

DG CATEGORIES AND EXCEPTIONAL COLLECTIONS

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ABSTRACT. A description of how to assign to a full exceptional collection on a variety X a DG category \mathcal{C} such that the bounded derived category of coherent sheaves on X is equivalent to the bounded derived category of \mathcal{C} is given in a 1990 work by Bondal and Kapranov, *Framed triangulated categories*. In this paper we show that the category \mathcal{C} can be chosen to have finite-dimensional spaces of morphisms. We describe how it behaves under mutations and present an algorithm allowing us to calculate it for full exceptional collections with vanishing Ext^k groups for $k > 1$.

1. INTRODUCTION

1.1. Motivation. Derived categories of coherent sheaves have been an object of vivid interest since their discovery in the 1960's. Full exceptional collections are one of the most important tools used to study them. When such a collection is strong the derived category is equivalent to a derived category of modules over some finite-dimensional noncommutative algebra; see [Bon89]. On the other hand, the theorem of Orlov [Orl92] describes how derived categories change while passing to projective bundles over X and under blow-ups. In particular, if derived categories of coherent sheaves on both X and a smooth $Z \subset X$ have full exceptional collections, then the derived category of $Y = \text{Bl}_Z X$ also does. Moreover, the exceptional collection on Y can be explicitly given in terms of exceptional collections on X and Z . However, if the collections on X and Z are strong the resulting collection on Y does not have to satisfy this condition. Therefore, it is natural to try to understand what description of $D^b(X)$ a full exceptional collection, not necessarily strong, gives.

The answer is given in [BK90]. Namely, there exists a DG category \mathcal{C}_σ such that $D^b(\text{Coh}(X))$ is equivalent to a bounded derived category of modules over \mathcal{C}_σ . It follows from the construction that \mathcal{C}_σ has finitely many objects.

However, as \mathcal{C}_σ is constructed as a subcategory of the category of complexes of injective sheaves on X , we a priori do not know anything about the dimension of the spaces of morphisms between any two objects in it. The possibility that the space of morphisms can be infinitely dimensional prevents us, for example, from considering canonical objects such as $E \otimes \text{Hom}(E, F)$, which are crucial for defining the braid group action on the set of exceptional collections; see [Bon89], [BK90]. The main goal of this paper is to show that for smooth projective varieties, the DG category \mathcal{C}_σ can be chosen to have finite-dimensional spaces of morphisms.

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1.2. Main result. Using the theory of A_∞ categories, we prove the following fact.

Theorem 1.1. *Let X be a smooth projective variety and let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on X . Then there exists an ordered, finite DG category $\widetilde{\mathcal{C}}_\sigma$ such that $D^b(X)$ is equivalent to $D^b(\widetilde{\mathcal{C}}_\sigma)$.*

Here, we say that a category $\widetilde{\mathcal{C}}$ is *ordered* if one can partially order objects of $\widetilde{\mathcal{C}}$ in such a way that there are no nontrivial morphisms from C_1 to C_2 if $C_2 \prec C_1$. Furthermore, a category $\widetilde{\mathcal{C}}$ is *finite* if all spaces of morphisms in $\widetilde{\mathcal{C}}$ are finite dimensional.

Thus, $\widetilde{\mathcal{C}}_\sigma$ can be presented as a path algebra of a DG quiver, a quiver with grading and differential on arrows.

Knowing that the DG category $\widetilde{\mathcal{C}}_\sigma$ can be chosen to be finite we then describe how it can be calculated in some particular cases. One method uses the, already mentioned, fact that the set of full exceptional collections on X admits an action of the braid group. As it has been stated in the original paper [BK90], the action of the braid group can be lifted to associated DG categories. It allows us to describe the category $\widetilde{\mathcal{C}}_\sigma$ for any collection σ which can be mutated to a strong one.

Another way to calculate the DG category $\widetilde{\mathcal{C}}_\sigma$ is to use universal extensions introduced by Hille and Perling in [HP11]. This method leads to the following result.

Theorem 1.2. *Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on a smooth projective variety X such that $\text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k \neq 0, 1$ and any i, j . Then the DG category $\widetilde{\mathcal{C}}_\sigma$ can be calculated by means of universal extensions.*

From the above theorem and the discussion preceding we conclude the following fact.

Corollary 1.3. *Let X be a smooth projective variety and let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on X which either can be mutated to a strong exceptional collection or satisfies the condition $\text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k > 1$ and all i, j . Then the bounded derived category of coherent sheaves on X is equivalent to the homotopy category of modules over a finite and ordered DG category \mathcal{C}_σ . The DG category \mathcal{C}_σ can be presented as a DG category associated to a finite DG quiver which can be explicitly calculated.*

In particular, the assumptions of the above corollary are satisfied by full exceptional collections of line bundles on smooth rational surfaces constructed by Hille and Perling in [HP11]. As an example we present a DG quiver of a full exceptional collection on \mathbb{P}^2 blown up along a subscheme of degree 2 supported at one point.

Full strong exceptional collections have also been used in [AKO06], [HI13] and [Per10] to describe noncommutative deformations of varieties. Knowledge of the DG categories of exceptional collections, which are not strong, can lead to a generalisation of this idea. As an example we present a noncommutative deformation from \mathbb{P}^2 blown up in two different points to \mathbb{P}^2 blown up along a subscheme of degree 2 supported at one point.

2. ENHANCED TRIANGULATED CATEGORIES

Let us recall definitions and facts about enhanced triangulated categories and exceptional collections after [Bon89], [BK90], [BLL04] and [Kel94].

2.1. Exceptional collections and mutations. Let \mathcal{D} be a \mathbb{C} -linear triangulated category. An object $E \in \mathcal{D}$ is called *exceptional* if $\mathcal{D}(E, E) = \mathbb{C}$ and $\mathcal{D}(E, E[i]) = 0$ for $i \neq 0$. An ordered collection $\sigma = \langle E_1, \dots, E_n \rangle$ of exceptional objects is called an *exceptional collection* if $\mathcal{D}(E_i, E_j[k]) = 0$ for $i > j$ and all k . An exceptional collection σ is *strong* if we also have $\mathcal{D}(E_i, E_j[k]) = 0$ for $k \neq 0$ and all i, j . It is *full* if the smallest strictly full subcategory of \mathcal{D} containing E_1, \dots, E_n is equal to \mathcal{D} .

