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DIOPHANTINE AND COHOMOLOGICAL DIMENSIONS

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Abstract. We give explicit linear bounds on the p-cohomological dimension of a field in terms of its Diophantine dimension. In particular, we show that for a field of Diophantine dimension at most 4, the 3-cohomological dimension is less than or equal to the Diophantine dimension.

Introduction

Let F be a field and fix a prime integer p not equal to the characteristic of F. Recall that a field F is said to have the C_d property if every homogeneous polynomial of degree n over F in more than n^d variables has a nontrivial zero. The Diophantine dimension of F, denoted $\mathrm{ddim}(F)$ is defined to be the least integer d such that F has the C_d property. We define the p-cohomological dimension of F, denoted $\mathrm{cd}_p(F)$ to be the least integer d such that the Galois cohomology groups $H^d(L, \mathbb{Z}/p\mathbb{Z})$ vanish for every L/F finite. One says that F has cohomological dimension d, written $\mathrm{cd}(F) = d$ if d is the maximum value of $\mathrm{cd}_p(F)$ taken over all p.

Although there are well-known examples of Ax [Ax65] of fields of cohomological dimension 1 which do not have the C_d property for any d, it is quite possible that having a bound on the Diophantine dimension will give one a bound on the p-cohomological dimension. In [Ser94, page 99], after outlining why the Milnor Conjectures would imply $\mathrm{cd}_2(F) \leq \mathrm{ddim}(F)$, Serre states "Il est probable que ce résultat est également valable pour $p \neq 2$". On the other hand, it was not known until the present whether or not a field with $\mathrm{ddim}(F)$ finite must have $\mathrm{cd}_p(F)$ finite in general for $p \neq 2$.

In this paper we give explicit bounds on the p-cohomological dimensions in terms of the Diophantine dimension, and show in particular that $\mathrm{cd}_p(F)$ grows at most linearly with respect to $\mathrm{ddim}(F)$ (see Example 1.15). We do this by constructing explicit hypersurfaces which serve as splitting varieties for symbols in Milnor K-theory, although we choose to do this using the language of *forms* (as in Definition 1.1). Although splitting varieties in such contexts were constructed as part of the work towards the Bloch-Kato conjecture (in fact, "generic" splitting varieties – see for example [SJ06]), by expressing our splitting varieties as hypersurfaces, we are able to conclude that a bound on the Diophantine dimension of the field

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forces high degree symbols to all be trivial and hence results in a bound on the cohomological dimension of the field. Such hypersurfaces were already known for all p in degrees at most 3 (see Examples 1.5, 1.6, and Theorem 1.12), in the case $p = 3$ in degrees at most 4 (see Example 1.7) and in the case $p = 2$ for cohomology classes of all degrees (see Proposition 1.10). In this paper, we construct splitting hypersurfaces for all p and all degrees, though in general we make no claim that they are *generic* splitting varieties.

To sharpen our bounds for the cohomological dimension of fields, particularly of small Diophantine dimension, and to provide a common framework for our arguments, we make use of norm forms on certain finite dimensional nonassociative algebras. These forms generalize the familiar examples of norm forms on field extensions and central simple algebras. A particular class of algebras which are equipped with norm forms with the properties we need are *strictly power associative, finite dimensional, principally division algebras*. Basic definitions and facts that we require about these algebras are described in Section 2.

Throughout, we use the notation $K_n^M(F)$ to denote the nth Milnor K-group of F, and $k_n(F) = K_n^M(F)/pK_n^M(F)$. By the Bloch-Kato conjecture/norm residue isomorphism theorem [Voe11, Wei09], we may identify $k_n(F) = H^n(F, \mu_p^{\otimes n})$. We will use the notation $\{a_1, \dots, a_n\}$ to denote a symbol in the group $K_n^M(F)$, and $(a_1, \dots, a_n)$ to denote the corresponding symbol in the group $k_n(F)$. In the case that $\rho \in F$ is a primitive pth root of unity, we will use the notation $D_{(a,b)}$ to denote the symbol algebra generated by elements x, y and satisfying $x^p = a, y^p = b, xy = \rho yx$. Finally, if V is an F-vector space and L/F a field extension, we write V_L to denote the L-space $V \otimes_F L$.

