PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 6, June 2015, Pages 2427–2437
S 0002-9939(2015)12416-1
Article electronically published on February 3, 2015

LATTICE PROPERTY OF p-ADMISSIBLE WEIGHTS

TERO KILPELÄINEN, PEKKA KOSKELA, AND HIROAKI MASAOKA

(Communicated by Jeremy T. Tyson)

Abstract. We show that, for large p's, the maximum of two p-admissible weights remains p-admissible in the terminology of nonlinear potential theory. We also give examples showing that in general, the minimum may fail to remain p-admissible.

1. Introduction

Let $1 < p < \infty$ be fixed. Following [8, Ch. 20], we say that a locally integrable nonnegative function w on $\mathbf{R}^n$, $n \geq 1$, is *p-admissible* if it is the density of a doubling measure μ that supports a p-Poincaré inequality. More precisely we require that there exist positive constants C_d and C_P so that for each ball $B(x,r)$ and every Lipschitz function u on $\mathbf{R}^n$ we have that

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)) \tag{1.1}$$

and

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \Big(⨍_{B(x,r)} |\nabla u|^p \, d\mu\Big)^{1/p} . \tag{1.2}$$

Here, and in what follows, we use the notation

$$\mu(A) = \int_A w(x) \, dx$$

and, for any integrable function v,

$$v_A = \frac{1}{\mu(A)} \int_A v \, d\mu = ⨍_A v \, d\mu$$

For the significance of the class of p-admissible weights we refer, e.g., to [1, 4, 6, 8].

A core class of p-admissible weights is formed by the class of Muckenhoupt A_p-weights [4, 8]. Since the A_p-weights form a lattice:

$$w_1 \wedge w_2 \in A_p$$

and

$$w_1 \vee w_2 \in A_p$$

whenever $w_1, w_2 \in A_p$ (see Appendix below), it is natural to inquire if the same feature is shared by the entire class of p-admissible weights. It is rather surprising to us that this issue does not seem to have been addressed in the literature, not even

Received by the editors December 21, 2012 and, in revised form, November 16, 2013.
2010 *Mathematics Subject Classification.* Primary 46E35.

for A_p-weights. In this note, we discuss this question by establishing the following result:

1.3. **Theorem.** *The class of p-admissible weights on $\mathbf{R}$ is a lattice.*

In $\mathbf{R}^n$, $n \geq 2$, the minimum $w_1 \wedge w_2$ of two p-admissible weights w_1 and w_2 may fail to be p-admissible. Further, there is q_0 (depending on w_1) so that the maximum $w_1 \vee w_3$ is q-admissible whenever $q \geq q_0$ and w_3 is q-admissible.

Let us briefly comment on the proof of Theorem 1.3. First of all, in dimension one, w is p-admissible if and only if $w \in A_p$ ([2] also see [3]), and hence the lattice property is that of the A_p class.

Secondly, in higher dimensions, we have been able to solve the problem only partially. The example in the negative direction necessarily deals with non-A_p weights that are p-admissible. For $1 < p < n$, prime examples of such weights are of the form

$$w = J_f^{1-p/n},$$

where f is a quasiconformal self-homeomorphism of $\mathbf{R}^n$, $n \geq 2$. In our construction, we employ a planar quasiconformal mapping that generates a singular measure on the real line and simply use $w \equiv 1$ as our second weight. Higher dimensional cases are handled via a lifting procedure. This approach only applies for sufficiently small p; see Example 3.9. It would be interesting to see similar examples for all values of p.

Our proof for the positive direction in the case of the maximum uses a Hölder estimate for Sobolev functions in terms of the gradient. It would be interesting to dispense with it.

2. Toolbox

In this section we collect auxiliary results that will be used in our proof of Theorem 1.3.

Our first lemma reduces the p-Poincaré inequality into a more checkable condition. The result relies on Mazya's truncation argument [10] and a chaining argument; see, e.g., [6, p. 10 and Corollary 9.8].

2.1. **Lemma.** *Let w be a nonnegative, locally integrable function on $\mathbf{R}^n$ such that the associate measure μ with $d\mu = w\,dx$ is doubling (i.e. satisfies (1.1)). Suppose, further, that there is a constant c so that the estimate*

$$\begin{aligned} &\min\left(\mu(\{y \in B(x,r) : u(y) = 0\}), \mu(\{y \in B(x,r) : u(y) = 1\})\right) \\ &\qquad \leq c r^p \int_{B(x,4\sqrt{n}r)} |\nabla u|^p \, d\mu \end{aligned}$$

holds for all Lipschitz functions u and every ball $B(x,r)$. Then w is p-admissible.