Let $\langle E, F \rangle$ be an exceptional pair in \mathcal{D} . Then $\langle L_E F, E \rangle$ and $\langle F, R_F E \rangle$ are also exceptional pairs for $L_E F$ and $R_F E$ defined in [Bon89] by means of distinguished triangles in \mathcal{D} :

$$\begin{aligned} L_E F &\rightarrow \mathcal{H}om_{\mathcal{D}}(E, F) \otimes E \rightarrow F \rightarrow L_E F[1], \\ E &\rightarrow \mathcal{H}om_{\mathcal{D}}(E, F)^* \otimes F \rightarrow R_F E \rightarrow E[1]. \end{aligned}$$

Here, $\mathcal{H}om_{\mathcal{D}}(E, F)$ denotes a complex of \mathbb{C} -vector spaces with trivial differential; $\mathcal{H}om_{\mathcal{D}}(E, F) = \bigoplus \mathcal{D}(E, F[k])$. For an element $E \in \mathcal{D}$ and a complex V^\bullet the tensor product is defined by $E \otimes V^\bullet = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{i=0}^{\dim V^k} E[-k]$.

For an exceptional collection $\sigma = \langle E_1, \dots, E_n \rangle$ the i -th left mutation $L_i \sigma$ and the i -th right mutation $R_i \sigma$ are defined by

$$\begin{aligned} L_i \sigma &= \langle E_1, \dots, E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_n \rangle, \\ R_i \sigma &= \langle E_1, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \dots, E_n \rangle. \end{aligned}$$

Mutations define an action of the braid group on the set of full exceptional collections.

2.2. Enhanced triangulated categories.

Definition 2.1. A *DG category* is a preadditive category \mathcal{C} in which abelian groups $\mathcal{C}(A, B)$ are endowed with a \mathbb{Z} -grading and a differential ∂ of degree one. The composition of morphisms

$$\mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

is a morphism of complexes and for any object $C \in \mathcal{C}$ the identity morphism id_C is a closed morphism of degree zero.

For an element x in a graded vector space we will denote by $|x|$ the grading of x . By $\mathcal{C}^i(A, B)$ we will denote morphisms of degree i .

A DG category \mathcal{C} is *ordered* if there exists a partial order \preceq on the set of objects such that $\mathcal{C}(A, B) = 0$ for $B \prec A$. It is *finite* if the set of objects of \mathcal{C} is finite and for any $C_1, C_2 \in \mathcal{C}$ the vector space $\mathcal{C}(C_1, C_2)$ is finite dimensional.

To a DG category \mathcal{C} we associate three categories. The graded category \mathcal{C}^{gr} is obtained from \mathcal{C} by forgetting the differentials on morphisms while the *homotopy category* $H(\mathcal{C})$ has the same objects as \mathcal{C} and morphisms given by the cohomology groups of morphisms in \mathcal{C} . A further restriction to the zeroth cohomology gives a preadditive category $H^0(\mathcal{C})$.

A morphism $s : C \rightarrow C'$ in \mathcal{C} is a *homotopy equivalence* if $H(s)$ is an isomorphism. Then we say that C and C' are *homotopy equivalent*.

A *DG functor* between two DG categories \mathcal{C} and \mathcal{C}' is an additive functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ which preserves the grading and differential on morphisms. DG functors between DG categories form a DG category.

Let $F, G : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be DG functors. To construct a DG category $\text{DG-Fun}(\mathcal{C}_1, \mathcal{C}_2)$ we put $\text{DG-Fun}(\mathcal{C}_1, \mathcal{C}_2)^k(F, G)$ to be the set of natural transformations $t : F^{\text{gr}} \rightarrow G^{\text{gr}}[k]$ (i.e. for $C \in \mathcal{C}_1$ we have $t_C \in \mathcal{C}_2^k(F(C), G(C))$). The differential ∂ is defined pointwise; for $t_C : F(C) \rightarrow G[k](C)$ we have $(\partial(t))_C = \partial(t_C) : F(C) \rightarrow G[k+1](C)$.

The category of contravariant DG functors is denoted by $\text{DG-Fun}^\circ(\mathcal{C}, \mathcal{C}')$. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a *quasi-equivalence* if $H(F) : H(\mathcal{C}) \rightarrow H(\mathcal{C}')$ is an equivalence.

The category $\text{DGVect}_{\mathbb{C}}$ of complexes of vector spaces with homogeneous morphisms $f : V^\bullet \rightarrow W^\bullet$, $f(V^i) \subset W^{i+k}$ and a differential

$$\partial(f)^i = \partial_W f^{i+1} - (-1)^{|f|} f^{i+1} \partial_V$$

is a DG category.

A *right DG module* M over a DG category \mathcal{C} is an element of $\text{DG-Fun}^\circ(\mathcal{C}, \text{DGVect}_{\mathbb{C}})$. After [Kel94] we define a derived category $D(\mathcal{C})$ as a localization of $H^0(\text{DG-Fun}^\circ(\mathcal{C}, \text{DGVect}_{\mathbb{C}}))$ with respect to the class of homotopy equivalences. By $D^b(\mathcal{C})$ we will denote the subcategory of $D(\mathcal{C})$ formed by compact objects. The Yoneda embedding gives a functor $h : \mathcal{C} \rightarrow D^b(\mathcal{C})$ which assigns to every $C \in \mathcal{C}$ a free module $h_C = \mathcal{C}(-, C)$.

For a DG category \mathcal{C} we define the category $\widehat{\mathcal{C}}$ of formal shifts. The objects of $\widehat{\mathcal{C}}$ are $C[n]$ where $C \in \mathcal{C}$ and $n \in \mathbb{N}$. For elements $C_1[k]$ and $C_2[n]$ of $\widehat{\mathcal{C}}$ we put $\text{Hom}_{\widehat{\mathcal{C}}}(C_1[k], C_2[n]) = \text{Hom}_{\mathcal{C}}^{l+n-k}(C_1, C_2)$. For appropriate sign convention see [BLL04].

Let B, C be objects of a DG category \mathcal{C} and let $f \in \mathcal{C}(B, C)$ be a closed morphism. Assume that $B[1]$ is also an object of \mathcal{C} ; i.e. there exists an object B' and closed morphisms $t : B \rightarrow B'$, $t' : B' \rightarrow B$ of degree 1 and -1 respectively such that $t't = \text{id}_B$ and $tt' = \text{id}_{B'}$. An object $D \in \mathcal{C}$ is called a *cone* of f if there exist morphisms

$$B' \xrightarrow{i} D \xrightarrow{p} B', \quad C \xrightarrow{j} D \xrightarrow{s} C,$$

such that

$$\begin{aligned} pi = 1, \quad sj = 1, \quad si = 0, \quad pj = 0, \quad ip + js = 1, \\ \partial(j) = 0, \quad \partial(p) = 0, \quad \partial(i) = jf, \quad \partial(s) = -fp. \end{aligned}$$

It is proved in [BLL04] that the cone of closed degree zero morphism is uniquely defined up to a DG isomorphism.

One can formally add cones of closed morphisms to a DG category \mathcal{C} by considering the category $\mathcal{C}^{\text{pre-tr}}$ of one-sided twisted complexes over \mathcal{C} .

