PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 7, July 2015, Pages 2873–2886
S 0002-9939(2015)12240-X
Article electronically published on March 18, 2015

MAHLER MEASURE AND THE WZ ALGORITHM

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(Communicated by Matthew A. Papanikolas)

Abstract. We use the Wilf-Zeilberger method to prove identities between Mahler measures of polynomials. In particular, we offer a new proof of a formula due to Lalín, and we show how to translate the identity into a formula involving elliptic dilogarithms. This work settles a challenge problem proposed by Kontsevich and Zagier in their paper, "Periods".

1. Introduction

In this paper we use the Wilf-Zeilberger algorithm to prove relations between Mahler measures of polynomials. The (logarithmic) Mahler measure of an n-variable Laurent polynomial, $P(x_1, \ldots, x_n)$, is defined by

$$\mathrm{m}(P) := \int_0^1 \ldots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right)\right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.$$

We are primarily interested in the following special function:

$$m(\alpha) := \mathrm{m}\left(\alpha + x + \frac{1}{x} + y + \frac{1}{y}\right),$$

because there are many conjectural formulas relating special values of $m(\alpha)$ to values of L-functions attached to elliptic curves. Deninger hypothesized that $m(1)$ should be a rational multiple of $L(E_{15}, 2)/\pi^2$, where E_{15} is a conductor 15 elliptic curve [9]. Boyd used numerical calculations to make the constant explicit [7]:

$$m(1) = \frac{15}{4\pi^2} L(E_{15}, 2). \tag{1.1}$$

The second author and Zudilin proved formula (1.1) quite recently [23]. We note that Boyd's paper contains dozens of additional formulas for $m(\alpha)$, and most of those remain open.

It is usually much easier to prove identities between Mahler measures, than to prove formulas relating them to L-functions. One important intermediate step in the proof of (1.1), is to show that

$$11m(1) = m(16). \tag{1.2}$$

Formula (1.2) has been the subject of several questions and papers. The first proof of (1.2) is due to Lalín [16]. She used the fact that Mahler measures can be interpreted as values of regulator maps on K_2 groups of elliptic curves. The connection to algebraic K-theory was outlined and exploited by Rodriguez-Villegas

Received by the editors March 6, 2013 and, in revised form, March 29, 2013.
2010 *Mathematics Subject Classification.* Primary 33C20, 33F10; Secondary 19F27.
The second author was supported by the National Science Foundation award DMS-0803107.

in [19]. An equivalent version of formula (1.2) appears in the paper, "Periods", by Kontsevich and Zagier [14]. They asked if the relation

$$6m(1) = m(5) \tag{1.3}$$

can be proved with elementary calculus [14, pg. 9], [24, pg. 56]. Mahler measures are examples of *periods* - numbers which can be expressed as multiple integrals of algebraic functions, over domains described by algebraic equations. Kontsevich and Zagier conjectured that any relation between periods should be provable with only "the rules of calculus". They suggested finding an elementary proof of (1.3) as a challenge problem. The equivalence of (1.2) and (1.3) follows easily from a result of Kurokawa and Ochiai [15]:

$$m(1) + m(16) = 2m(5). \tag{1.4}$$

In the first section of this paper we present an elementary proof of (1.2), answering the Kontsevich-Zagier challenge. The most difficult part of the proof is to establish formula (2.12) below, and we accomplish this using the Wilf-Zeilberger method.

In the second portion of the paper, we present several new q-series expansions for Mahler measures. If $\varphi(q)$ denotes the standard theta function

$$\varphi(q) := \sum_{n\in\mathbb{Z}} q^{n^2}, \tag{1.5}$$

then

$$m\left(4\frac{\varphi^2(q)}{\varphi^2(-q)}\right) = \frac{4}{\pi}\sum_{n\in\mathbb{Z}} D(iq^n), \tag{1.6}$$

where $D(z)$ is the Bloch-Wigner dilogarithm. These results provide an easy way to translate between Mahler measures and elliptic dilogarithms. In Theorem 3.3 we translate an exotic relation due to Bertin into an identity between Mahler measures [4]. We conclude by briefly comparing our new results to Ramanujan's formulas for $1/\pi$.

2. An application of the WZ method

We begin with a brief review of the WZ method. We say that $F(n,k)$ is hypergeometric, if $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k. Two hypergeometric functions are called a WZ-pair if they satisfy the following functional equation:

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{2.1}$$

Wilf and Zeilberger proved that if $F(n,k)$ satisfies (2.1), then it is always possible to determine $G(n,k)$ (see [18] and [27]). Their algorithm has been implemented in `Maple` and `Mathematica`.

