\lambda I - T) \cap K(\lambda I - T) = \{0\}$.

An operator $T \in L(X)$ is said to be a *Fredholm operator* (*upper semi-Fredholm*, *lower semi-Fredholm*, respectively), if $\dim \ker(T) < \infty$ and $\operatorname{codim} T(X) < \infty$ (if

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dim $\ker(T) < \infty$ and $T(X)$ is closed, if $\operatorname{codim} T(X) < \infty$, respectively). An operator $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T$ is a Fredholm operator for every $\lambda \in \mathbb{C} \setminus \{0\}$. The spectrum $\sigma(T)$ of a Riesz operator is at most countable and has no non-zero cluster point. Furthermore, each non-zero element of the spectrum is an eigenvalue. Examples of Riesz operators are quasi-nilpotent operators and compact operators; see [8]. Moreover, the spectral subspaces associated with non-zero elements of the spectrum are finite dimensional. It is well known that the classes of semi-Fredholm operators are stable under Riesz commuting perturbations.

In general, the sum of two commuting operators with SVEP need not have SVEP; see [4]. However, the SVEP is stable under commuting quasi-nilpotent perturbations (see [1, Corollary 2.12]), and in the very recent article ([4]) it was questioned if that is also true for the localized SVEP. In this paper we give a positive answer to this question; actually we show much more:

Theorem 0.3. *Let X be a Banach space, $T, R \in L(X)$, where R is a Riesz operator such that $TR = RT$. If $\lambda \in \mathbf{C}$, then T has SVEP at λ if and only if $T - R$ has SVEP at λ. In particular, the SVEP is stable under Riesz commuting perturbations.*

Proof. Without loss of generality we may assume that $\lambda = 0$. Suppose T does not have SVEP at 0. We show that $T - R$ has does not have SVEP at 0. Since T does not have SVEP at 0, then $\ker T \cap K(T) \neq \{0\}$, by Lemma 0.2, so there exists a sequence of vectors $(x_i)_{i=0.1,\dots}$ of X such that $x_0 \neq 0$, $Tx_0 = 0$, $Tx_i = x_{i-1} \quad (i \geq 1)$ and $\sup_{i\geq 1} \|x_i\|^{1/i} < \infty$.

Let $K := \sup_{i\geq 1} \|x_i\|^{1/i}$. Fix an ε, $0 < \varepsilon < \frac{1}{2K}$. Let X_1 and X_2 be the spectral subspaces of R corresponding to the parts of the spectrum $\{z \in \sigma(R) : |z| < \varepsilon\}$ and $\{z \in \sigma(R) : |z| \geq \varepsilon\}$, respectively. So $X = X_1 \oplus X_2$, $\dim X_2 < \infty$, $RX_j \subset X_j \quad (j = 1, 2)$, $\sigma(R|X_1) \subset \{z : |z| < \varepsilon\}$ and $\sigma(R|X_2) \subset \{z : |z| \geq \varepsilon\}$. Let P be the corresponding spectral projection onto X_2 with kernel equal to X_1.

Since $TR = RT$, we have $TX_j \subset X_j \quad (j = 1, 2)$. We have $TPx_0 = PTx_0 = 0$, and

$$TPx_i = PTx_i = Px_{i-1} \quad (i \geq 1).$$

We claim that $Px_i = 0$ for all i. To see this, suppose that $Px_i \neq 0$ for some $i \geq 0$. From $TPx_{i+1} = Px_i \neq 0$ we then deduce that $Px_{i+1} \neq 0$, and by induction it then follows that $Px_n \neq 0$ for all $n \geq i$.

Let $k \geq 1$ be the smallest integer for which $Px_k \neq 0$. Then

$$TPx_k = Px_{k-1} = 0.$$

For all $n \geq k$ we have

$$\begin{aligned} T^{n-k}Px_n &= T^{n-k-1}(TPx_n) = T^{n-k-1}Px_{n-1} = \dots. \\ &= TPx_{k+1} = Px_k \neq 0, \end{aligned}$$

so $Px_n \notin \ker (T|X_2)^{n-k+1}$, for all $n \geq k$. Furthermore,

$$T^{n-k+1}Px_n = TT^{n-k}Px_n = TPx_k = Px_{k-1} = 0,$$

so $Px_n \in \ker (T|X_2)^{n-k+1}$. This implies that $T|X_2$ has infinite ascent, which is impossible, since $\dim X_2 < \infty$. Therefore, $Px_i = 0$, and hence $x_i \in \ker P = X_1$, for all $i \geq 0$.