Definition 2.2. A *one-sided twisted complex* over a DG category \mathcal{C} is an expression $(\bigoplus_{i=1}^n C_i[r_i], q_{i,j})$ where the C_i 's are objects of \mathcal{C} , $r_i \in \mathbb{Z}$, $n \geq 0$ and $q_{i,j} \in \mathcal{C}^1(C_i[r_i], C_j[r_j])$ such that $q_{i,j} = 0$ for $i \geq j$ and $\partial q + q^2 = 0$.

One-sided twisted complexes over \mathcal{C} form a DG category $\mathcal{C}^{\text{pre-tr}}$ with the morphism space between $C = (\bigoplus C_i[r_i], q)$ and $C' = (\bigoplus C'_j[r'_j], q')$ given by the set of matrices $f = (f_{i,j})$, $f_{i,j} : C_i[r_i] \rightarrow C'_j[r'_j]$. With a differential defined as

$$\partial(f) = (\partial f_{i,j}) + q'f - (-1)^{\text{deg} f} f q$$

the category $\mathcal{C}^{\text{pre-tr}}$ is a DG category. Its zeroth homotopy category is \mathcal{C}^{tr} .

Let $C = (\bigoplus C_i[r_i], q_{i,j})$ be an object of $\mathcal{C}^{\text{pre-tr}}$. The shift of C is defined as $C[1] = (\bigoplus C_i[r_i + 1], -q_{i,j})$. Clearly, the category $\mathcal{C}^{\text{pre-tr}}$ is closed under shifts.

Let $f : C \rightarrow D$ be a closed morphism of degree zero in $\mathcal{C}^{\text{pre-tr}}$, where $C = (\bigoplus_{i=1}^n C_i[r_i], q_{i,j})$, $D = (\bigoplus_{i=1}^m D_i[s_i], p_{i,j})$. The cone of f is a twisted complex $\text{Cone}(f) = (\bigoplus_{i=1}^{n+m} E_i[t_i], u_{i,j})$, where

$$E_i = \begin{cases} C_i & \text{for } i \leq n, \\ D_{i-n} & \text{for } i > n, \end{cases}$$

$$t_i = \begin{cases} r_i + 1 & \text{for } i \leq n, \\ s_i & \text{for } i > n, \end{cases} \quad u_{i,j} = \begin{cases} q_{i,j} & \text{for } i, j \leq n, \\ f_{i,j-n} & \text{for } i \leq n < j, \\ p_{i-n,j-n} & \text{for } i, j > n. \end{cases}$$

The convolution functor $\text{Tot} : (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}} \rightarrow \mathcal{C}^{\text{pre-tr}}$ establishes a quasi-equivalence between $(\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$ and $\mathcal{C}^{\text{pre-tr}}$. Let $C_i = (\bigoplus_{j=1}^{n_i} D_j^i[r_j^i], q_{j,k}^i)$ be objects of $\mathcal{C}^{\text{pre-tr}}$ and let $C = (\bigoplus_{i=1}^n C_i[r_i], q_{ij})$ be a twisted complex in $(\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$.

Then the convolution $\text{Tot}(C)$ is equal to $(\bigoplus_{i=1}^n \bigoplus_{j=1}^{n_j^i} D_j^i[r_j^i + r_i], q_{j,k}^i + q_{ij})$.

The DG category \mathcal{C} is *pretriangulated* if the embedding $H^0(\mathcal{C}) \rightarrow \mathcal{C}^{\text{tr}}$ is an equivalence. The category $H^0(\mathcal{C})$ for a pretriangulated category is triangulated.

A triangulated category \mathcal{D} is *enhanced* if it has an *enhancement* — a pretriangulated category \mathcal{C} such that \mathcal{D} is equivalent to $H^0(\mathcal{C})$.

The category \mathcal{C}^{tr} need not to be Karoubian. As shown in [BLL04] the category $D^b(\mathcal{C})$ is the Karoubization of \mathcal{C}^{tr} .

A standard example of an enhanced triangulated category is the derived category $D(\mathcal{A})$ of an abelian category \mathcal{A} with enough injectives I . Its enhancement is the category of complexes of injective sheaves $\text{Kom}(I)$.

With the above definitions we are ready to state the following theorem.

Theorem 2.3 (Theorem 1 of [BK90]). *Let $\tilde{\mathcal{C}}$ be a pretriangulated category, E_1, \dots, E_n objects of $\tilde{\mathcal{C}}$ and $\mathcal{C} \subset \tilde{\mathcal{C}}$ the full DG subcategory on the objects E_i . Then the smallest triangulated subcategory of $H^0(\tilde{\mathcal{C}})$ containing E_1, \dots, E_n is equivalent to \mathcal{C}^{tr} as a triangulated category.*

It follows that a full exceptional collection $\sigma = \langle E_1, \dots, E_n \rangle$ in an enhanced triangulated category \mathcal{D} leads to an equivalence of \mathcal{D} and $\mathcal{C}_\sigma^{\text{tr}}$ for some DG category \mathcal{C}_σ . The category \mathcal{C}_σ is a subcategory of the enhancement of \mathcal{D} and thus $H^i(\mathcal{C}_\sigma(E, F)) = \mathcal{D}(E, F[i])$ for any elements E, F of σ .

Remark 2.4 (see [BLL04]). If an enhanced triangulated category \mathcal{D} is Karoubian, then a full exceptional collection σ leads to an equivalence of \mathcal{D} and $D^b(\mathcal{C}_\sigma)$.

3. PROPERTIES OF DG CATEGORIES ASSOCIATED TO FULL EXCEPTIONAL COLLECTIONS

From now on let X be a smooth projective variety. Let $D^b(X) = D^b(\text{Coh}(X))$ denote the bounded derived category of the category of coherent sheaves on X . It is an enhanced Karoubian triangulated category. Its enhancement is, for example, given by the DG category $\underline{\text{Kom}}^b(I)$ of complexes of injective sheaves bounded from below and with finitely many nonzero coherent cohomology sheaves.

Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection of coherent sheaves on X . By Theorem 2.3 there exists a DG category \mathcal{C}_σ with objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ such that $D^b(X)$ is equivalent to $\mathcal{C}_\sigma^{\text{tr}}$. The category $D^b(X)$ is Karoubian so $D^b(X)$ and $D^b(\mathcal{C}_\sigma)$

are equivalent. As σ is an exceptional collection the category $H(\mathcal{C}_\sigma)$ is ordered and finite.