Adding dummy variables allows us to lift weights to higher dimensions:

2.2. **Lemma.** *Let w be a p-admissible weight on $\mathbf{R}^n$. Then the weight $\hat{w}$,*

$$\hat{w}(x_1, \ldots x_n, x_{n+1}) = w(x_1, \ldots, x_n)$$

is p-admissible on $\mathbf{R}^{n+1}$.

Proof. Write μ and $\hat{\mu}$ for the associated measures with $d\mu = w dx$ on $\mathbf{R}^n$ and $d\hat{\mu} = \hat{w} dx$ on $\mathbf{R}^{n+1}$, respectively. Regarding the doubling condition (1.1), simply notice that

$$\hat{\mu}(B^{n+1}(x,2r)) \leq 4r\mu(\pi(B^{n+1}(x,2r))) \leq c(C_d,\mu,n)\hat{\mu}(B^{n+1}(x,r))\,,$$

where π is the projection from $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$ and C_d is the doubling constant of μ; notice that the cylinder

$$B^n(\pi(x),r/2)\times]x_{n+1}-r/2,x_{n+1}+r/2[$$

is contained in the ball $B^{n+1}(x,r)$.

Towards the p-Poincaré inequality (1.2), fix a Lipschitz function u and a ball $B^{n+1}(x,r)$. Write

$$E = \{y \in B^{n+1}(x,r) : u(y) = 0\}\,, \quad F = \{y \in B^{n+1}(x,r) : u(y) = 1\}\,.$$

Set

$$E_G = \{z \in \pi(E) : \text{ there is } s \in]x_{n+1}-r, x_{n+1}+r[\text{ with } u(z,s) > \frac{1}{3}\}\,.$$

If $z \in E_G$, then

$$\begin{aligned}\frac{1}{3} &\leq \int_{]x_{n+1}-r,x_{n+1}+r[} |\nabla u(z,t)|\,dt \\ &\leq (\int_{]x_{n+1}-r,x_{n+1}+r[} |\nabla u(z,t)|^p\,dt)^{1/p}(2r)^{1-1/p}\,,\end{aligned}$$

and hence

$$r^p \int_{B(x,\sqrt{n}r)} |\nabla u|^p\,d\hat{\mu} \geq 2^{1-p}3^{-p}\mu(E_G)r\,.$$

Suppose that

$$\mu(E_G) \geq \frac{1}{2}\mu(\pi(E)).$$

Since

$$\hat{\mu}(E) \leq 2r\mu(\pi(E))\,,$$

it would follow that

$$r^p \int_{B(x,\sqrt{n}r)} |\nabla u|^p\,d\hat{\mu} \geq 2^{1-p}3^{-p}\mu(E_G)r \geq 2^{-(1+p)}3^{-p}\hat{\mu}(E)\,.$$

Hence the estimate assumed in Lemma 2.1 and hence also our claim would follow. Thus we may assume that

$$\mu(E_G) \leq \frac{1}{2}\mu(\pi(E))\,.$$

Analogously, defining

$$F_G = \{z \in \pi(F) : \text{ there is } s \in]x_{n+1}-r, x_{n+1}+r[\text{ with } u(z,s) < \frac{2}{3}\}\,,$$

we may assume that

$$\mu(F_G) \leq \frac{1}{2}\mu(\pi(F))\,.$$

Thus, we are reduced to the case

$$\mu(\pi(E)\setminus E_G) \geq \frac{1}{2}\mu(\pi(E)) \text{ and } \mu(\pi(F)\setminus F_G) \geq \frac{1}{2}\mu(\pi(F))\,.$$

Now by truncating u appropriately, the definition of $\hat{w}$, the p-Poincaré inequality for μ on (copies of) $\mathbf{R}^n$, and the Fubini theorem yield

$$r^p \int_{B(x,\sqrt{n}r)} |\nabla u|^p \, d\hat{\mu} \geq c(C_P, \mu, p) r \min\left(\mu(\pi(E)), \mu(\pi(F))\right)$$
$$\geq c(C_P, \mu, p) \min\left(\hat{\mu}(E), \hat{\mu}(F)\right) ,$$

and the claim follows from Lemma 2.1. □

The following result due to Tukia [11] gives us the building block for our construction for the negative part in Theorem 1.3.

