PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 6, June 2015, Pages 2395–2401
S 0002-9939(2015)12550-6
Article electronically published on February 3, 2015

INVOLUTIONS AND FREE PAIRS OF BICYCLIC UNITS IN INTEGRAL GROUP RINGS OF NON-NILPOTENT GROUPS

J. Z. GONÇALVES AND D. S. PASSMAN

(Communicated by Pham Huu Tiep)

Abstract. If $^*\colon G \to G$ is an involution on the finite group G, then * extends to an involution on the integral group ring $\mathbb{Z}[G]$. In this paper, we consider whether bicyclic units $u \in \mathbb{Z}[G]$ exist with the property that the group $\langle u, u^* \rangle$, generated by u and u^*, is free on the two generators. If this occurs, we say that (u, u^*) is a free bicyclic pair. It turns out that the existence of u depends strongly upon the structure of G and on the nature of the involution. The main result here is that if G is a non-nilpotent group, then for any involution, $\mathbb{Z}[G]$ contains a free bicyclic pair.

1. Introduction

Let $\mathbb{Z}[G]$ denote the integral group ring of the finite group G over the ring of rational integers $\mathbb{Z}$. If B is a subgroup of G, we let $\widehat{B} \in \mathbb{Z}[G]$ denote the sum of the elements of B in $\mathbb{Z}[G]$. Since $(1-b)\widehat{B} = \widehat{B}(1-b) = 0$ for any $b \in B$, we see that group ring elements of the form $(1-b)a\widehat{B}$, with $a \in G$, have square 0. Hence $1+(1-b)a\widehat{B}$ is a unit in the ring $\mathbb{Z}[G]$ with inverse $1-(1-b)a\widehat{B}$. When $B = \langle b \rangle$ is cyclic, elements of the form $u = 1+(1-b)a\widehat{B}$ are known as bicyclic units. It is easy to see that $1+(1-b)a\widehat{B} = 1$ if and only if $b^a = a^{-1}ba \in B$ and hence if and only if $a \in \mathfrak{N}_G(B)$, the normalizer of B. In particular, if G is a Dedekind group, namely a group with all subgroups normal, then $\mathbb{Z}[G]$ has no nontrivial bicyclic units.

Now suppose that $^*\colon G \to G$ is an involution, that is, an antiautomorphism of order 2. Then * extends to an involution of $\mathbb{Z}[G]$. In particular, if u is a unit of $\mathbb{Z}[G]$, then so is u^*, and we are interested in the nature of the subgroup $\langle u, u^* \rangle$ of the unit group that is generated by these two elements. If $^*\colon G \to G$ is the inverse map, then it was shown in [4] that, for every nontrivial bicyclic unit u, the group $\langle u, u^* \rangle$ is free of rank 2, namely (u, u^*) is a free bicyclic pair. Certainly, it is of interest to see whether free bicyclic pairs exist for other involutions, and that was the theme of [1], where groups G with $|G : \mathfrak{Z}(G)| = 4$ were considered. Somewhat later, in [2] we studied different classes of groups. For example, Theorem 2.8 of that paper asserts

Theorem 1.1. *Let G be a finite nonabelian group that admits an involution *. If all Sylow subgroups of G are abelian, then $\mathbb{Z}[G]$ contains a free bicyclic pair (u, u^*).*

Received by the editors January 26, 2014.
2010 *Mathematics Subject Classification.* Primary 16S34, 20D15, 20E05.
Key words and phrases. Integral group ring, non-nilpotent group, dihedral group, bicyclic unit, free bicyclic pair.
This research was supported in part by the grant CNPq 303.756/82-5 and by Fapesp-Brazil, Proj. Tematico 00/07.291-0.

We note that the group G above cannot be nilpotent since otherwise it would be isomorphic to the direct product of its Sylow subgroups and hence be abelian. In fact integral group rings of non-Dedekind nilpotent groups can fail to have free bicyclic pairs as was shown in [1] and [2]. On the other hand, this is really the only obstruction. Indeed, our main result here is

Theorem 1.2. *Let G be a finite non-nilpotent group that admits an involution *. Then $\mathbb{Z}[G]$ contains a free bicyclic pair (u, u^*).*

This will be proved in Section 2, while Section 3 will consider some nilpotent and related examples. We close this section by quoting the rather obvious [2, Lemma 1.1], namely

Lemma 1.3. *The map $^*\colon G \to G$ is an involution if and only if * is equal to σ followed by the inverse map, where $\sigma\colon G \to G$ is an automorphism of order 1 or 2.*

Since the inverse map stabilizes all subgroups H of G, we see that $H \subseteq G$ is *-stable if and only if it is σ-stable.