Remark 3.1. If a DG category \mathcal{C} has only zeroth cohomology, then it is quasi-equivalent to $H(\mathcal{C}) = H^0(\mathcal{C})$. Indeed, for $C_1, C_2 \in \mathcal{C}$ let $\mathcal{C}(C_1, C_2) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}^n(C_1, C_2)$ with differential

$$\partial_{C_1, C_2}^n : \mathcal{C}^n(C_1, C_2) \rightarrow \mathcal{C}^{n+1}(C_1, C_2)$$

and let \mathcal{C}_I be a DG category such that $\text{ob } \mathcal{C}_I = \text{ob } \mathcal{C}$ and

$$\mathcal{C}_I(C_1, C_2) = \bigoplus_{n < 0} \mathcal{C}^n(C_1, C_2) \oplus \ker \partial_{C_1, C_2}^0.$$

Then the natural inclusion functor $\mathcal{C}_I \rightarrow \mathcal{C}$ is a quasi-equivalence. Let us also set

$$J_{C_1, C_2} = \text{Im}(\partial_{C_1, C_2}^{-1}) \oplus \left(\bigoplus_{n < 0} \mathcal{C}^n(C_1, C_2) \right)$$

for any C_1 and C_2 in \mathcal{C} and consider the category $\mathcal{C}_{I/J}$ with $\text{ob } \mathcal{C}_{I/J} = \text{ob } \mathcal{C}$ and

$$\mathcal{C}_{I/J}(C_1, C_2) = \mathcal{C}_I(C_1, C_2) / J_{C_1, C_2}.$$

Then $\mathcal{C}_{I/J}$ is isomorphic to $H(\mathcal{C})$ and the natural functor $\mathcal{C}_I \rightarrow \mathcal{C}_{I/J}$ is a quasi-equivalence.

It shows that if σ is a strong exceptional collection, then the DG category \mathcal{C}_σ is quasi-equivalent to an ordinary category.

3.1. Finiteness. Let $D = D^b(\text{Coh}(X))$ and let us consider its enhancement given by the category $\underline{\text{Kom}}^b(I)$ of complexes of injective sheaves with finitely many nonzero coherent cohomology. Then, calculating the category \mathcal{C}_σ requires taking injective resolutions. This suggests that the category \mathcal{C}_σ can have infinitely-dimensional spaces of morphisms. However, it is A_∞ -quasi-equivalent to an ordered DG category with finite-dimensional morphisms between objects. To prove this let us first recall definitions and properties of A_∞ categories after [Kel06] and [LH03].

3.1.1. A_∞ categories.

Definition 3.2. An A_∞ category \mathcal{A} over \mathbb{C} consists of

- objects $\text{ob}(\mathcal{A})$,
- for any two $A, B \in \text{ob}(\mathcal{A})$ a \mathbb{Z} -graded \mathbb{C} -vector space $\mathcal{A}(A, B)$,
- for any $n \geq 1$ and a sequence $A_0, A_1, \dots, A_n \in \text{ob}(\mathcal{A})$ a graded map:

$$m_n : \mathcal{A}(A_{n-1}, A_n) \otimes \dots \otimes \mathcal{A}(A_0, A_1) \rightarrow \mathcal{A}(A_0, A_n)$$

of degree $2 - n$ such that for any n

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}) = 0.$$

When these formulae are applied to elements additional signs appear because of the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y).$$

An A_∞ algebra is an A_∞ category with one object.

An A_∞ category \mathcal{A} is *ordered* if there exists a partial order \preceq on the set $\text{ob}(\mathcal{A})$ such that $\mathcal{A}(A, A') = 0$ for $A' \prec A$. It is *finite* if $\text{ob}(\mathcal{A})$ is a finite set and $\mathcal{A}(A, A')$ is finite dimensional for any A and A' .

The operation m_1 gives for any $A, B \in \text{ob}(\mathcal{A})$ a structure of a complex on $\mathcal{A}(A, B)$. Using m_1 , analogously as for DG categories, we can associate a graded category $H(\mathcal{A})$ to an A_∞ category.

The A_∞ category is *minimal* if the operation m_1 is trivial.

Any DG category is an A_∞ category with trivial m_i 's for $i > 2$.

For any set S there exists an A_∞ category $\mathbb{C}\{S\}$. The objects of $\mathbb{C}\{S\}$ are elements of S and

$$\mathbb{C}\{S\}(s_1, s_2) = \begin{cases} \mathbb{C} & \text{if } s_1 = s_2, \\ 0 & \text{otherwise.} \end{cases}$$

All operations m_n in $\mathbb{C}\{S\}$ are trivial.

Definition 3.3. A functor of A_∞ categories $F : \mathcal{A} \rightarrow \mathcal{B}$ is a map $F_0 : \text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathcal{B})$ and a family of graded maps

$$F_n : \mathcal{A}(A_{n-1}, A_n) \otimes \dots \otimes \mathcal{A}(A_0, A_1) \rightarrow \mathcal{B}(F_0(A_0), F_0(A_n))$$

of degree $1 - n$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} F_{r+1+t}(\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}) = \sum_{i_1+\dots+i_r=n} (-1)^p m_r(F_{i_1} \otimes \dots \otimes F_{i_r}),$$

where $p = (r - 1)(i_1 - 1) + (r - 2)(i_2 - 1) + \dots + 2(i_{r-2} - 1) + (i_{r-1} - 1)$.

Composition of functors is given by

$$(F \circ G)_n = \sum_{i_1+\dots+i_s=n} F_s \circ (G_{i_1} \otimes \dots \otimes G_{i_s}).$$

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called an A_∞ -*quasi-equivalence* if F_1 induces an equivalence between $H(\mathcal{A})$ and $H(\mathcal{B})$.

An A_∞ category \mathcal{A} is *strictly unital* if for any object $A \in \text{ob}(\mathcal{A})$ there exists a morphism $1_A \in \mathcal{A}(A, A)$ of degree 0 such that for any A' in \mathcal{A} and any morphisms $\phi \in \mathcal{A}(A, A')$, $\psi \in \mathcal{A}(A', A)$ we have $m_2(\phi, 1_A) = \phi$ and $m_2(1_A, \psi) = \psi$. Moreover, for $n \neq 2$ the operation m_n equals 0 if any of its arguments is equal to 1_A .

In particular, for any set S the category $\mathbb{C}\{S\}$ is strictly unital.

The category \mathcal{A} is *homologically unital* if there exist units for the homotopy category $H(\mathcal{A})$.

Proposition 3.4 (Theorem 3.2.1.1 of [LH03]). *A minimal homologically unital A_∞ category is A_∞ -quasi-equivalent to a minimal strictly unital A_∞ category.*

Remark 3.5. A minimal A_∞ category is equal to its homotopy category. Hence, an A_∞ -quasi-isomorphism F given by the above proposition satisfies $F_1 = \text{id}$.

An A_∞ category \mathcal{A} is *augmented* if there exists a strict unit preserving functor $\epsilon : \mathbb{C}\{\text{ob}(\mathcal{A})\} \rightarrow \mathcal{A}$. Then \mathcal{A} decomposes as $\mathcal{A} = \mathbb{C}\{\text{ob}(\mathcal{A})\} \oplus \bar{\mathcal{A}}$.