2.3. **Lemma.** *Let $0 < s < 1$. There is a quasiconformal mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ and a set $E_s \subset \mathbf{R}$ with*

$$f(\mathbf{R}) = \mathbf{R}, \quad \dim_H(E_s) \leq s \quad \text{and} \quad \dim_H(f(\mathbf{R} \setminus E_s)) \leq s .$$

Here and in what follows $\dim_H(E)$ refers to the Hausdorff dimension of the set E.

2.1. **Sets of (p, μ)-capacity zero.** We need to recall some facts of sets of (p, μ)-capacity zero. For a more thorough discussion the reader is referred to [8].

Suppose that $\Omega \subset \mathbf{R}^n$ is open. The (p, μ)-*capacity* $\mathrm{cap}_{p,\mu}(E, \Omega)$ of any set $E \subset \Omega$ is defined as follows: the (p, μ)-capacity of a compact set $K \subset \Omega$ is

$$\mathrm{cap}_{p,\mu}(K, \Omega) = \inf \left\{ \int_\Omega |\nabla \varphi|^p \, d\mu : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } K \right\}.$$

The (p, μ)-capacity of an open set $U \subset \Omega$ is then

$$\mathrm{cap}_{p,\mu}(U, \Omega) = \sup \left\{ \mathrm{cap}_{p,\mu}(K, \Omega) : K \text{ compact}, K \subset U \right\};$$

and for an arbitrary set $E \subset \Omega$

$$\mathrm{cap}_{p,\mu}(E, \Omega) = \inf \left\{ \mathrm{cap}_{p,\mu}(U, \Omega) : U \text{ open}, E \subset U \right\}.$$

A set E is said to be of (p, μ)-*capacity zero* if

$$\mathrm{cap}_{p,\mu}(E \cap \Omega, \Omega) = 0 \quad \text{for all open } \Omega.$$

The definition seems a bit complicated, but for bounded sets E, one needs only one bounded open set $\Omega \supset E$ to find out if E is of (p, μ)-capacity zero [8, Lemma 2.9]. Moreover, the capacity is subadditive in E, so that E is of (p, μ)-capacity zero if and only if it is a countable union of sets of (p, μ)-capacity zero.

We shall employ the fact that a bounded set E is of (p, μ)-capacity zero as soon as we find Lipschitz functions η_j (or more generally, quasi-continuous functions from the corresponding weighted Sobolev space $W^{1,p}(\mathbf{R}^n; \mu)$), vanishing outside a fixed ball, such that $\max_j \eta_j \geq 1$ on E and

$$\sum_{j=1}^{\infty} \int_{\mathbf{R}^n} |\nabla \eta_j|^p \, d\mu < \varepsilon ,$$

whenever $\epsilon > 0$ is a given number; see [8].

2.4. **Lemma.** *Suppose that $1 < p < n$ and that $f : \mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal. Let $w(x) = J_f(x)^{1-p/n}$ and $E \subset \mathbf{R}^n$. If $\dim_H(f(E)) < n - p$, then E is of (p, μ)-capacity zero; recall $d\mu = w dx$.*

Proof. Recall that w is p-admissible. Let $\varepsilon > 0$. Since $\dim_H(f(E)) < n - p$, we may cover $f(E)$ with balls $B(x_j, r_j)$ such that

$$\sum_{j=1}^{\infty} r_j^{n-p} < \varepsilon\,.$$

Next choose Lipschitz functions η_j with compact supports in $B(x_j, 2r_j)$ such that $|\nabla\eta_j| < C/r_j$, $\eta_j = 1$ on $B(x_j, r_j)$. Then

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla(\eta_j \circ f)|^p \, d\mu &\le \int_{\mathbf{R}^n} |Df|^p |\nabla\eta_j \circ f|^p J_f(x)^{1-p/n} dx \\ &\le c\int_{\mathbf{R}^n} |\nabla\eta_j \circ f|^p J_f(x) dx \\ &= c\int_{B(x_j,2r_j)} |\nabla\eta_j|^p \, dy \\ &\le c r_j^{n-p}\,.\end{aligned}$$

Since $\max(\eta_j \circ f) \ge 1$ on E and

$$\sum_{j=1}^{\infty} \int_{\mathbf{R}^n} |\nabla(\eta_j \circ f)|^p \, d\mu \le c \sum_{j=1}^{\infty} r_j^{n-p} < c\varepsilon\,,$$

we have by referring to discussion above that E is of (p, μ)-capacity zero. $\square$

3. New admissible weights from the old ones

In what follows we use the notation that μ_j stands for the measure with density w_j. Also if $B(x, r)$ is a ball, then $\lambda B = B(x, \lambda r)$ for $\lambda > 0$.