2. Non-nilpotent groups

Our goal here is to prove Theorem 1.2. The proof will proceed by induction on $|G|$ using [2, Lemma 1.11] which is given below.

Lemma 2.1. *Let G admit an involution * and assume that either*

i. G has a *-stable subgroup H such that $\mathbb{Z}[H]$ has a free bicyclic pair, or
ii. G has a *-stable normal subgroup N such that the group ring $\mathbb{Z}[G/N]$ contains a free bicyclic pair.

Then $\mathbb{Z}[G]$ has a free bicyclic pair (u, u^*).

Furthermore, as will soon be apparent, we do not construct any bicyclic units in the proof. Rather we reduce the group to situations where earlier results can apply. In particular, we will use Theorem 1.1 and also [2, Corollary 1.10], namely

Lemma 2.2. *Suppose $a \in G$ satisfies $\langle a^*a\rangle \neq \langle aa^*\rangle$ and set $b = (aa^*)^{-1}$. If $B = \langle b\rangle$ and $\mu = (1-b)a\widehat{B} \in \mathbb{Z}[G]$, then $(1+\mu, 1+\mu^*)$ is a free pair.*

We need one more preliminary result.

Lemma 2.3. *Suppose G is a non-nilpotent group, but that all proper characteristic subgroups and characteristic homomorphic images of G are nilpotent. If G has a proper characteristic subgroup, then there are two distinct primes p and q, such that G is the semidirect product $G = P \rtimes Q$, where $P \neq 1$ is an elementary abelian p-group, $Q \neq 1$ is an elementary abelian q-group, and Q acts faithfully on P.*

Proof. Let K be the given proper characteristic subgroup of G. Then K is nilpotent, so $\mathbb{O}_p(K) \neq 1$ for some prime p. Hence $\mathbb{O}_p(G) \neq 1$ and since $G/\mathbb{O}_p(G)$ is nilpotent, it follows that G has a normal, and hence characteristic, Sylow p-subgroup P.

Since G is not nilpotent, there exists a Sylow q-subgroup Q of G that is not normal. But G/P is nilpotent, so PQ is a normal Hall $\{p,q\}$-subgroup of G. In particular, $PQ \operatorname{char} G$. Furthermore, Q is not normal in PQ since otherwise $Q \operatorname{char} PQ$ and hence $Q \operatorname{char} G$. Thus PQ is not nilpotent, and minimality implies that $G = PQ = P \rtimes Q$. If the Frattini subgroup $\Phi(P) \neq 1$, then $G/\Phi(P)$ is nilpotent and Q centralizes $P/\Phi(P)$. Thus Q centralizes P and $\mathfrak{N}_G(Q) \supseteq PQ = G$, a contradiction. It follows that $\Phi(P) = 1$ and P is an elementary abelian p-group.

Finally, note that $\mathbb{O}_q(G) = 1$ since otherwise $G/\mathbb{O}_q(G)$ is nilpotent and $Q \lhd G$. Now suppose that N is any normal nilpotent subgroup of G. Then $\mathbb{O}_q(N) \subseteq \mathbb{O}_q(G) = 1$, so N is a p-group and therefore $N \subseteq P$. In particular, since $\mathfrak{C}(P)$ is clearly nilpotent, we have $\mathfrak{C}(P) = P$ and hence Q acts faithfully on P. Furthermore, since any proper characteristic subgroup of G is nilpotent, we see that $G/P \cong Q$ has no proper characteristic subgroups. Thus $\Phi(Q) = 1$ and Q is an elementary abelian q-group. □

With this, we can now offer the

Proof of Theorem 1.2. If the result is false, let G be a non-nilpotent group of minimal order admitting an involution * such that $\mathbb{Z}[G]$ has no free bicyclic pairs. Using Lemma 1.3, we write * as the inverse map followed by an automorphism σ of G having order 1 or 2. Recall that subgroups of G are *-stable if and only if they are σ-stable. In view of Lemma 2.1 and the minimality of $|G|$, we can assume that every σ-stable proper subgroup of G is nilpotent. Furthermore, if $M \neq 1$ is a σ-stable normal subgroup of G, then G/M is nilpotent. We proceed in a series of three steps.