3.1.2. Minimal model. Any A_∞ category \mathcal{A} is A_∞ -quasi-equivalent to its homotopy category $H(\mathcal{A})$.

Theorem 3.6 ([Kad82]). *If \mathcal{A} is an A_∞ category, then $H(\mathcal{A})$ admits an A_∞ category structure such that*

- (1) $m_1 = 0$ and m_2 is induced from $m_2^{\mathcal{A}}$ and
- (2) there is an A_∞ -quasi-equivalence $\mathcal{A} \rightarrow H(\mathcal{A})$.

Moreover, this structure is unique up to a nonunique A_∞ -equivalence.

The A_∞ category $H(\mathcal{A})$ is called the *minimal model* of \mathcal{A} .

Remark 3.7. Let \mathcal{C} be a DG category. Its minimal model $H(\mathcal{C})$ is A_∞ -quasi-equivalent to a strictly unital A_∞ category. Indeed, the category \mathcal{C} is strictly unital and hence homologically unital. Its homotopy category is also homologically unital and Proposition 3.4 guarantees that there exists a strictly unital minimal category A_∞ -quasi-equivalent to $H(\mathcal{C})$.

3.1.3. *The universal DG category.* For any augmented A_∞ category \mathcal{A} there exists a DG category $U(\mathcal{A})$ and an A_∞ -quasi-isomorphism $\mathcal{A} \rightarrow U(\mathcal{A})$. To define the category $U(\mathcal{A})$ we need the following definitions (see further [LH03]).

Definition 3.8. A DG cocategory \mathcal{B} consists of

- the set of objects $B_i \in \text{ob}(\mathcal{B})$,
- for any pair of objects $B_i, B_j \in \text{ob}(\mathcal{B})$ a complex of \mathbb{C} -vector spaces $\mathcal{B}(B_i, B_j)$ with a differential d^{ij} of degree 1 and
- a coassociative cocomposition — a family of linear maps

$$\Delta : \mathcal{B}(B_i, B_j) \rightarrow \sum_{B_k \in \text{ob}(\mathcal{B})} \mathcal{B}(B_k, B_j) \otimes \mathcal{B}(B_i, B_k).$$

These data have to satisfy the condition

$$\Delta \circ d = (d \otimes \text{id} + \text{id} \otimes d) \circ \Delta.$$

For any set S the A_∞ category $\mathbb{C}\{S\}$ is also a DG cocategory.

A functor Φ between DG cocategories \mathcal{B} and \mathcal{B}' preserves the grading and differentials on morphisms and satisfies the condition

$$\Delta \circ \Phi = (\Phi \otimes \Phi) \circ \Delta.$$

A DG cocategory \mathcal{B} is *counital* if it admits a counit — a functor $\eta : \mathcal{B} \rightarrow \mathbb{C}\{\text{ob}(\mathcal{B})\}$. The category \mathcal{B} is *coaugmented* if it is counital and admits a coaugmentation functor $\varepsilon : \mathbb{C}\{\text{ob}(\mathcal{B})\} \rightarrow \mathcal{B}$ such that the composition $\eta\varepsilon$ is the identity on $\mathbb{C}\{\text{ob}(\mathcal{B})\}$.

Let \mathcal{B} be a coaugmented DG cocategory. Denote by $\bar{\mathcal{B}}$ a cocategory with the same objects as \mathcal{B} and morphisms $\bar{\mathcal{B}}(B_i, B_j) = \ker \varepsilon$.

For an augmented A_∞ category \mathcal{A} one can define its *bar* DG cocategory $B_\infty(\mathcal{A})$. Recall that as an augmented category \mathcal{A} can be written as $\bar{\mathcal{A}} \oplus \mathbb{C}\{\text{ob}(\mathcal{A})\}$. Then $B_\infty(\mathcal{A}) = T^c(S\bar{\mathcal{A}})$ is a tensor cocategory of the suspension of $\bar{\mathcal{A}}$. Here $S\bar{\mathcal{A}}$ denotes the category $\bar{\mathcal{A}}$ with a shift in the spaces of morphisms $(S\bar{\mathcal{A}})^n(A, A') = \bar{\mathcal{A}}^{n+1}(A, A')$. $S\bar{\mathcal{A}}$ is not an A_∞ category, however the operations m_n in $\bar{\mathcal{A}}$ define graded maps of degree 1 in $S\bar{\mathcal{A}}$:

$$b_n : S\bar{\mathcal{A}}(A_{n-1}, A_n) \otimes \dots \otimes S\bar{\mathcal{A}}(A_0, A_1) \rightarrow S\bar{\mathcal{A}}(A_0, A_n),$$

$$b_n = -s \circ m_n \circ \omega^{\otimes n},$$

where $s : V \rightarrow SV$ is a suspension of a graded vector space V and $\omega = s^{-1}$.

The cocategory $B_\infty(\mathcal{A})$ has the same objects as \mathcal{A} and

$$B_\infty(\mathcal{A})(A, A') = S\bar{\mathcal{A}}(A, A') \oplus \bigoplus_{A_1 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_1, A') \otimes S\bar{\mathcal{A}}(A, A_1)$$

$$\oplus \bigoplus_{A_1, A_2 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_2, A') \otimes S\bar{\mathcal{A}}(A_1, A_2) \otimes S\bar{\mathcal{A}}(A, A_1) \oplus \dots$$

for $A \neq A'$. In the case $A = A'$ we have to add 1_A in degree 0 to the above sum. To simplify the notation we shall write $(\alpha_n, \dots, \alpha_1)$ for $\alpha_n \otimes \dots \otimes \alpha_1$. The differential in $B_\infty(\mathcal{A})$ is given by

$$d(\alpha_n, \dots, \alpha_1) = \sum_{k=1}^n \sum_{l=1}^{n-k+1} (-1)^{|\alpha_{l-1}| + \dots + |\alpha_1|} (\alpha_n, \dots, \alpha_{l+k}, b_k(\alpha_{l+k-1}, \dots, \alpha_l), \alpha_{l-1}, \dots, \alpha_1),$$

for $(\alpha_n, \dots, \alpha_1) \in B_\infty(\mathcal{A})(A, A')$. The cocomposition is given by

$$\Delta(\alpha_n, \dots, \alpha_1) = 1_{A'} \otimes (\alpha_n, \dots, \alpha_1) + (\alpha_n, \dots, \alpha_1) \otimes 1_A + \sum_{l=1}^{n-1} (\alpha_n, \dots, \alpha_{l+1}) \otimes (\alpha_l, \dots, \alpha_1).$$

With these definitions $B_\infty(\mathcal{A})$ is an augmented DG cocategory.

Remark 3.9. Let \mathcal{A} be an ordered and finite A_∞ category such that $\bar{A}(A, A) = 0$ for any object $A \in \text{ob}\mathcal{A}$. Then the DG cocategory $B_\infty(\mathcal{A})$ also satisfies these conditions; i.e. is ordered, finite and $\overline{B_\infty(\mathcal{A})}(A, A) = 0$ for any object A .