Let us consider WZ-pairs where F and G are meromorphic functions of n and k. If we sum both sides of (2.1) from $n = 0$ to $n = \infty$, the left-hand side of the equation telescopes, and we have

$$-F(0,k) + \lim_{n\to\infty} F(n,k) = \sum_{n=0}^{\infty} G(n,k+1) - \sum_{n=0}^{\infty} G(n,k).$$

In instances where $F(0,k)=0$, and $\lim_{n\to\infty} F(n,k)=0$, this becomes

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,k+1).$$

It follows immediately that the series is periodic with respect to k. If the series also converges uniformly, and j is an integer, then we can write

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} \lim_{j\to\infty} G(n,k+j).$$

If $\lim_{j\to\infty} G(n,k+j)$ is independent of k, then we can conclude that for unrestricted k:

$$\sum_{n=0}^{\infty} G(n,k) = \text{constant}. \tag{2.2}$$

We use this method to prove Theorem 2.2 below. Theorem 2.2 is the key result we need to establish the relation between $m(1)$ and $m(16)$.

In order to apply the WZ method to equation (1.2), we need to relate the Mahler measures to hypergeometric functions. We use several of the identities summarized in [21]. If $r \in (0,1]$, results from [15] and [20] show that

$$m\left(\frac{4}{r}\right) = \log\left(\frac{4}{r}\right) - \sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{(r/4)^{2n}}{2n}, \tag{2.3}$$

$$m\left(4r\right) = 4\sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{(r/4)^{2n+1}}{2n+1}. \tag{2.4}$$

Both of these sums depend upon the same binomial coefficients. Therefore, if we define s by

$$s := \frac{m\left(4/r\right)}{m\left(4r\right)},$$

we can form a linear combination of (2.3) and (2.4) to obtain

$$\log\left(\frac{4}{r}\right) = rs + \sum_{n=1}^{\infty} \frac{(2(1+rs)n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \left(\frac{r}{4}\right)^{2n}. \tag{2.5}$$

It follows that $(r,s) \in \bar{\mathbb{Q}}^2$ and $r \in (0,1]$, *if and only if* (2.5) also gives an explicit formula for an algebraic hypergeometric series. By linearity, explicit cases of (2.5) immediately imply formulas for s. Notice that (2.5) diverges when $|r| > 1$.

Theorem 2.1. *The following formulas are true:*

$$2\log(2) = 1 + \sum_{n=1}^{\infty} \frac{(4n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{4n}}, \tag{2.6}$$

$$3\log(2) = 2 + \sum_{n=1}^{\infty} \frac{(6n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{6n}}, \tag{2.7}$$

$$8\log(2) = \frac{11}{2} + \sum_{n=1}^{\infty} \frac{(15n+2)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{8n}}. \tag{2.8}$$

Furthermore, (2.7) *is equivalent to*

$$4m(2) = m(8), \tag{2.9}$$

and (2.8) *is equivalent to*

$$11m(1) = m(16). \tag{2.10}$$

So far we have not been able to prove equation (2.9) with the WZ method. This is surprising, because the K-theoretic proof of the relation between $m(2)$ and $m(8)$ [17] is much easier than the K-theoretic proof of the relation between $m(1)$ and $m(16)$. In order to prove (2.10), we must first prove equation (2.12) below. It is interesting to note that `Mathematica` can recognize (2.6), but not (2.7) or (2.8). While it is possible to derive (2.6) from Dougall's theorem [25], it seems that equations (2.7) and (2.8) are not as easily accessible.

Theorem 2.2. *The following identities are true:*

$$\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(4n+2x+1)}{(2n+1)(n+x)} \frac{\left(\frac{1}{2}+x\right)_n}{(1+x)_n} \binom{2n}{n} \frac{1}{2^{2n}}, \tag{2.11}$$

$$4\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(2(2n+1)^2(15n+2)+xP(n,x))}{(2n+1)(2n+x)(2n+x+1)^2} \frac{\left(\frac{1}{2}+x\right)_n^2}{\left(1+\frac{x}{2}\right)_n \left(\frac{1+x}{2}\right)_n} \binom{2n}{n} \frac{1}{2^{6n}}, \tag{2.12}$$

where $P(n,x) = (2n+1)(86n+19)+4x(20n+7)+12x^2$. *The Pochhammer symbol is given by* $(x)_m := \Gamma(x+m)/\Gamma(x)$.