We will also have use of the Milnor Conjecture, proved in [OVV07] (see also Merkur′ev's survey in [Mil10] for a more detailed explanation of the details of the proof in arbitrary characteristic not 2), which asserts an isomorphism between the Milnor K-groups modulo 2 and the quotients of the powers of the fundamental ideals $e_n : I^n(F)/I^{n-1}(F) \to K_n^M(F)/2K_n^M(F) \cong H^n(F, \mathbb{Z}/2\mathbb{Z})$, where $I(F)$ is the ideal of the Witt group consisting of classes of even dimensional quadratic forms. This isomorphism is described as mapping the class of the Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$ to the symbol $(a_1, \dots, a_n)$.

1. Homogeneous forms and Milnor K-theory

Definition 1.1. By a *form* of degree d and dimension n, we mean a homogeneous degree d polynomial function $f \in F[V^*]$ where V is an n-dimensional vector space. We will also refer to f as a degree d form on V. We say that f is *isotropic* if there is a $v \in V \setminus \{0\}$ such that $f(v) = 0$.

Definition 1.2. Suppose that $\alpha \in k_n(F)$, and let V be a vector space. We say that a form N on V

- *neutralizes* α if $\alpha \cup (N(v))$ is trivial in $k_{n+1}(L)$ for every field extension L/F and every choice of $v \in V_L$ with $N(v) \neq 0$,
- *splits* α if for every field extension L/F, we have N_L isotropic implies $\alpha_L = 0$.

Where $\cup$ denotes the product in the ring $k(F) = \oplus_n k_n(F)$. We note that to say that a form N is a splits α implies that the projective hypersurface defined by N is a splitting variety for α (see for example [SJ06], although the authors there require splitting varieties to be smooth).

A particularly useful source of forms from the point of view of symbols in Milnor K-theory are those which come from finite dimensional strictly power associative principally division algebras.

Definition 1.3. Let A be a unital finite dimensional strictly power associative algebra, and let $N_A \in F[A^*]$ be the reduced norm form (see Section 2 for definitions). As in Section 2, we say that A is principally division, if for every $a \in A$, the subalgebra $F[a]$ generated by a is a division algebra. We say that A is *adapted to a class* $\alpha \in k_n(F)$ if N_A is degree p and if for every field extension L/F, A_L is principally division if and only if $\alpha_L \neq 0$.

Lemma 1.4. *Suppose that A is a unital finite dimensional strictly power associative algebra which is adapted to $\alpha \in k_n(F)$. Then N_A neutralizes and splits α.*

Proof. To check that N_A neutralizes and splits α, we must consider all field extensions L/F and check the conditions of the definition of neutralizing and splitting. Since the hypothesis is also preserved by such field extensions, however, it suffices to consider the case $F = L$.

Let $d = N_A(x)$ for $x \in A$. We claim that $\alpha \cup (d) = 0$. If $x \in F \subset A$, then $N_A(x) = x^p = d$ and consequently $\alpha \cup (d) = p\alpha \cup (x) = 0$. On the other hand, if $x \in A \setminus F$, then $F(x)$ is a degree p subfield of A which by assumption concerning the characteristic of F is separable. After passing to a prime-to-p extension, which will not affect the triviality of $\alpha \cup (d)$ due to the standard restriction-corestriction argument, we may in fact assume that $F(x)/F$ is a cyclic Kummer extension, say $E = F(x) = F(y)$ where $y \in A$ with $y^d = b \in F$.