Analogously, to an augmented DG cocategory \mathcal{B} via a *cobar* construction one can assign a DG category $\Omega(\mathcal{B})$. Let \mathcal{B} be a DG cocategory with a differential d and cocomposition Δ . Its cobar DG category is equal to $T(S^{-1}\bar{\mathcal{B}})$. Here $S^{-1}\bar{\mathcal{B}}$ denotes the shift of the cocategory \mathcal{B} and $T(S^{-1}\bar{\mathcal{B}})$ is the tensor DG category of it. As before, the spaces of morphisms in $T(S^{-1}\bar{\mathcal{B}})$ are given by

$$\begin{aligned} \Omega(\mathcal{B})(B, B') &= S^{-1}\bar{\mathcal{B}}(B, B') \oplus \bigoplus_{B_1 \in \text{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_1, B') \otimes S^{-1}\bar{\mathcal{B}}(B, B_1) \\ &\oplus \bigoplus_{B_1, B_2 \in \text{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_2, B') \otimes S^{-1}\bar{\mathcal{B}}(B_1, B_2) \otimes S^{-1}\bar{\mathcal{B}}(B, B_1) \oplus \dots \end{aligned}$$

for $B \neq B'$ and again one has to add 1_B in degree 0 for $\Omega(\mathcal{B})(B, B)$. The composition in $\Omega(\mathcal{B})$ is defined by concatenation and the differential ∂ on the spaces of morphisms is

$$\partial = \sum 1 \otimes \dots \otimes 1 \otimes (d + \Delta) \otimes 1 \otimes \dots \otimes 1.$$

Remark 3.10. If an ordered and finite DG cocategory \mathcal{B} satisfies the condition $\bar{\mathcal{B}}(B, B) = 0$, then the category $\Omega(\mathcal{B})$ is also finite, ordered and $\overline{\Omega(\mathcal{B})}(B, B) = 0$.

For an augmented A_∞ category \mathcal{A} its universal DG category $U(\mathcal{A})$ is defined as $\Omega(B_\infty(\mathcal{A}))$. There is a natural map $\mathcal{A} \rightarrow U(\mathcal{A})$. It is proved in [LH03] that this map extends to a functor and is an A_∞ -quasi-equivalence. Moreover, for an A_∞ -quasi-equivalence ϕ the functor $U(\phi)$ is a quasi-equivalence of DG categories.

3.1.4. A_∞ modules.

Definition 3.11. An A_∞ module over an A_∞ category \mathcal{A} is an A_∞ functor $M : \mathcal{A} \rightarrow \text{DGVect}_{\mathbb{C}}$. A morphism of modules $G : M \rightarrow N$ is given by a family

$\{G_A : M_0(A) \rightarrow N_0(A)\}_{A \in \text{ob}\mathcal{A}}$ of morphisms in $\text{DGVect}_{\mathbb{C}}$ such that the diagrams

$$\begin{CD} M_0(A_0) @>M_n(\alpha_{n-1} \otimes \dots \otimes \alpha_0)>> M_0(A_n) \\ @V G_{A_0} VV @VV G_{A_n} V \\ N_0(A_0) @>N_n(\alpha_{n-1} \otimes \dots \otimes \alpha_0)>> N_0(A_n) \end{CD}$$

commute for any $n \in \mathbb{N}$, $A_i \in \text{ob}\mathcal{A}$ and $\alpha_i \in \mathcal{A}(A_i, A_{i+1})$.

Following [LH03], the category of A_∞ modules over an A_∞ category \mathcal{A} will be denoted as $\text{Mod}_\infty^{\text{strict}}\mathcal{A}$.

The morphism $G : M \rightarrow N$ of modules is an A_∞ -quasi-isomorphism if G_A is a quasi-isomorphism of complexes for any $A \in \text{ob}\mathcal{A}$.

The derived category $D_\infty(\mathcal{A})$ of an A_∞ category \mathcal{A} is defined as a localization of the category $\text{Mod}_\infty^{\text{strict}}\mathcal{A}$ with respect to the class of A_∞ -quasi-isomorphisms. For A_∞ -quasi-equivalent A_∞ categories the derived categories are equivalent.

Remark 3.12 (Lemme 2.4.3.2 of [LH03]). For a DG category \mathcal{C} the derived categories $D(\mathcal{C})$ and $D_\infty(\mathcal{C})$ are equivalent. Hence, for any augmented A_∞ category \mathcal{A} the category $D_\infty(\mathcal{A})$ is equivalent to $D(U(\mathcal{A}))$.

Theorem 3.13. *Let \mathcal{C} be a DG category with finitely many objects. Assume that $H(\mathcal{C})$ is an ordered and finite graded category such that $H(\mathcal{C})(C, C) = \mathbb{C}$ for any object C of \mathcal{C} . Then there exists an ordered and finite DG category $\tilde{\mathcal{C}}$ such that $D(\mathcal{C})$ is equivalent to $D(\tilde{\mathcal{C}})$.*

Proof. Theorem 3.6 guarantees existence of a minimal model of \mathcal{C} — an A_∞ category $H(\mathcal{C})$. By Remark 3.7 we can assume that $H(\mathcal{C})$ is strictly unital. As $H(\mathcal{C})(C, C) = \mathbb{C}$ for any C the category $H(\mathcal{C})$ is an augmented ordered A_∞ category. We have

$$D = D(\mathcal{C}_\sigma) = D_\infty(H(\mathcal{C}_\sigma)) = D(\mathcal{U}(H(\mathcal{C}_\sigma))).$$

Remarks 3.9 and 3.10 show that the category $\mathcal{U}(H(\mathcal{C}_\sigma))$ is ordered and finite. □

As a corollary we get

Theorem 3.14. *Let X be a smooth projective variety and let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on X . Then there exists an ordered, finite DG category $\tilde{\mathcal{C}}_\sigma$ such that $D^b(X)$ is equivalent to $D^b(\tilde{\mathcal{C}}_\sigma)$.*

Proof. By Theorem 2.3 there exists a DG category \mathcal{C}_σ such that $D^b(X) = D^b(\mathcal{C}_\sigma)$. As sheaves $\mathcal{E}_1, \dots, \mathcal{E}_n$ are exceptional and the category $\text{Coh}(X)$ is Ext-finite, the category \mathcal{C}_σ satisfies the conditions of Theorem 3.13 and there exists a DG category $\tilde{\mathcal{C}}_\sigma$ such that categories $D(\mathcal{C}_\sigma)$ and $D(\tilde{\mathcal{C}}_\sigma)$ are equivalent. As $D^b(\mathcal{C}_\sigma) \subset D(\mathcal{C}_\sigma)$ is the subcategory of compact objects, it follows that $D^b(\mathcal{C}_\sigma) = D^b(\tilde{\mathcal{C}}_\sigma)$. □

The category $\tilde{\mathcal{C}}_\sigma$ is A_∞ -quasi-equivalent to the category \mathcal{C}_σ and hence $H^k(\tilde{\mathcal{C}}_\sigma(\mathcal{E}_i, \mathcal{E}_j)) = \text{Ext}_X^k(\mathcal{E}_i, \mathcal{E}_j)$.