Proof. We begin by proving (2.11). Consider the following WZ-pair:

$$\begin{aligned} F(n,k) &= -\frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_k^2}{(1+k)_n (1)_n (1)_k^2} \cdot \frac{n}{2(n+k)}, \\ G(n,k) &= \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_k^2}{(1+k)_n (1)_n (1)_k^2} \cdot \frac{k(4n+2k+1)}{2(n+k)(2n+1)}. \end{aligned} \tag{2.13}$$

It is easy to see that $F(0,k) = 0$, and $\lim_{n\to\infty} F(n,k) = 0$. Since $\sum_{n=0}^{\infty} G(n,k)$ converges uniformly, we conclude from the previous discussion that

$$\begin{aligned} \sum_{n=0}^{\infty} G(n,k) &= \sum_{n=0}^{\infty} \lim_{j\to\infty} G(n,k+j) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \binom{2n}{n} \frac{1}{2^{2n}} = \frac{\arcsin(1)}{\pi} = \frac{1}{2}. \end{aligned}$$

Rearrange the formula, and let $k \to x$ to complete the proof of (2.11).

The proof of (2.12) is similar. Consider the following WZ-pair:

$$\begin{aligned} F(n,k) &= -U(n,k) \cdot \frac{4n}{2n+k}, \\ G(n,k) &= U(n,k) \cdot \frac{2(15n+2)(2n+1)^2 + kP(n,k)}{(2n+k+1)^2(2n+k)(2n+1)} \cdot \frac{k}{2}, \end{aligned} \tag{2.14}$$

where

$$U(n,k) = \frac{1}{16^n} \frac{\left(\frac{1}{2}+k\right)_n^2 \left(\frac{1}{2}\right)_n}{\left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n (1)_n} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2},$$

and $P(n,k)$ is given in the statement of the theorem. Then, $F(0,k) = 0$, $\lim_{n\to\infty} F(n,k) = 0$, and $\sum_{n=0}^{\infty} G(n,k)$ then converges uniformly, so we have

$$\begin{aligned}\sum_{n=0}^{\infty} G(n,k) &= \sum_{n=0}^{\infty} \lim_{j\to\infty} G(n,k+j) \\ &= \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \binom{2n}{n} \frac{1}{2^{4n}} = \frac{12 \arcsin(1/2)}{\pi} = 2.\end{aligned}$$

Rearranging the final result and relabeling k as x completes the proof of (2.12). □

Proof of Theorem 2.1. The shortest proof of (2.6) follows from using the definition of s, to show that $s = 1$ when $r = 1$. An alternative proof follows from using (2.11) to show that

$$\begin{aligned}2 + 2\sum_{n=1}^{\infty} \frac{(4n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{4n}} &= \lim_{x\to 0} \left(\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2(x+1/2)} - \frac{1}{x} \right) \\ &= 4\log(2).\end{aligned}$$

Similarly, (2.8) follows from using (2.12), to show that

$$\begin{aligned}11 + 2\sum_{n=1}^{\infty} \frac{(15n+2)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{8n}} &= \lim_{x\to 0} \left(4\pi \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2(x+1/2)} - \frac{4}{x} \right) \\ &= 16\log(2).\end{aligned}$$

□

We can also use our newly discovered WZ-pairs to obtain some formulas for $\zeta(3)$.

Theorem 2.3. *The following identities are true:*

$$\zeta(3) = \frac{2}{7} \sum_{n=0}^{\infty} \frac{(4n+3)16^n}{(2n+1)^3(n+1)\binom{2n}{n}^2}, \tag{2.15}$$

$$\zeta(3) \stackrel{?}{=} \frac{4}{7} \sum_{n=0}^{\infty} \frac{(3n+2)4^n}{(2n+1)^3(n+1)\binom{2n}{n}^2}, \tag{2.16}$$

$$\zeta(3) = \frac{1}{16} \sum_{n=0}^{\infty} \frac{(30n+19)}{(2n+1)^3(n+1)\binom{2n}{n}^2}. \tag{2.17}$$

Proof. The idea is that shifting the summation in formulas (2.6), (2.7), and (2.8) by $n \to n - 1/2$ changes them into formulas for $\zeta(3)$ [12]. The summand in (2.7) becomes

$$\frac{(6n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{6n}} \to \frac{2(3n+2)4^n}{\pi^2(2n+1)^3(n+1)\binom{2n}{n}^2}.$$

To make this observation rigorous, we prove (2.17) and (2.15) by using the WZ-pairs from Theorem 2.2. A rigorous proof of (2.16) should be easy to construct by first finding a WZ proof of (2.7).