We start with a lemma for sums.

3.1. **Lemma.** *Let w_1 and w_2 be p-admissible and let $w = w_1 + w_2$. Suppose further that*

$$\frac{1}{\mu(B)^2} \int_B \int_B |u(x) - u(y)| d\mu_1(x) d\mu_2(y) \le Cr\Big(\fint_{2B} |\nabla u|^p \, d\mu\Big)^{1/p} \tag{3.2}$$

for all Lipschitz functions u and all balls $B = B(z, r)$; here $\mu = \mu_1 + \mu_2$. Then w is p-admissible.

Proof. The doubling property (1.1) for the sum measure μ immediately follows from the corresponding doubling property with weights w_1 and w_2; indeed,

$$\mu(2B) = \mu_1(2B) + \mu_2(2B) \le C_{D1}\mu_1(B) + C_{D2}\mu_2(B) \le C\mu(B).$$

Towards the Poincaré inequality (1.2), let u_B, u_{B1}, and u_{B2} stand for the averages of u over B with respect to measures μ, μ_1, and μ_2, respectively. In light of [7, Theorem 9.5] it suffices to find the estimate

$$\fint_B |u - u_B| d\mu \le Cr\Big(\fint_{2B} |\nabla u|^p \, d\mu\Big)^{1/p}\,,$$

where the constant C is independent of u and B. To reach this, we first observe that

$$
\begin{aligned}
\fint_B |u-u_B|\,d\mu \le & \fint_B\fint_B |u(x)-u(y)|\,d\mu(x)d\mu(y)\\
\le & \Big(\frac{\mu_1(B)}{\mu(B)}\Big)^2 \fint_B\fint_B |u(x)-u(y)|\,d\mu_1(x)d\mu_1(y)\\
& + \frac{2}{\mu(B)^2}\int_B\int_B |u(x)-u(y)|\,d\mu_1(x)d\mu_2(y)\\
& + \Big(\frac{\mu_2(B)}{\mu(B)}\Big)^2 \fint_B\fint_B |u(x)-u(y)|\,d\mu_2(x)d\mu_2(y).
\end{aligned}
\tag{3.3}
$$

Now we use (3.2) to estimate the second term on the right-hand side:

$$\frac{2}{\mu(B)^2}\int_B\int_B |u(x)-u(y)|\,d\mu_1(x)d\mu_2(y) \le Cr\Big(\fint_{2B} |\nabla u|^p\,d\mu\Big)^{1/p}.$$

Hence by (3.3) we need only to find an estimate for the terms

$$\Big(\frac{\mu_j(B)}{\mu(B)}\Big)^2 \fint_B\fint_B |u(x)-u(y)|\,d\mu_j(x)d\mu_j(y)\,, \quad j=1,2.$$

To this end, we obtain by using the Poincaré inequality that

$$
\begin{aligned}
& \fint_B\fint_B |u(x)-u(y)|\,d\mu_j(x)d\mu_j(y)\\
& \le \fint_B\fint_B |u(x)-u_{Bj}|\,d\mu_j(x)d\mu_j(y) + \fint_B\fint_B |u_{Bj}-u(y)|\,d\mu_j(x)d\mu_j(y)\\
& \le \fint_B |u(x)-u_{Bj}|\,d\mu_j(x) + \fint_B |u_{Bj}-u(y)|\,d\mu_j(y)\\
& \le 2C_{pj} r\Big(\fint_B |\nabla u|^p\,d\mu_j\Big)^{1/p} \le \Big(\frac{\mu(B)}{\mu_j(B)}\Big)^{1/p} 2C_{pj} r\Big(\fint_B |\nabla u|^p\,d\mu\Big)^{1/p}\\
& \le \Big(\frac{\mu(B)}{\mu_j(B)}\Big)^2 2C_{pj} r\Big(\fint_B |\nabla u|^p\,d\mu\Big)^{1/p},
\end{aligned}
$$

where we also used the simple fact that $\mu(B)\ge\mu_j(B)$. This completes the proof. $\square$

3.4. **Lemma.** *Let w_1 be p-admissible. If w_2 is a function with*

$$\frac{1}{c_0}w_1 \le w_2 \le c_0 w_1$$

for a constant $c_0>0$, then w_2 is also p-admissible.