Now, A_E contains $E \otimes_F E$ which contains zerodivisors. Since E/F is separable, E is principally generated over F, say $E = F[a]$, and hence $E \otimes_F E = E[a \otimes 1]$ is also principally generated. It follows that A_E is not principally division. Since N_A is adapted to α, this implies in turn that α_E is split. But now, by [LMS07, Theorem 6] (see also [Voe03, Proposition 5.2]), we have the exact sequence:

$$k_{n-1}(E) \xrightarrow{N_{E/F}} k_{n-1}(F) \xrightarrow{\cup(b)} k_n(F) \longrightarrow k_n(E),$$

which tells us that $\alpha = \beta \cup (b)$ for some $\beta \in k_{n-1}(F)$. But now, we claim that $\alpha \cup (d) = 0$. In fact this will come from the fact that $(b) \cup (d) = (b, d) = 0$. Again, by the exact sequence above, this amounts to saying that $d \in N_{E/F}(k_1(E)) = N_{E/F}(E^*)$. But $d = N_{E/F}(x)$ for $x \in E^*$ by hypothesis.

Next, suppose that $N_A(x) = 0$. We need to show that $\alpha = 0$ in this case. We will do this by showing that A is not principally division, which would imply $\alpha = 0$ by our hypothesis on A.

Arguing by contradiction, suppose A is principally division. Since the restriction of N_A to F is the pth power operation on F, and F is a field, it follows that $x \in A \setminus F$ and hence $F(x) = E$ is a degree p separable field extension. Since x satisfies its characteristic polynomial $\chi_x(T)$ of degree p, it follows that $\chi_x(T)$ is in fact the minimal polynomial of x as an element of E. Since E is a field, $\chi_x(T)$ must be irreducible. But, since $N_A(x) = 0$ is the constant coefficient of this polynomial, we find that $\chi_x(T) = Tf(T)$ for some monic polynomial $f(T)$ contradicting the irreducibility of $\chi_x(T)$. $\square$

Example 1.5 (Kummer extensions)**.** If $L = F(\sqrt[p]{a})$, then L is adapted to (a). This follows from the fact that L/F is a division algebra exactly when $a \notin (F^*)^p$, which in turn happens exactly when $(a) \neq 0$ in $k_1(F^*)$.

Example 1.6 (Symbol algebras)**.** Assume $\mu_p \subset F$, and let $\alpha = (a,b) \in k_2(F) = \mathrm{Br}(F)_p$ (by the Theorem of Merkur′ev and Suslin [MS82]). Let $D = D_{(a,b)}$ be the corresponding symbol algebra. Then D is adapted to α. Indeed, since D has degree p, it is either a split algebra or a division algebra. Further, it is split if and only if $(a,b) = 0 \in k_2(F) = \mathrm{Br}(F)_p$, and it is division if and only if its norm form is anisotropic, and as we see in Lemma 2.2, it is division if and only if it is principally division.

Example 1.7 (First Tits process Albert algebras)**.** Consider the case $p = 3$. Suppose that $\mu_3 \in F$ and let $D = D_{(a,b)}$ be the symbol algebra as before. Choose $c \in F^*$ and let A be the Albert algebra given by the first Tits process applied to D and c (see, for example, [PR96, section 2.5], or [KMRT98, Section IX.39]).

We claim that A is adapted to (a,b,c). This follows from [PR96, Theorem 1.8 and Proposition 2.6], where it is shown that A is division if and only if $(a,b,c) \neq 0$, and Lemma 2.3, where we show that A is principally division if and only if it is division.

We will have use of the following relations, which are well known over any field F; for example [Ker09, Lemmas 2.2, 2.4].

Lemma 1.8. *The following identities hold in $K_2^M(F)$ for any field F:*

1. $\{a, 1-a\} = 0$,
2. $\{a, -a\} = 0$,
3. $\{a, b\} = -\{b, a\}$,
4. $\{a, a\} = \{a, -1\}$,
5. $\{a, b\} = \{a+b, -b/a\} = \{-a/b, a+b\}$.