To graphically present the DG category $\tilde{\mathcal{C}}_\sigma$ we introduce the following definition.

A *DG quiver* is a quiver Q endowed with a \mathbb{Z} -grading on the set of arrows Q_1 and a structure of a DG algebra on the path algebra $\mathbb{C}Q$ such that the following conditions are satisfied:

- The \mathbb{Z} -grading of $\mathbb{C}Q$ is compatible with the grading of Q_1 .
- For any vertex $q \in Q_0$ the trivial path $e_q \in \mathbb{C}Q$ is closed of degree 0.

The DG category $\widetilde{\mathcal{C}}_\sigma$ has finite-dimensional spaces of morphisms and so one can associate to it a DG quiver Q_σ . If the collection σ is strong, the DG quiver coincides with the quiver introduced in [Bon89].

4. CALCULATING THE DG CATEGORY $\widetilde{\mathcal{C}}_\sigma$

We present two algorithms allowing us to calculate the DG category $\widetilde{\mathcal{C}}_\sigma$. The first one goes back to [BK90] and uses mutations of exceptional collections. The second one is based on universal extensions defined by Hille and Perling in [HP11].

4.1. Action of the braid group. We have seen that the braid group acts on the set of exceptional collections. This action lifts to associated DG categories.

Twisted complexes provide a description of the categories $\widetilde{\mathcal{C}}_{L_i\sigma}$ and $\widetilde{\mathcal{C}}_{R_i\sigma}$ by means of $\widetilde{\mathcal{C}}_\sigma$. To see this we need to define a tensor product of a twisted complex with a complex of vector spaces.

Let \mathcal{C} be a finite DG category, let $C \in \mathcal{C}^{\text{pre-tr}}$ be a twisted complex and let V^\bullet be a finite-dimensional complex of vector spaces with the differential $\partial^i : V^i \rightarrow V^{i+1}$. $C \otimes V$ is defined as $(\bigoplus_{\{i|V^i \neq 0\}} C[-i]^{\oplus \dim V^i}, q_{i,j}) \in (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$. The morphisms $q_{i,i+1}$ are induced by the differential ∂^i tensored with the identity on C and $q_{i,j} = 0$ for $j \neq i + 1$.

Now, let $C, D \in \mathcal{C}^{\text{pre-tr}}$ be twisted complexes. There exist closed morphisms of degree 0:

$$\begin{aligned} \phi : C \otimes \text{Hom}_{\mathcal{C}^{\text{pre-tr}}}(C, D) &\rightarrow D, \\ \psi : C &\rightarrow \text{Hom}_{\mathcal{C}^{\text{pre-tr}}}(C, D)^* \otimes D. \end{aligned}$$

The morphism $\phi_{i,0} : C[-i]^{\oplus \dim \text{Hom}^i(C,D)} \rightarrow D$ is given by morphisms of degree i between C and D ; ψ is defined analogously. Now we define new twisted complexes over \mathcal{C} :

$$L_C D = \text{Tot}(\text{Cone}(\phi)[-1]), \quad R_D C = \text{Tot}(\text{Cone}(\psi)).$$

Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on a smooth projective variety X and let $\widetilde{\mathcal{C}}_\sigma$ be the category described in Theorem 3.13. Let E_1, \dots, E_n denote the objects of $\widetilde{\mathcal{C}}_\sigma$. We define two full subcategories of $\widetilde{\mathcal{C}}_\sigma^{\text{pre-tr}}$: $\widetilde{\mathcal{C}}_\sigma^{L_i}$ with objects $E_1, \dots, E_i, L_{E_i} E_{i+1}, E_{i+2}, \dots, E_n$ and $\widetilde{\mathcal{C}}_\sigma^{R_i}$ with $E_1, \dots, E_{i-1}, R_{E_{i+1}} E_i, E_{i+1}, \dots, E_n$.

Proposition 4.1. *Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on X and let $\widetilde{\mathcal{C}}_\sigma$ be a finite DG category with objects E_1, \dots, E_n such that $H^k \widetilde{\mathcal{C}}_\sigma(E_i, E_j) = \text{Ext}_X^k(\mathcal{E}_i, \mathcal{E}_j)$ and $D^b(X)$ is equivalent to $D^b(\widetilde{\mathcal{C}}_\sigma)$. Then the categories $\widetilde{\mathcal{C}}_\sigma^{L_i}$ and $\widetilde{\mathcal{C}}_\sigma^{R_i}$ satisfy analogous conditions for the collections $L_i\sigma$ and $R_i\sigma$ respectively.*

Proof. As the category $\widetilde{\mathcal{C}}_\sigma$ is finite, mutations of twisted complexes over $\widetilde{\mathcal{C}}_\sigma$ are well defined. Furthermore, the category $D^b(X)$ is equivalent to $\widetilde{\mathcal{C}}_\sigma^{\text{tr}}$. Under this

equivalence $L_{\mathcal{E}_i} \mathcal{E}_{i+1}$ corresponds to $L_{E_i} \bar{E}_{i+1}$. Hence the category $\widetilde{\mathcal{C}}_\sigma^{L_i}$ is the DG category described by Theorem 2.3. By construction it is also finite. \square

4.2. Universal extensions. Let E, F be objects of a \mathbb{C} -linear abelian category. Note that $\text{End}(\text{Ext}^1(E, F)) = \text{Ext}^1(E, F \otimes \text{Ext}^1(E, F)^*)$. As $\text{id} \in \text{End}(\text{Ext}^1(E, F))$ there exists a distinguished element $\widetilde{\text{id}}$ of $\text{Ext}^1(E, F \otimes \text{Ext}^1(E, F)^*)$.

Following [HP11] we say that an object \bar{E} is the *universal extension of E by F* if \bar{E} is the extension of E by $F \otimes \text{Ext}^1(E, F)^*$ corresponding to $\widetilde{\text{id}} \in \text{Ext}^1(E, F \otimes \text{Ext}^1(E, F)^*)$. \bar{E} fits into the short exact sequence

$$(4.1) \quad 0 \longrightarrow F \otimes \text{Ext}^1(E, F)^* \longrightarrow \bar{E} \longrightarrow E \longrightarrow 0.$$

This short exact sequence gives the long exact sequence

$$0 \longrightarrow \text{Hom}(E, F) \longrightarrow \text{Hom}(\bar{E}, F) \longrightarrow \text{Hom}(F, F) \otimes \text{Ext}^1(E, F) \longrightarrow \dots$$

This shows that $\text{Ext}^1(\bar{E}, F) = 0$ and the groups $\text{Ext}^n(\bar{E}, F)$ and $\text{Ext}^n(E, F)$ are isomorphic for $n > 1$ if $\text{Ext}^n(F, F) = 0$ for $n > 0$.