Shift both sides of (2.1) by $\frac{1}{2}$ and y, and then sum the equation from $n=0$ to $n=\infty$. Under the hypothesis that $\lim_{n\to\infty} F\left(n+\frac{1}{2},k+y\right)=0$, we have

$$F\left(\frac{1}{2},k+y\right)=-\sum_{n=0}^{\infty}G\left(n+\frac{1}{2},k+y+1\right)+\sum_{n=0}^{\infty}G\left(n+\frac{1}{2},k+y\right).$$

The right-hand side of the formula telescopes with respect to k. Sum both sides of the equation from $k=0$ to $k=\infty$:

$$\sum_{k=0}^{\infty}F\left(\frac{1}{2},k+y\right)=-\lim_{k\to\infty}\sum_{n=0}^{\infty}G\left(n+\frac{1}{2},k+y\right)+\sum_{n=0}^{\infty}G\left(n+\frac{1}{2},y\right).$$

In the cases we consider, all three sums converge uniformly, the limits are independent of y, and $G\left(n+\frac{1}{2},0\right)=0$. Differentiating with respect to y, and using the notation $F^*(n,k)=\frac{\partial}{\partial k}F(n,k)$, and $G^*(n,k)=\frac{\partial}{\partial k}G(n,k)$, reduce the formula to

$$\sum_{k=0}^{\infty}F^*\left(\frac{1}{2},k\right)=\sum_{n=0}^{\infty}G^*\left(n+\frac{1}{2},0\right). \tag{2.18}$$

Notice that $G^*\left(n+\frac{1}{2},0\right)=\lim_{y\to 0}\frac{G\left(n+\frac{1}{2},y\right)}{y}$.

If we use F and G from (2.13), we obtain

$$\sum_{n=0}^{\infty}G^*\left(n+\frac{1}{2},0\right)=\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{(4n+3)16^n}{(2n+1)^3(n+1)\binom{2n}{n}^2}.$$

The sum involving F^* is easy to evaluate. Notice that

$$F\left(\frac{1}{2},k\right)=-\frac{2}{\pi^2(2k+1)^2},$$

and therefore

$$\sum_{k=0}^{\infty}F^*\left(\frac{1}{2},k\right)=\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}=\frac{7\zeta(3)}{\pi^2}.$$

Formula (2.15) follows from substituting these results into (2.18).

In order to prove (2.17), we use the WZ-pair given in (2.14). Observe that

$$\sum_{n=0}^{\infty}G^*\left(n+\frac{1}{2},0\right)=\frac{1}{4\pi^2}\sum_{n=0}^{\infty}\frac{(30n+19)}{(2n+1)^3(n+1)\binom{2n}{n}^2},$$

which matches the right-hand side of (2.17) up to a constant. To evaluate the F^*-sum, notice that

$$F\left(\frac{1}{2},k\right)=\frac{-2}{\pi^2(k+1)^2},$$

and therefore

$$\sum_{k=0}^{\infty}F^*\left(\frac{1}{2},k\right)=\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(k+1)^3}=\frac{4\zeta(3)}{\pi^2}.$$

Substituting the last two results into (2.18) completes the proof of (2.17). □

Notice that Gosper first proved equation (2.17) [11], and Batir proved (2.15) using log-sine integrals (combine formulas 3 and 4 on page 664 of [1]). Formula (2.16) is numerically true, but it remains open.

We conclude this section by showing that it is also possible to find WZ-pairs when (2.5) diverges. If we consider the WZ-pair

$$\begin{aligned} F(n,k) =& \frac{n}{(2n+k)^2} \cdot \frac{\left(\frac{1}{2}\right)_n \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}{(1)_n (1+k)_n^2} \cdot 16^n, \\ G(n,k) =& -\frac{P(n,k)}{n(2n+k)^2(1+2n+k)} \cdot \frac{\left(\frac{1}{2}\right)_n \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}{(1)_n (1+k)_n^2} \cdot 16^n, \end{aligned}$$

where $P(n,k) = 3k^3 + k^2(20n+3) + kn(43n+12) + n^2(30n+11)$, then it is possible to obtain a finite summation identity,

$$\begin{aligned} \sum_{n=1}^{m-1} \frac{30n+11}{(2n)(2n+1)} \binom{2n}{n}^2 =& -4 + 6 \sum_{n=1}^{m-1} \frac{1}{n} \binom{2n}{n} \\ &+ \frac{1}{2m} \binom{2m}{m}^2 {}_4F_3\left(\begin{smallmatrix} 1,1,2m,2m \\ m+1,m+1,2m+1 \end{smallmatrix}; 1\right), \end{aligned} \tag{2.19}$$

which holds for $m \in \mathbb{N}$. This formula corresponds to the values $(r,s) = (4, 1/11)$, but notice that equation (2.10) already shows that $s = 1/11$ when $r = 4$.