Step 1. If $H \neq 1$ is a σ-stable proper subgroup of G, then $\mathfrak{N}_G(H)$ is nilpotent.

Proof. Notice that $\mathfrak{N}_G(H)$ is also σ-stable. In particular, if $\mathfrak{N}_G(H)$ is not nilpotent, then $\mathfrak{N}_G(H) = G$ and $H \lhd G$. Also $H \neq G$, by assumption, so H is nilpotent and there exists a prime p with $\mathbb{O}_p(H) \neq 1$. Thus $\mathbb{O}_p(G) \neq 1$ and G has a proper characteristic subgroup. It follows that G satisfies all the hypotheses of the preceding lemma since any characteristic subgroup of G is σ-stable. Thus, by Lemma 2.3, $G = P \rtimes Q$, where P is an elementary abelian p-group, Q is an elementary abelian q-group, $p \neq q$, and Q acts nontrivially on P. In particular, G satisfies the hypothesis of Theorem 1.1, and hence $\mathbb{Z}[G]$ has a free bicyclic pair, contrary to our assumption. Thus $\mathfrak{N}_G(H)$ is proper and hence nilpotent. □

Step 2. If the prime p divides $|G|$, then σ stabilizes a Sylow p-subgroup of G.

Proof. We first show that σ stabilizes a nontrivial p-subgroup of G. To this end, let $a \neq 1$ be a p-element of G. If $a^\sigma = a$, then σ stabilizes the p-subgroup $\langle a \rangle \neq 1$ and we are done. Thus we may assume that $a^\sigma \neq a$ and hence that $a^* \neq a^{-1}$. In particular, $b = a^*a$ is a nonidentity element of G. Of course, b is not necessarily a p-element. If $\langle aa^* \rangle \neq \langle a^*a \rangle$, then Lemma 2.2 implies that $\mathbb{Z}[G]$ has a free bicyclic pair, contrary to our assumption. Thus, we may assume that $\langle aa^* \rangle = \langle a^*a \rangle = B$. Now $(aa^*)^* = aa^*$, so $B = \langle b \rangle$ is a nonidentity cyclic *-stable subgroup of G and Step 1 implies that $\mathfrak{N}_G(B)$ is nilpotent and σ-stable. In particular, σ stabilizes the characteristic Sylow p-subgroup of $\mathfrak{N}_G(B)$ and we need only show that this subgroup is nontrivial. But observe that $a^{-1}(aa^*)a = a^*a \in B$, so a is a p-element in $\mathfrak{N}_G(B)$. Thus σ stabilizes a nontrivial p-subgroup of G, as required.

Next, let P be a p-subgroup of G of maximal order stabilized by σ. Then $P \neq 1$, by the above, and hence $\mathfrak{N}_G(P) \supseteq P$ is nilpotent by Step 1. In particular, σ stabilizes the unique Sylow p-subgroup of $\mathfrak{N}_G(P)$ and this group contains P. Thus, by maximality, P is a Sylow p-subgroup of $\mathfrak{N}_G(P)$. Since normalizers grow in p-groups, this implies that P is indeed a Sylow p-subgroup of G. □

Step 3. Final contradiction.

Proof. Since G is not nilpotent, its order is divisible by at least two primes, so we can choose p to be an odd prime divisor of $|G|$. By Step 2, σ stabilizes a Sylow p-subgroup $P \neq 1$ of G, and of course $P \neq G$. In particular, σ stabilizes both $\mathfrak{Z}(P) \neq 1$, the center of P, and $\mathbb{T}(P) \neq 1$, the Thompson subgroup of P. By Step 1, $\mathfrak{N}_G(\mathfrak{Z}(P))$ and $\mathfrak{N}_G(\mathbb{T}(P))$ are both nilpotent and hence so is $\mathfrak{C}_G(\mathfrak{Z}(P))$. Thus, by Thompson's Theorem [3, Theorem 7.1], G has a normal p-complement N. Then $N \neq 1$, $N \neq G$ and N is characteristic in G, so σ-stable. Thus, by Step 1, $\mathfrak{N}_G(N) = G$ is nilpotent, certainly a contradiction. □

With this final contradiction, the theorem is proved. □

We remark that the above proof is reasonably constructive modulo dealing with some aspects of the group theory. Indeed, Lemmas 2.1 and 2.2 offer concrete constructions, while the group $G = P \rtimes Q$ which we handled with Theorem 1.1 can be dealt with more explicitly using [2, Examples 2.5 and 2.7].