Moreover, if $\text{Ext}^n(F, E) = 0$ for $n \geq 0$ and $\text{Hom}(F, F) = \mathbb{C}$ and $\text{Ext}^q(F, F) = 0$ for $q > 0$, then the long exact sequence

$$0 \longrightarrow \text{Hom}(F, F) \otimes \text{Ext}^1(E, F)^* \longrightarrow \text{Hom}(F, \bar{E}) \longrightarrow \text{Hom}(F, E) \longrightarrow \dots$$

shows that $\text{Hom}(F, \bar{E}) = \text{Ext}^1(E, F)^*$.

Thus, if F is exceptional and $\text{Ext}^n(F, E) = 0$ for all n , the objects \bar{E}, F and morphisms between them determine E — as a cone of the canonical morphism

$$F \otimes \mathcal{H}om(F, \bar{E}) \xrightarrow{\text{can}} \bar{E} \longrightarrow E.$$

Assume that (E, F) is an exceptional pair in an enhanced triangulated category with the enhancement $\widetilde{\mathcal{C}}$ and denote by \mathcal{C} the DG subcategory of $\widetilde{\mathcal{C}}$ with objects \bar{E} and F . Then the cone of the canonical morphism $F \otimes \mathcal{H}om(F, \bar{E}) \rightarrow \bar{E}$ in $\mathcal{C}^{\text{pre-tr}}$ corresponds to E . Hence, as in the case of mutations, knowing the DG subcategory of $\widetilde{\mathcal{C}}$ with objects \bar{E} and F we can calculate the DG subcategory with objects E and F .

Theorem 4.2. *Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on a smooth projective variety X such that $\text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k \neq 0, 1$ and any i, j . The DG category $\widetilde{\mathcal{C}}_\sigma$ can be calculated by means of universal extensions.*

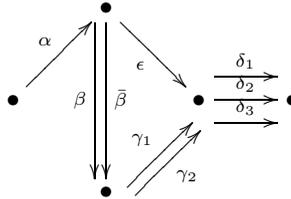
Proof. Following [HP11] we define $\mathcal{E}_i(1) = \mathcal{E}_i(2) = \dots = \mathcal{E}_i(i) = \mathcal{E}_i$ and for $j > i$ we put $\mathcal{E}_i(j)$ to be the universal extension of $\mathcal{E}_i(j-1)$ by \mathcal{E}_j :

$$0 \longrightarrow \mathcal{E}_j \otimes \text{Ext}^1(\mathcal{E}_i(j-1), \mathcal{E}_j)^* \longrightarrow \mathcal{E}_i(j) \longrightarrow \mathcal{E}_i(j-1) \longrightarrow 0.$$

Then $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{E}_i(n)$ is a tilting object — it generates $D^b(X)$ and $\text{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$. Moreover, as the \mathcal{E}_i 's are exceptional and there are no morphisms from \mathcal{E}_j to $\mathcal{E}_i(j-1)$, the objects $\mathcal{E}_1(n), \dots, \mathcal{E}_n(n)$ determine the \mathcal{E}_i 's. Thus, as described above, the endomorphism algebra of $\bigoplus_{i=1}^n \mathcal{E}_i(n)$ determines the DG structure of the collection $\langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$. \square

On X the exceptional collection $\sigma = \langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H_X), \mathcal{O}_X(2H_X) \rangle$ is full and the exceptional collection $\tau = \langle \mathcal{O}_Y, \mathcal{O}_Y(F_2), \mathcal{O}_Y(F_1), \mathcal{O}_Y(H_Y), \mathcal{O}_Y(2H_Y) \rangle$ is full on Y .

Let $t \in \mathbb{C}$ be a parameter and consider a DG quiver $Q(t)$:



with $\bar{\beta}$ in degree 1, remaining arrows in degree 0, differentials

$$\partial(\beta) = t\bar{\beta}, \quad \partial(\epsilon) = \gamma_1\bar{\beta}$$

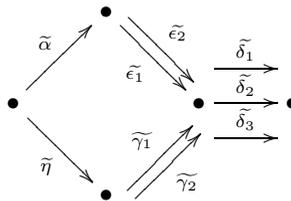
and relations

$$\begin{aligned} \gamma_2\bar{\beta} &= 0, & \bar{\beta}\alpha &= 0, \\ \delta_1\gamma_2 &= \delta_2\gamma_1, & \delta_1\epsilon &= \delta_3\gamma_1\beta, & \delta_2\epsilon &= \delta_3\gamma_2\beta, \\ \delta_1\gamma_1\bar{\beta} &= 0, & \delta_2\gamma_1\bar{\beta} &= 0. \end{aligned}$$

Let $D(t)$ denote the DG algebra of paths of this DG quiver.

For $t = 0$, $Q(t)$ is the DG quiver of the full collection σ on X .

For $t \neq 0$, $D(t)$ has cohomology groups concentrated in degree 0 and hence is quasi-isomorphic to the path algebra of the following quiver:



with

$$\begin{aligned} \tilde{\alpha} &= \alpha, & \tilde{\eta} &= \beta\alpha, & \tilde{\epsilon}_1 &= \gamma_1\beta - t\epsilon, \\ \tilde{\epsilon}_2 &= \gamma_2\beta, & \tilde{\gamma}_1 &= \gamma_1, & \tilde{\gamma}_2 &= \gamma_2, \\ \tilde{\delta}_1 &= \delta_1, & \tilde{\delta}_2 &= \delta_2, & \tilde{\delta}_3 &= \delta_1 - t\delta_3. \end{aligned}$$

Relations between morphisms are:

$$\begin{aligned} \tilde{\epsilon}_2\tilde{\alpha} &= \gamma_2\beta\alpha = \tilde{\gamma}_2\tilde{\eta}, \\ \tilde{\delta}_1\tilde{\gamma}_2 &= \delta_1\gamma_2 = \delta_2\gamma_1 = \tilde{\delta}_2\tilde{\gamma}_1, \\ \tilde{\delta}_2\tilde{\epsilon}_1 &= \delta_2\gamma_1\beta - t\delta_2\epsilon = \delta_1\gamma_2\beta - t\delta_3\gamma_2\beta = \tilde{\delta}_3\tilde{\epsilon}_2. \end{aligned}$$

Hence, for $t \neq 0$ the quiver $Q(t)$ is quasi-isomorphic to the quiver of the full collection τ on Y , and the path algebra $D(t)$ is an example of a noncommutative deformation of X to Y .

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