3. Connections with the elliptic dilogarithm

In Section 2 we used the WZ method to establish a relation between Mahler measures. In this section, we show that our techniques provide a new way to prove relations between elliptic dilogarithms. Briefly recall the definitions of $m(\alpha)$ and $n(\alpha)$:

$$\begin{aligned} m(\alpha) :=& \mathrm{m}\left(\alpha + x + x^{-1} + y + y^{-1}\right), \\ n(\alpha) :=& \mathrm{m}\left(x^3 + y^3 + 1 - \alpha x y\right). \end{aligned}$$

We examined $m(\alpha)$ in the previous section, and $n(\alpha)$ has been studied in papers such as [20], [21] and [17]. We begin by proving new q-series expansions for both functions.

Theorem 3.1. *Suppose that $q \in (-1,1)$, and let $D(z) = \Im\left(\mathrm{Li}_2(z) + \log|z| \log(1-z)\right)$ denote the Bloch-Wigner dilogarithm. The following formulas are true:*

$$\frac{4}{\pi} \sum_{n \in \mathbb{Z}} D\left(iq^n\right) = m\left(4 \frac{\varphi^2(q)}{\varphi^2(-q)}\right), \tag{3.1}$$

$$\frac{9}{2\pi} \sum_{n \in \mathbb{Z}} D\left(e^{2\pi i/3} q^n\right) = n\left(3 \frac{a(q)}{b(q)}\right), \tag{3.2}$$

$$\frac{9}{\pi} \sum_{n \in \mathbb{Z}} D\left(e^{\pi i/3} q^n\right) = 2n\left(3 \frac{a(q)}{b(q)}\right) + n\left(3 \frac{a(q^2)}{b(q^2)}\right). \tag{3.3}$$

The signature-3 theta functions are given by

$$a(q) := \sum_{(n,m) \in \mathbb{Z}^2} q^{n^2+mn+m^2}, \qquad b(q) := \sum_{(n,m) \in \mathbb{Z}^2} \omega^{n-m} q^{n^2+mn+m^2},$$

where $\omega = e^{2\pi i/3}$ [3].

Proof. First notice that (3.2) implies (3.3). By elementary functional equations for the Bloch-Wigner dilogarithm, $\frac{1}{2}D(z^2) = D(z) + D(-z)$, and $D(z) = -D(\frac{1}{z})$, it is possible to obtain

$$D\left(e^{\pi i/3}q^n\right) = D\left(e^{2\pi i/3}q^{-n}\right) + \frac{1}{2}D\left(e^{2\pi i/3}q^{2n}\right).$$

Summing over n shows that (3.2) implies (3.3).

To prove (3.1) and (3.2), we use an idea described in [21, Section 8]. Consider the following formula from [22] (Rodriguez-Villegas first proved a version of this formula in [20]):

$$\frac{\pi^2}{32x} m\left(4\sqrt{1-\frac{\varphi^4(-q)}{\varphi^4(q)}}\right) = \sum_{\substack{n=0\\k=0}}^{\infty} \frac{(-1)^n(2n+1)}{\left((2n+1)^2+x^2(2k+1)^2\right)^2},$$

where $q = e^{-\pi x}$, and $x > 0$. Express the sum as an integral, and then apply the involution for the weight-1/2 theta function:

$$\begin{aligned}
&= \frac{\pi^2}{16}\int_0^\infty u\left(\sum_{k=0}^{\infty} e^{-\pi(k+1/2)^2x^2u}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\pi(n+1/2)^2u}\right)\mathrm{d}u\\
&= \frac{\pi^2}{32x}\int_0^\infty \sqrt{u}\left(\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi k^2}{x^2u}}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\pi(n+1/2)^2u}\right)\mathrm{d}u\\
&= \frac{\pi}{8x}\sum_{k=-\infty}^{\infty}(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\left(\frac{\pi|k|}{x}+\frac{1}{(2n+1)}\right)e^{-\pi(2n+1)|k|/x}\\
&= \frac{\pi}{8x}\sum_{k=-\infty}^{\infty}(-1)^k D\left(ie^{-\pi|k|/x}\right).
\end{aligned}$$

If we let $x \to 1/x$ and then use the following identity:

$$\frac{\varphi^4(-q)}{\varphi^4(q)} = 1 - \frac{\varphi^4(-e^{-\pi/x})}{\varphi^4(e^{-\pi/x})},$$

it is easy to see that

$$m\left(4\frac{\varphi^2(-q)}{\varphi^2(q)}\right) = \frac{4}{\pi}\sum_{k\in\mathbb{Z}}(-1)^k D\left(iq^{|k|}\right) = \frac{4}{\pi}\sum_{k\in\mathbb{Z}} D\left(i(-q)^k\right).$$

The second step uses $(-1)^k D\left(iq^{|k|}\right) = D\left(i(-q)^k\right)$. Formula (3.1) then follows from sending $q \to -q$. A different proof can be constructed by differentiating (3.1) with respect to q and then applying the formulas of Ramanujan.