Proof. The doubling property (1.1) follows immediately. For the Poincaré one needs to observe that

$$
\begin{aligned}
\fint_B |u-u_{B_2}|\,d\mu_2 \le c\fint_B |u-u_{B_1}|\,d\mu_2 \le cc_0^2 \fint_B |u-u_{B_1}|\,d\mu_1\\
\le cc_0^2 C_{P_1} r\Big(\fint_B |\nabla u|^p\,d\mu_1\Big)^{1/p} \le Cr\Big(\fint_B |\nabla u|^p\,d\mu_2\Big)^{1/p},
\end{aligned}
$$

as desired. $\square$

3.5. **Lemma.** *Let w_1 and w_2 be p-admissible. Suppose further that for all balls $B = B(z,r)$*

$$|u(x) - u(y)| \leq Cr(⨍_{2B} |\nabla u|^p \, d\mu_1)^{1/p}, \quad x, y \in B, \tag{3.6}$$

for all Lipschitz functions u. Then $w = w_1 + w_2$ is p-admissible.

Proof. The claim follows from Lemma 3.1 once we notice that the condition (3.2) follows from the oscillation estimate (3.6). Indeed,

$$\begin{aligned}
\frac{1}{\mu(B)^2} \int_B \int_B & |u(x) - u(y)| d\mu_1(x) d\mu_2(y) \\
&\leq Cr \frac{\mu_1(B)\mu_2(B)}{\mu(B)^2} (⨍_{2B} |\nabla u|^p \, d\mu_1)^{1/p} \\
&\leq Cr (⨍_{2B} |\nabla u|^p \, d\mu)^{1/p} ,
\end{aligned}$$

since by the doubling property

$$\begin{aligned}
&\frac{\mu_1(B)\mu_2(B)}{\mu(B)^2} (\frac{1}{\mu_1(2B)})^{1/p} \\
&= (\frac{\mu_1(B)}{\mu(B)})^{1-1/p} (\frac{\mu_1(B)\mu(2B)}{\mu_1(2B)\mu(B)})^{1/p} \frac{\mu_2(B)}{\mu(B)} (\frac{1}{\mu(2B)})^{1/p} \\
&\leq C (\frac{1}{\mu(2B)})^{1/p}.
\end{aligned}$$

□

3.7. *Remark.* Condition (3.6) is the Hölder estimate given by the Sobolev embedding theorem if $w_1 = 1$ and $p > n$. Thus $1+w_2$ and $1 \vee w_2$ are both p-admissible whenever w_2 is p-admissible and $p > n$.

3.8. **Lemma.** *Let w_1 be p_0-admissible. There is $q_0 > 1$ such that for all $p \geq q_0$ the sum $w_1 + w_2$ and the maximum $w_1 \vee w_2$ are p-admissible whenever w_2 is p-admissible.*

Proof. Since any q-admissible weight is p-admissible for all $p \geq q$ [8, Thm. 1.8], it suffices, by Lemmas 3.4 and 3.5, to observe that the Hölder estimate (3.6) holds for some exponent q_0 depending on the doubling constant of w_1; see [6, Thm. 5.1]. □

If $n = 1$, then the class of p-admissible weights coincides with that of A_p-weights [2] and the claim follows because the class of A_p-weights forms a lattice; see Appendix below.

We conclude the proof of Theorem 1.3 by giving counterexamples.

3.9. **Example.** Fix $1 < p < 2$. First let $n = 2$. For a fixed $0 < s < 2 - p$, Lemma 2.3 provides us with a quasiconformal mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ and a set $E_s \subset \mathbf{R}$ so that

$$\dim_H(E_s) \leq s \quad \text{and} \quad \dim_H(f(\mathbf{R} \setminus E_s)) \leq s \, .$$

Then the weight $w_1 = J_f^{1-p/2}$ is p-admissible [8, Ch. 15] and $\mathbf{R} \setminus E_s$ is of (p,μ_1)-capacity zero (Lemma 2.4); here $\mu_1 = w_1 dx$. Since

$$\dim_H(E_s) \le s < 2 - p,$$

E_s is of (p, dx)-capacity zero (see the argument at the end of the proof of Lemma 2.4).