Let us consider the restriction $R_1 = R|X_1$. We have $r(R_1) < \varepsilon$, so there exists j_0 such that $\|R_1^j\| \leq \varepsilon^j$ for all $j \geq j_0$.

Set $y_0 := \sum_{i=0}^{\infty} R^i x_i$. Similarly, for $k \geq 1$ let

$$y_k := \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k} x_i.$$

This definition is correct, since

$$\sum_{i=k}^{\infty} \binom{i}{k} \|R^{i-k} x_i\| \leq \sum_{i=k}^{\infty} 2^i \|R_1^{i-k}\| K^i$$
$$\leq \sum_{i=k}^{j_0+k} 2^i K^i \|R_1^{i-k}\| + \sum_{i=j_0+k+1}^{\infty} 2^i K^i \varepsilon^{i-k} < \infty.$$

Moreover, for $k \geq 2j_0$ we have

$$\|y_k\| \leq \sum_{i=k}^{2k-1} 2^i K^i \|R_1^{i-k}\| + \sum_{i=2k}^{\infty} (2K)^i \varepsilon^{i-k}$$
$$\leq k \max\{(2K)^k, (2K)^{2k-1} \|R_1\|^{k-1}\} + \frac{(2K)^{2k} \varepsilon^k}{1 - 2K\varepsilon}.$$

Thus,

$$\|y_k\|^{1/k} \leq k^{1/k} \big(\max\{(2K)^k, (2K)^{2k-1} \|R_1\|^{k-1}\}\big)^{1/k} + \Big(\frac{(2K)^{2k} \varepsilon^k}{1 - 2K\varepsilon}\Big)^{1/k}$$
$$\leq k^{1/k} \max\{2K, (2K)^{\frac{2k-1}{k}} \|R_1\|^{\frac{k-1}{k}}\} + \frac{4K^2 \varepsilon}{1 - 2K\varepsilon},$$

from which we obtain $\limsup_{k\to\infty} \|y_k\|^{1/k} < \infty$.

We also have

$$(T - R) y_0 = \sum_{i=1}^{\infty} R^i x_{i-1} - \sum_{i=0}^{\infty} R^{i+1} x_i = 0.$$

Now, for $k \geq 1$ we have

$$(T - R) y_k = \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k} x_{i-1} - \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k+1} x_i$$
$$= x_{k-1} + \sum_{i=k}^{\infty} R^{i-k+1} x_i \left(\binom{i+1}{k} - \binom{i}{k} \right) = y_{k-1}.$$

It remains to show that not all of the y_k's are equal to zero. Suppose on the contrary that $y_k = 0 \quad (k \geq 0)$ and let $j_1 \geq j_0$. Then we have

$$\sum_{k=0}^{j_1} (-1)^k R^k y_k = \sum_{i=0}^{\infty} \alpha_i R^i x_i,$$

where, if we let $\nu := \min\{i, j_1\}$, we have

$$\alpha_i = \sum_{k=0}^{\nu} (-1)^k \binom{i}{k} \quad \text{for every } i = 0, 1, \ldots.$$

Clearly, $\alpha_0 = 1$. For $1 \leq i \leq j_1$ we obtain

$$\alpha_i = \sum_{k=0}^{i} (-1)^k \binom{i}{k} = 0.$$

For $i > j_1$ we have $|\alpha_i| \leq 2^i$, so

$$0 = \sum_{k=0}^{j_1} (-1)^k R^k y_k = x_0 + \sum_{i=j_1+1}^{\infty} \alpha_i R^i x_i$$

and

$$\|x_0\| \leq \sum_{i=j_1+1}^{\infty} 2^i \|R_1^i\| \|x_i\| \leq \sum_{i=j_1+1}^{\infty} 2^i \varepsilon^i K^i = \frac{(2K\varepsilon)^{j_1+1}}{1-2K\varepsilon}.$$

Letting $j_1 \to \infty$ yields $\|x_0\| = 0$, a contradiction. Therefore, $\ker(T-R) \cap K(T-R) \neq \{0\}$, and this implies, again by Lemma 0.2, that $T-R$ does not have SVEP at 0.

By symmetry we then conclude that T has SVEP at 0 if and only if $T-R$ has SVEP at 0. □

Theorem 0.3 considerably improves the results of Theorem 2.8 and Theorem 2.9 of [4], where the stability of SVEP at λ, under commuting Riesz perturbations, was proved under some additional assumptions on $\lambda I - T$. It also improves Theorem 2.4 and Corollary 2.5 of [4], and answers positively a question raised after this corollary, concerning quasi-nilpotent operators. Note that in Corollary 2.5 of [4] it was assumed that

$$H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\},$$

and this assumption is stronger than assuming the SVEP of T at λ; see [2].