A proof of (3.2) can be obtained by looking at the following sum:

$$\sum_{(n,k)\in\mathbb{Z}^2}\frac{(3k+1)}{\left((3k+1)^2+x^2(2n+1)^2\right)^2}.$$

If the series is transformed into an integral of theta functions, then the involution for the weight-1/2 theta function leads to a dilogarithm sum, and the involution for the weight-3/2 theta function leads to Rodriguez-Villegas's q-series for $n(\alpha)$ (see formula (2-10) in [17]). □

For certain values of q the left-hand sides of equations (3.1), (3.2), and (3.3) equal elliptic dilogarithms. To see this, briefly consider an elliptic curve

$$E : y^2 = 4x^3 - g_2 x - g_3.$$

Then E can be parameterized by $(x, y) = (\wp(u), \wp'(u))$, where $\wp(u)$ is the Weierstrass function. The periods of $\wp(u)$ are denoted ω and ω', and the period ratio $\tau = \frac{\omega'}{\omega}$ is assumed to have $\Im(\tau) > 0$. If $P = (\wp(u), \wp'(u))$ denotes an arbitrary point on E, and $q = e^{2\pi i\tau}$, then the elliptic dilogarithm is defined by

$$D^E(P) := \sum_{n\in\mathbb{Z}} D\left(e^{2\pi iu/\omega} q^n\right).$$

Since we will be interested only in torsion points, we can assume that $u = a\omega + b\omega'$, for some $(a, b) \in \mathbb{Q}^2$. For appropriate choices of E, the series expansions in Theorem 3.1 equal $D^E(P)$ at 3, 4 and 6-torsion points.

Now we focus on equation (3.1). In order to translate the right-hand side into elliptic dilogarithms, we must identify the relevant elliptic curves and torsion points. Let us set $\beta = 1 - \varphi^4(-q)/\varphi^4(q)$. The classical theory of elliptic functions shows that we can calculate q as a function of β:

$$q = e^{2\pi i\tau} = \exp\left(-\pi \frac{{}_2F_1\left(\begin{smallmatrix}\frac{1}{2},\frac{1}{2}\\ 1\end{smallmatrix}; 1-\beta\right)}{{}_2F_1\left(\begin{smallmatrix}\frac{1}{2},\frac{1}{2}\\ 1\end{smallmatrix}; \beta\right)}\right). \tag{3.4}$$

It is known that g_2 and g_3 are also functions of q [26]. In Ramanujan's notation we have $g_2 = \frac{4}{3}\pi^4 M(q)$, and $g_3 = \frac{8}{27}\pi^6 N(q)$ [2, pg. 126]. Applying formulas (13.3) and (13.4) in [2, pg. 127], we obtain

$$J(\tau) = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{\left(1 + 14\beta + \beta^2\right)^3}{108\beta(1-\beta)^4}. \tag{3.5}$$

This relation allows us to easily translate between β and (g_2, g_3). We have six choices of β for every g_2^3/g_3^2, so caution must be exercised to pick the correct β. We often checked our work numerically by computing q from g_2 and g_3 (with the `Mathematica` function "WeierstrassHalfPeriods"), and then comparing it to calculations using (3.4).

Theorem 3.2. *Let $E(k, \ell)$ denote the elliptic curve*

$$y^2 = 4x^3 - 27(k^4 - 16k^2 + 16)\ell^2 x + 27\left(k^6 - 24k^4 + 120k^2 + 64\right)\ell^3.$$

Formula (2.10) *is equivalent to*

$$11D^{E_1}(P_1) = 6D^{E_2}(P_2), \tag{3.6}$$

where $E_1 = E(5, 2)$, $E_2 = E(16, 1/2)$, $P_1 = (87, 1080)$, and $P_2 = (195, 432)$.

Formula (2.9) *is equivalent to*

$$5D^{E_3}(P_3) = 8D^{E_4}(P_4), \tag{3.7}$$

where $E_3 = E(8, 1/2)$, $E_4 = E(3\sqrt{2}, 1)$, $P_3 = (51, 216)$, and $P_4 = (33, 324)$.