Now let $w = w_1 \wedge 1$ and $\mu = w dx$. Then w is not p-admissible. If it were, then both E_s and $\mathbf{R} \setminus E_s$ would be of (p,μ)-capacity zero, and consequently, the whole line $\mathbf{R}$ would be of (p,μ)-capacity zero in $\mathbf{R}^2$ by subadditivity. However, at the presence of the Poincaré inequality, the sets of (p,μ)-capacity zero cannot separate the space [8, Lemma 2.46].

A counterexample for $n \ge 3$ follows by lifting the weights above by using Lemma 2.2 and reasoning similarly as above. The details are left to the reader.

4. Appendix

Recall that the Muckenhoupt class A_p, $p > 1$, consists of all locally integrable functions w with $0 < w < \infty$ a.e., for which there is a constant $c_{p,w}$ so that

$$\fint_B w\, dx \le c_{p,w} \Big(\fint_B w^{1/(1-p)}\, dx\Big)^{1-p}$$

for each ball B. Set

$$A_\infty = \bigcup_{p>1} A_p.$$

Recall that we set

$$\mu(E) = \mu_w(E) = \int_E w\, dx\,,$$

where w is a weight function. Now we have the following characterization [5, Theorem IV.2.11 and Corollary IV.2.13].

4.1. **Proposition.** *The following two conditions are equivalent:*

(1) $w \in A_\infty$.
(2) *The measure $\mu = \mu_w$ is doubling and there exist positive constants C and δ such that for each ball B and every measurable set $E \subset B$ it holds that*

$$\frac{\mu(E)}{\mu(B)} \le C \left(\frac{|E|}{|B|}\right)^\delta .$$

We also employ the following characterization from [5, Theorem IV.2.17].

4.2. **Proposition.** *The following two conditions are equivalent:*

(1) $w \in A_p$.
(2) $w \in A_\infty$ *and* $w^{-1/(p-1)} \in A_\infty$.

The above two propositions allow us to verify the lattice property for A_p.

4.3. **Proposition.** *The class of A_p-weights is a lattice.*

Proof. Suppose that $w_j \in A_p$, $j = 1, 2$. We first prove that $w_1 \vee w_2 \in A_p$. By the definition of A_p there exist positive constants $c_{p,j}$, $j = 1, 2$, with

$$\begin{aligned}\fint_B w_1 \vee w_2 \, dx &\le \fint_B w_1 \, dx + \fint_B w_2 \, dx \\ &\le c_{p,1}\Big(\fint_B w_1^{1/(1-p)} \, dx\Big)^{1-p} + c_{p,2}\Big(\fint_B w_2^{1/(1-p)} \, dx\Big)^{1-p} \\ &\le c_{p,12}\Big(\fint_B (w_1 \vee w_2)^{1/(1-p)} \, dx\Big)^{1-p},\end{aligned}$$

whenever B is a ball; here we wrote $c_{p,12} = 2\max\{c_{p,1}, c_{p,2}\}$. Hence

$$w_1 \vee w_2 \in A_p.$$

Next we prove that $w_1 \wedge w_2 \in A_p$. To this end we intend to use Proposition 4.2 and thus show that $w_1 \wedge w_2 \in A_\infty$ and $(w_1 \wedge w_2)^{-1/(p-1)} \in A_\infty$.

To establish the first, we verify condition (2) of Proposition 4.1 for $w_1 \wedge w_2$. To this end, we first observe that $w_1 \wedge w_2$ defines a doubling measure $\mu = w_1 \wedge w_2 \, dx$. Indeed, write

$$\mu_j(E) = \int_E w_j \, dx$$

and, for a fixed ball B, choose a set $A \subset B$ with $|A| \ge \frac{1}{2}|B|$ and that $w_1 \wedge w_2 = w_j$ on A, say $w_1 \wedge w_2 = w_1$ on A. Then

$$\mu_1(A) \ge c\Big(\frac{|A|}{|B|}\Big)^p \mu_1(B) \ge c2^{-p}\mu_1(B)$$

by the strong doubling property [8, 15.5] of A_p-weights. Consequently, μ is doubling, since