3. Nilpotent examples

In view of Theorem 1.2, the existence of free bicyclic pairs is reduced to the case of nilpotent groups, that is, to direct products of p-groups. Interesting nonabelian p-group examples were offered in [2] to show that free bicyclic pairs need not exist in these situations.

Recall that a finite p-group P is said to be extra-special if $\mathfrak{Z}(P) = P'$ has order p. Here, of course, $Z = \mathfrak{Z}(P)$ is the center of P, and P' is the commutator subgroup. Such groups can be described as a finite direct product of nonabelian groups of order p^3 with their centers identified. Indeed, there are just two isomorphism classes of such groups for any fixed order p^{2n+1}. If p is odd, we can take all but one of the n direct factors to be the nonabelian p-group of order p^3 and period p, while the last factor has either period p or p^2. When $p = 2$, we can take all but one of the n direct factors to be the dihedral group D_8 of order 8, while the last factor is either dihedral D_8 or quaternion Q_8.

It is easy to see that all such groups admit automorphisms σ of order 2 that invert P/Z, that is, act like the inverse map on this abelian quotient group. In particular, such groups admit involutions * that act like the identity on P/Z. The following is a more precise version of [2, Example 2.2].

Example 3.1. Let p be an odd prime and let P be an extra-special p-group with center Z. If * is an involution of P that is the identity on P/Z, then $\mathbb{Z}[P]$ has no free bicyclic pairs (u, u^*). Indeed, if $u = 1 + \mu$, then either $\mu\mu^* = 0$ or $\mu^*\mu = 0$.

It is a fairly simple matter to extend this result to nilpotent groups that are not p-groups by adjoining abelian factors. Specifically, we have

Example 3.2. Let p be an odd prime and let P be an extra-special p-group with center Z. Set $G = P \times Q$ where Q is any abelian p'-group and assume that * is an involution on G that is the identity on P/Z. Then $\mathbb{Z}[G]$ has no free bicyclic pairs.

Proof. Let $X = \langle x \rangle$ be a cyclic subgroup of G and write $x = bc$ with $b \in P$ and $c \in Q$. Then $X = B \times C$ where $B = \langle b \rangle \subseteq P$ and $C = \langle c \rangle \subseteq Q$. In particular, $\widehat{X} = \widehat{B}\cdot\widehat{C} = \widehat{C}\cdot\widehat{B}$ and thus $c\widehat{X} = \widehat{X}$. Now consider $\mu = (1 - x)g\widehat{X}$ with $g \in G$, and write $g = ad$ with $a \in P$ and $d \in Q$. Since c and d are central in G, we see first that $xg\widehat{X} = bgc\widehat{X} = bg\widehat{X}$ and then that $\mu = (1-b)g\widehat{X} = (1-b)a\widehat{B}\cdot d\widehat{C} = \beta\cdot\gamma$, where $\beta = (1-b)a\widehat{B}$ is used to construct a bicyclic unit in $\mathbb{Z}[P]$ and where $\gamma = d\widehat{C}$ is central in $\mathbb{Z}[G]$. But, by Example 3.1 applied to $\mathbb{Z}[P]$, we have either $\beta\beta^* = 0$ or $\beta^*\beta = 0$. Therefore, we conclude that either $\mu\mu^* = 0$ or $\mu^*\mu = 0$.

This implies easily that the units $u = 1+\mu$ and $u^* = 1+\mu^*$ do not generate a free group of rank 2. Indeed, if $\mu\mu^* = 0$, then $u^i(u^*)^i = 1 + i(\mu+\mu^*)$, so all these elements commute, a contradiction. Similarly, the group is not free if $\mu^*\mu = 0$. □

The situation for extra-special 2-groups is slightly different. The following is [2, Lemma 2.4].

Example 3.3. Let P be an extra-special 2-group with center Z and let * be an involution of P that is the identity on P/Z. Then $\mathbb{Z}[P]$ has a bicyclic unit u with (u, u^*) a free pair unless $|P| = 8$ and * moves all noncentral elements of order 2.