Proof. If we set $\beta = 1 - 16/k^2$, then we can rearrange (3.5) to obtain

$$\frac{g_2^3}{g_3^2} = \frac{27\left(16 - 16k^2 + k^4\right)^3}{\left(64 + 120k^2 - 24k^4 + k^6\right)^2}.$$

Therefore, for some choice of ℓ, we have

$$g_2 = 27\left(k^4 - 16k^2 + 16\right)\ell^2,$$
$$g_3 = -27\left(k^6 - 24k^4 + 120k^2 + 64\right)\ell^3.$$

In practice, we choose ℓ so that $E(k,\ell)$ has a rational 4-torsion point P. We can then use $4\varphi^2(q)/\varphi^2(-q) = k$, along with equation (3.1), to conclude that

$$m(k) = D^{E(k,\ell)}(P).$$

Now we prove (3.6). It is easy to check that $E(5,2)$ has a 4-torsion point $P_1 = (87, 1080) = \left(\wp\left(\frac{\omega}{4}\right), \wp'\left(\frac{\omega}{4}\right)\right)$. It follows from the definition of the elliptic dilogarithm, that

$$m(5) = \frac{4}{\pi} D^{E(5,2)}(P_1).$$

A result from [17] shows that $m(1) + m(16) = 2m(5)$, and therefore we have proved that

$$m(1) + m(16) = \frac{8}{\pi} D^{E(5,2)}(P_1). \tag{3.8}$$

When $k = 16$, it is easy to see that $E(16, 1/2)$ has the 4-torsion point $P_2 = (195, 432) = \left(\wp\left(\frac{\omega}{4}\right), \wp'\left(\frac{\omega}{4}\right)\right)$. Using the definition of the elliptic dilogarithm, we conclude that

$$m(16) = \frac{4}{\pi} D^{E(16,1/2)}(P_2). \tag{3.9}$$

Substituting (3.8) and (3.9) into (2.10) completes the proof of (3.6).

The proof of (3.7) is more or less identical, except we require the relation $m(2) + m(8) = 2m(3\sqrt{2})$. □

The key point of Theorem 3.2 is that we can start from elliptic dilogarithm identities such as (3.6) or (3.7), translate both sides into hypergeometric functions, and then prove the hypergeometric identities with the WZ method. It seems plausible that some additional formulas involving elliptic dilogarithms might be provable with the WZ method. Bloch and Grayson conjectured several identities of the form:

$$\sum_r a_r D^E(rP) \stackrel{?}{=} 0,$$

which they refer to as "exotic relations" [5]. Zagier later proposed restrictions that E should satisfy in order to possess such a relation [10]. Theorem 3.1 implies that certain exotic relations are equivalent to formulas for hypergeometric functions. We conclude this section by translating an exotic relation due to Bertin into an identity between Mahler measures [4].

Theorem 3.3. *Let E denote the elliptic curve*

$$y^2 = 4x^3 - 432x + 1188,$$

and let $P = (-6, 54)$. Bertin's exotic relation,

$$16D^E(P) - 11D^E(2P) = 0, \tag{3.10}$$

is equivalent to

$$16n\left(\frac{7+\sqrt{5}}{\sqrt[3]{4}}\right) - 8n\left(\frac{7-\sqrt{5}}{\sqrt[3]{4}}\right) = 19n\left(\sqrt[3]{32}\right). \tag{3.11}$$

Proof. If we notice that $P = (-6, 54) = \left(\wp\left(\frac{\omega-3\omega'}{6}\right), \wp'\left(\frac{\omega-3\omega'}{6}\right)\right)$, then (3.10) is equivalent to

$$16\sum_{n\in\mathbb{Z}} D\left(e^{\pi i/3}q^{n-1/2}\right) = 11\sum_{n\in\mathbb{Z}} D\left(e^{2\pi i/3}q^{n}\right).$$

By formulas (3.2) and (3.3), this amounts to showing that

$$0 = 16n\left(3\frac{a(\sqrt{q})}{b(\sqrt{q})}\right) - 19n\left(3\frac{a(q)}{b(q)}\right) - 8n\left(3\frac{a(q^2)}{b(q^2)}\right).$$

We translate the ratios of theta functions into algebraic numbers below.