$$\mu(2B) \le \mu_1(2B) \le c\mu_1(B) \le c\mu_1(A) = c\mu(A) \le c\mu(B).$$

Next, let C_j and δ_j be the positive constants associated to weights w_j, given us by Proposition 4.1. Set $\delta = \min\{\delta_1, \delta_2\}$, $C = C_1 + C_2$ and observe that by the previous estimation

$$\mu(B) \ge c\min(\mu_1(B), \mu_2(B)).$$

Therefore, for all measurable $E \subset B$

$$c\frac{\mu(E)}{\mu(B)} \le \frac{\mu_1(E)}{\mu_1(B)} + \frac{\mu_2(E)}{\mu_2(B)} \le C_1\left(\frac{|E|}{|B|}\right)^{\delta_1} + C_2\left(\frac{|E|}{|B|}\right)^{\delta_2} \le C\left(\frac{|E|}{|B|}\right)^{\delta}.$$

Hence by Proposition 4.1, we have that $w_1 \wedge w_2 \in A_\infty$.

To complete the proof, we verify the second condition part (2) of Proposition 4.2. To this end, write

$$w_j' = w_j^{-1/(p-1)}, \quad j = 1, 2.$$

Since $w_j' \in A_\infty$, there exist $p_j > 1$ with $w_j' \in A_{p_j}$. Set $q = \max\{p_1, p_2\}$. Then $w_j' \in A_q$ since $A_{p_j} \subset A_q$ (cf. [8, p. 298]).

By the first part of this proof

$$w_1' \vee w_2' \in A_q \subset A_\infty.$$

Since

$$(w_1 \wedge w_2)^{-1/(p-1)} = w_1' \vee w_2',$$

we infer that

$$(w_1 \wedge w_2)^{-1/(p-1)} \in A_\infty.$$

Hence Proposition 4.2 ensures us that $w_1 \wedge w_2 \in A_p$, as desired. $\square$

Acknowledgements

We thank the anonymous referee for pointing out a flaw in the first version of our manuscript.

The research for this paper was begun during the stay of the third author at the University of Jyväskylä and completed during the visit of the first and second authors at Kyoto Sangyo University. The authors wish to thank the institutions for the hospitality and acknowledge the support of Kyoto Sangyo University Research Grant (E1214). The second author also acknowledges the support of the Academy of Finland grant 131477.

References

[1] Anders Björn and Jana Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, vol. 17, European Mathematical Society (EMS), Zürich, 2011. MR2867756

[2] Jana Björn, Stephen Buckley, and Stephen Keith, *Admissible measures in one dimension*, Proc. Amer. Math. Soc. **134** (2006), no. 3, 703–705 (electronic), DOI 10.1090/S0002-9939-05-07925-6. MR2180887 (2006k:26021)

[3] Seng-Kee Chua and Richard L. Wheeden, *Sharp conditions for weighted 1-dimensional Poincaré inequalities*, Indiana Univ. Math. J. **49** (2000), no. 1, 143–175, DOI 10.1512/iumj.2000.49.1754. MR1777034 (2001h:26021)

[4] Eugene B. Fabes, Carlos E. Kenig, and Raul P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116, DOI 10.1080/03605308208820218. MR643158 (84i:35070)

[5] José García-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104. MR807149 (87d:42023)

[6] Piotr Hajłasz and Pekka Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101, DOI 10.1090/memo/0688. MR1683160 (2000j:46063)

[7] Juha Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. MR1800917 (2002c:30028)

[8] Juha Heinonen, Tero Kilpeläinen, and Olli Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original. MR2305115 (2008g:31019)

[9] Stephen Keith and Xiao Zhong, *The Poincaré inequality is an open ended condition*, Ann. of Math. (2) **167** (2008), no. 2, 575–599, DOI 10.4007/annals.2008.167.575. MR2415381 (2009e:46028)

[10] V. G. Maz′ja, *On the theory of the higher-dimensional Schrödinger operator* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1145–1172. MR0174879 (30 #5070)

[11] Pekka Tukia, *Hausdorff dimension and quasisymmetric mappings*, Math. Scand. **65** (1989), no. 1, 152–160. MR1051832 (92b:30026)

Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

E-mail address: tero.kilpelainen@jyu.fi

Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

E-mail address: pekka.j.koskela@jyu.fi

Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Kamigamo, Motoyama, Kita-ku, Kyoto 603-8555, Japan

E-mail address: masaoka@cc.kyoto-su.ac.jp