In particular, we see that there is involution * defined on the dihedral group $P = D_8$ of order 8 with the property that $\mathbb{Z}[P]$ does not have any free bicyclic pairs. Notice that if * is the identity on P/Z as above and if $x \in P$, then $x^* = xz$ for some $z \in \mathfrak{Z}(P)$, and hence x and x^* commute. Thus the involution given below in the following two examples is of a decidedly different nature. We include these examples to indicate how explicit constructions can be achieved.

Example 3.4. Let the group G admit an involution * and suppose that $x \in G$ is an element order 2 that does not commute with x^*. Assume that $w = xx^*$ has order $n \geq 4$. If $u = 1+\mu \in \mathbb{Z}[G]$ with $\mu = (1-x)x^*(1+x)$, then (u, u^*) is a free bicyclic pair in the integral group ring.

Proof. It suffices to assume that $G = \langle x, x^*\rangle$ so that G is a dihedral group. Indeed, if $w = xx^* \in G$, then $G = \langle w, x\rangle$ and $x^{-1}wx = x^*x = w^{-1}$. Since x and x^* do not commute, $w \neq w^{-1}$ and hence w has order $n \geq 3$, and $G \cong D_{2n}$. We now construct a homomorphism φ from $\mathbb{Z}[G]$ to $\mathbf{M}_2(\mathbb{C})$, the 2×2 matrix ring over the complex numbers $\mathbb{C}$, by defining

$$\varphi(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \varphi(w) = \begin{bmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon} \end{bmatrix},$$

where ε is a complex nth root of unity (not necessarily primitive) with complex conjugate $\bar{\varepsilon} = \varepsilon^{-1}$. Since $x^* = xw$, we also have

$$\varphi(x^*) = \begin{bmatrix} 0 & \bar{\varepsilon} \\ \varepsilon & 0 \end{bmatrix}.$$

Now $\mu = (1-x)x^*(1+x)$ and $\mu^* = (1+x^*)x(1-x^*)$, so

$$\varphi(\mu) = (\varepsilon - \bar{\varepsilon})\cdot \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \varphi(\mu*) = (\varepsilon - \bar{\varepsilon})\cdot \begin{bmatrix} -1 & \bar{\varepsilon} \\ -\varepsilon & 1 \end{bmatrix}.$$

Thus

$$\varphi(\mu\mu^*) = (\varepsilon - \bar{\varepsilon})^2\cdot \begin{bmatrix} 1+\varepsilon & -1-\bar{\varepsilon} \\ -1-\varepsilon & 1+\bar{\varepsilon} \end{bmatrix}$$

and this matrix has trace

$$T = \mathrm{Tr}(\varphi(\mu\mu^*)) = (\varepsilon - \bar{\varepsilon})^2\cdot(2 + \varepsilon + \bar{\varepsilon}).$$

In particular, if $\varepsilon = \cos\theta + i\sin\theta$, then $T = 8\sin^2\theta\cdot(1+\cos\theta) = 8(1-t^2)(1+t)$, where we set $t = \cos\theta$. Our goal is to show that ε can be chosen so that $|T| > 4$.

To this end, first note that $T = T(t)$ is a cubic in t with $T(-1) = T(1) = 0$ and with $T(t) \geq 0$ in the interval $[-1, 1]$. Furthermore, its derivative is given by $dT/dt = 8(1+t)(1-3t)$, so T is increasing in $[-1, 1/3]$ and decreasing in $[1/3, 1]$. With this, it is easy to check that $T(t) > 4$ in the interval $[-0.4, 0.8]$. Next, since $\varepsilon^n = 1$, θ must equal $2\pi r/n$ for some integer r and we choose r so that θ is close to $\pi/2$. Indeed, we can write $n = 4k + a$ where $a = -1, 0, 1$ or 2 and we choose $r = k$. Then $t = \cos\theta = \sin(\pi/2 - \theta)$ and $\pi/2 - \theta = \pi/2 - 2k\pi/n = \pi a/2n$. Since $n \geq 4$, the values of a imply that $-\pi/8 \leq \pi/2 - \theta \leq \pi/4$ and hence

$$t = \sin(\pi/2 - \theta) \in [-0.39, 0.71] \subseteq [-0.4, 0.8],$$

as required.