It is well known that the following inverse relation holds [6]:

$$\beta = \frac{c^3(q)}{a^3(q)}, \qquad q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\1\end{smallmatrix};1-\beta\right)}{{}_2F_1\left(\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\1\end{smallmatrix};\beta\right)}\right),$$

where $a(q)$ and $b(q)$ appear in Theorem 3.2, and $c(q)$ is given by

$$c(q) := \sum_{(n,m)\in\mathbb{Z}^2} q^{(n+1/3)^2+(m+1/3)(n+1/3)+(m+1/3)^2}.$$

It is also known that $a^3(q) = b^3(q) + c^3(q)$ [6]. By formulas (4.6) and (4.8) in [3, pg. 107], and the values $g_2 = \frac{4}{3}\pi^4 M(q) = 432$, and $g_3 = \frac{8}{27}\pi^6 N(q) = -1188$, we have

$$\frac{6912}{6971} = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{(1+8\beta)^3}{64\beta(1-\beta)^3}.$$

Therefore $\beta = \frac{5}{32}$. Since $\frac{a(q)}{b(q)} = \frac{1}{\sqrt[3]{1-\beta}}$, we have

$$\frac{a(q)}{b(q)} = \frac{1}{\sqrt[3]{1-\beta}} = \frac{\sqrt[3]{32}}{3}.$$

Finally, if we write $\frac{a^3(\sqrt{q})}{b^3(\sqrt{q})} = \frac{1}{1-\alpha}$ and $\frac{a^3(q^2)}{b^3(q^2)} = \frac{1}{1-\gamma}$, then α and γ are conjugate zeros of a second-degree modular polynomial with respect to β [17, pg. 94]:

$$27\alpha\beta(1-\alpha)(1-\beta) - (\alpha+\beta-2\alpha\beta)^3 = 0.$$

With the aid of a computer, we obtain $\frac{a(\sqrt{q})}{b(\sqrt{q})} = \frac{7+\sqrt{5}}{3\sqrt[3]{4}}$, and $\frac{a(q^2)}{b(q^2)} = \frac{7-\sqrt{5}}{3\sqrt[3]{4}}$. □

The hypergeometric form of $n(\alpha)$ is due to Rodriguez-Villegas (see formula (2-36) in [17]). If $|\alpha|$ is sufficiently large, then

$$n(\alpha) = \Re\left(\log\alpha - \frac{2}{\alpha^3}{}_4F_3\left(\begin{smallmatrix}\frac{4}{3},\frac{5}{3},1,1\\2,2,2\end{smallmatrix};\frac{27}{\alpha^3}\right)\right). \tag{3.12}$$

If follows that (3.11) can be rewritten as a series identity. Unfortunately, it seems doubtful that a WZ proof of (3.11) is possible. The arguments of the hypergeometric functions are irrational numbers, while virtually all of the known WZ proofs deal with rational hypergeometric functions [13].

4. Conclusion

We conclude by comparing our new results to Ramanujan's formulas for $1/\pi$. One of Ramanujan's major insights was to find formulas such as

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (a+bn) \binom{2n}{n}^3 \frac{z^n}{2^{6n}}, \tag{4.1}$$

where a, b, and z are parameterized by logarithmic, modular, and quasi-modular functions [6], [8]:

$$\begin{aligned} z =& 4\frac{\varphi^4(-q)}{\varphi^4(q)} \left(1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right), \\ a =& \frac{1}{\pi\varphi^4(q)} \left(1 + \frac{8\log|q|}{\varphi(q)} \sum_{n=1}^{\infty} n^2 q^{n^2}\right), \\ b =& \frac{\log|q|}{\pi} \left(1 - 2\frac{\varphi^4(-q)}{\varphi^4(q)}\right). \end{aligned}$$

We can produce infinitely many irrational algebraic triplets (a, b, z) which make (4.1) valid. The parameters are simultaneously algebraic whenever $q = e^{2\pi i\tau}$, with τ a quadratic irrational in the upper half plane (some additional restrictions on τ are necessary to ensure that the infinite series in (4.1) converges). In fact, it is known that z is algebraic whenever τ is the period ratio of an elliptic curve, but the values of a and b are only algebraic if the elliptic curve has complex multiplication (and this is less than obvious).

Now consider the fact that we detected only three algebraic pairs (r, s) for which

$$\log\left(\frac{4}{r}\right) = rs + \sum_{n=1}^{\infty} \frac{(2(1+rs)n+1)}{(2n)(2n+1)} \binom{2n}{n}^2 \left(\frac{r}{4}\right)^{2n}.$$

We can use formula (3.1) to deduce q-parameterizations for r and s:

$$r = \frac{\varphi^2(-q)}{\varphi^2(q)}, \qquad s = \frac{\mathscr{L}(i,q)}{\mathscr{L}(i,-q)}, \tag{4.2}$$

where

$$\varphi(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad \mathscr{L}(i,q) = \sum_{n\in\mathbb{Z}} D(iq^n).$$

It follows that r is a modular function and s is not. It seems that a small miracle has to occur for r and s to be algebraic simultaneously. We conjecture that r and s are algebraically independent for almost all values of $q = e^{2\pi i\tau}$, where τ is the period ratio of an elliptic curve. Verifying this conjecture will likely require a careful investigation of divisors of elliptic curves.

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