Denote by $\sigma_e(T)$ the essential Fredholm spectrum of T, i.e., the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not Fredholm. Let $r_e(T)$ denote the essential spectral radius of T, i.e., $r_e(T) := \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. A closer look at the proof of Theorem 0.3 shows that it was not necessary to assume that $r_e(R) = 0$, i.e., R is a Riesz operator. It is sufficient to assume for the proof that $r_e(R)$ is small enough, in order to have the spectral decomposition $X = X_1 \oplus X_2$, with X_2 finite dimensional.

Remark 0.4. Every Riesz operator is *meromorphic*, i.e., every non-zero $\lambda \in \sigma(T)$ is a pole of the resolvent of T. Meromorphic operators have the same structure of the spectrum as Riesz operators, i.e., every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue, and the spectrum is at most countable and has no non-zero cluster point. A simple example shows that the result of Theorem 0.3 cannot be extended to meromorphic operators. Denote by L the backward shift on $\ell_2(\mathbb{N})$ and let $\lambda_0 \notin \sigma(L) = \mathbf{D}$, $\mathbf{D}$ be the closed unit disc. It is known that L does not have SVEP at 0. Since L has SVEP at λ_0, then $T := \lambda_0 I - L$ has SVEP at 0, while $T - \lambda_0 I = -L$, does not have SVEP at 0, and, obviously, $\lambda_0 I$ is meromorphic.

The result of Theorem 0.3 also permits an alternative proof of a well-known result of Rakočević ([10]) concerning the stability of semi-Browder spectra under commuting Riesz perturbations. Let $p(T)$ denote the *ascent* of an operator $T \in L(X)$, i.e., $p(T)$ is the smallest non-negative integer p for which $\ker T^p = \ker T^{p+1}$, if such an integer exists, and otherwise $p(T) = \infty$. Analogously, let $q(T)$ be the *descent* of an operator T; i.e., $q(T)$ is the smallest non-negative integer q for which $R^q(T) = R^{q+1}(T)$ if such an integer exists, and otherwise $q(T) = \infty$. Note that if $\lambda I - T$ is (upper or lower) semi-Fredholm, then

$$T \text{ has SVEP at } \lambda \Leftrightarrow p(\lambda I - T) < \infty,$$

and dually

$$T^* \text{ has SVEP at } \lambda \Leftrightarrow q(\lambda I - T) < \infty;$$

see [1, Theorem 3.16 and Theorem 3.17]. Recall that $T \in L(X)$ is said to be an *upper (lower) semi-Browder operator* if T is upper (lower) semi-Fredholm with finite ascent $p(T)$ (finite descent $q(T)$). $T \in L(X)$ is said to be a *Browder operator* if T is both upper and lower semi-Browder. Denote by $\sigma_{ub}(T)$, $\sigma_{lb}(T)$, and $\sigma_b(T)$ the corresponding spectra.

Corollary 0.5. *The spectra $\sigma_{ub}(T)$, $\sigma_{lb}(T)$, and $\sigma_b(T)$ are stable under Riesz commuting perturbations.*

Proof. Let $\lambda \notin \sigma_{ub}(T)$. Then $\lambda I - T$ is upper semi-Browder, so $p(\lambda I - T) < \infty$ and this is equivalent to saying that T has SVEP at λ. By Theorem 0.3, $T + R$ has SVEP at λ for every commuting Riesz operator R, and since $\lambda I - (T + R)$ is upper semi-Fredholm it then follows that $p(\lambda I - (T + R) < \infty$, so $\lambda I - (T + R)$ is upper semi-Browder. The converse follows by symmetry, so $\sigma_{ub}(T) = \sigma_{ub}(T + R)$. The stability of $\sigma_{lb}(T)$, and $\sigma_b(T)$ is proved by duality, using the well-known fact that T is Riesz if and only if its dual T^* is Riesz. ☐

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Facoltà di Ingegneria, DIEETCAM, Viale delle Scienze, I-90128 Palermo, Italy
E-mail address: pietro.aiena@unipa.it

Institute of Mathematics, Czech Academy of Sciences, Zitna 25, 11567, Praha 1, Czech Republic
E-mail address: muller@math.cas.cz