We conclude that $T = \mathrm{Tr}(\varphi(\mu\mu^*)) > 4$. Note that $\mu^2 = 0$, so $\varphi(\mu\mu^*)$ is a singular matrix and has an eigenvalue equal to 0. Thus the second eigenvalue is equal to the trace T, so it is real and larger than 4. [2, Lemma 1.2] now implies that $\varphi(1+\mu)$ and $\varphi(1+\mu^*)$ generate a free group of rank 2 in the units of $\mathbf{M}_2(\mathbb{C})$, and hence $u = 1 + \mu$ and $u^* = 1 + \mu^*$ form a free bicyclic pair in $\mathbb{Z}[G]$. □

It remains to handle $n = 3$. Since the same argument applies to all odd integers n, we include this more general construction.

Example 3.5. Let the group G admit an involution * and suppose that $x \in G$ is an element order 2 that does not commute with x^*. Assume that $w = xx^*$ has odd order $n \geq 3$ and write $n = 2k + 1$. Then $y = w^k x$ is a *-stable element of order 2 in G. If $u = 1 + \mu \in \mathbb{Z}[G]$ with $\mu = (1-y)x(1+y)$, then (u, u^*) is a free bicyclic pair in the integral group ring.

Proof. Again, it suffices to assume that $G = \langle x, x^* \rangle$, a dihedral group of order $2n$. Here $w = xx^*$ is *-stable and generates the normal cyclic subgroup W of G of order $n = 2k + 1$. In particular, if $y = w^k x$, then y has order 2 and

$$y^* = x^* w^k = x^*(w^{-1})^{k+1} = w^{k+1} x^* = w^k x = y.$$

Recall that the group ring trace map $\mathrm{tr}\colon \mathbb{Z}[G] \to \mathbb{Z}$ reads off the identity coefficient of each element of $\mathbb{Z}[G]$. It is linear and satisfies $\mathrm{tr}\,\alpha\beta = \mathrm{tr}\,\beta\alpha$ for all $\alpha, \beta \in \mathbb{Z}[G]$. In particular,

$$\begin{aligned} \mathrm{tr}\,\mu\mu^* &= \mathrm{tr}(1-y)x(1+y)\cdot(1+y)x^*(1-y) \\ &= \mathrm{tr}(1-y)^2 x(1+y)^2 x^* = 4\,\mathrm{tr}(1-y)x(1+y)x^* \\ &= 4\,\mathrm{tr}(xx^* + xyx^* - yxx^* - yxyx^*), \end{aligned}$$

since $(1-y)^2 = 2(1-y)$ and $(1+y)^2 = 2(1+y)$. Now $xx^* = w \neq 1$ and $xyx^*, yxx^* \in G \setminus W$. On the other hand, $yxyx^* = w^k x\cdot x\cdot w^k x\cdot x^* = w^{2k+1} = 1$.

Thus $\mathrm{tr}\,\mu\mu^* = -4$ and we conclude from [2, Lemma 1.3] that $u = 1 + \mu$ and $u^* = 1 + \mu^*$ form a free bicyclic pair. □

References

[1] A. Dooms, E. Jespers, and M. Ruiz, *Free groups and subgroups of finite index in the unit group of an integral group ring*, Comm. Algebra **35** (2007), no. 9, 2879–2888, DOI 10.1080/00927870701404259. MR2356305 (2008h:16024)

[2] J. Z. Gonçalves and D. S. Passman, *Involutions and free pairs of bicyclic units in integral group rings*, J. Group Theory **13** (2010), no. 5, 721–742, DOI 10.1515/JGT.2010.019. MR2720200 (2011i:16030)

[3] I. Martin Isaacs, *Finite group theory*, Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, 2008. MR2426855 (2009e:20029)
[4] Zbigniew S. Marciniak and Sudarshan K. Sehgal, *Constructing free subgroups of integral group ring units*, Proc. Amer. Math. Soc. **125** (1997), no. 4, 1005–1009, DOI 10.1090/S0002-9939-97-03812-4. MR1376998 (97f:16057)

Department of Mathematics, University of São Paulo, São Paulo, 05389-970, Brazil
E-mail address: jz.goncalves@usp.br

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706
E-mail address: passman@math.wisc.edu