And therefore the corresponding identities hold for symbols in $k_2(F) = K_2^M(F)/pK_2^M(F)$ for any p.

Proof of Lemma 1.8.

1. This is one of the defining relations of $K_2^M(F)$.
2. If $a = 1$, this is clear. Otherwise, we may write $-a = (1-a)/(1-a^{-1})$, and so we have:
$$\{a, -a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = -\{a, 1-a^{-1}\} = \{a^{-1}, 1-a^{-1}\} = 0.$$
3. From the previous identity, we have
$$\{a, b\} + \{b, a\} = \{a, -ab\} + \{b, -ab\} = \{ab, -ab\} = 0.$$
4. We compute
$$\{a, a\} - \{a, -1\} = \{a, a\} + \{a, -1\} = \{a, -a\} = 0.$$
5. Since $a/(a+b) + b/(a+b) = 1$, we have
$$\begin{aligned} 0 &= \{a/(a+b), b/(a+b)\} = \{a, b\} - \{a, a+b\} - \{a+b, b\} + \{a+b, a+b\} \\ &= \{a, b\} - \{a+b, a^{-1}\} - \{a+b, b\} - \{a+b, -1\} = \{a, b\} - \{a+b, -b/a\} \end{aligned}$$
and $\{a+b, -b/a\} = -\{-b/a, a+b\} = \{-a/b, a+b\}$. □

Proposition 1.9. *Suppose that $\alpha = \beta \cup (a)$, and that N is a degree p form defined on a vector space V which neutralizes and splits β. Then the form $N' = N \oplus -aN$ on $V \oplus V$ splits and neutralizes α.*

Proof. Suppose that $v, w \in V$. We first show that N' splits α. Suppose that we have $v, w \in V$ such that $N(v) - aN(w) = 0$. We must show in this case that $\alpha = 0$. Note that we may assume $N(v), N(w) \neq 0$ since otherwise, using the fact that N splits β, we would have $\beta = 0$ and hence $\alpha = 0$ as well. But now we may write

$$\alpha = \beta \cup (a) = \beta \cup (N(w)) + \beta \cup (a) = \beta \cup (aN(w)) = \beta \cup (N(v)) = 0.$$

Now, to see that N neutralizes α, let $v, w \in V$. We must show that $\alpha \cup (N(v) - aN(w)) = 0$. If either $N(v)$ or $N(w)$ are zero, then $\beta = 0$ since N splits β, and hence $\alpha \cup (N(v) - aN(w)) = 0$ as well. Hence we may assume that $N(w), N(v) \neq 0$. Then we have:

$$\alpha \cup (N(v) - aN(w)) = \beta \cup (a) \cup (N(v) - aN(w)).$$

Since N neutralizes β, we may write $\beta \cup (a) = \beta \cup (N(w)N(v)^{-1}a)$, and combining this with the above gives

$$\alpha \cup (N(v) - aN(w)) = \beta \cup (N(w)N(v)^{-1}a) \cup (N(v) - aN(w)).$$

By Lemma 1.8(5), we may write

$$(-aN(w)) \cup (N(v)) = (N(w)N(v)^{-1}a) \cup (N(v) - aN(w)),$$

and so we may use this to rewrite our expression to obtain:

$$\alpha \cup (N(v) - aN(w)) = \beta \cup (-aN(w)) \cup (N(v)) = \beta \cup (N(v)) \cup (-a^{-1}N(w)^{-1}) = 0,$$

as desired, using the fact that $\beta \cup (N(v)) = 0$ since N neutralizes β. ☐

Of course, in the case $p = 2$, the Pfister forms give examples of forms which neutralize and split symbols. The contents of the following proposition were noted by Serre (see [Ser94, page 99]):

Proposition 1.10. *Suppose that $p = 2$ and $\alpha = (a_1, \ldots, a_n) \in k_n(F)$ is a symbol. Let $q = \langle\langle a_1, \ldots, a_n \rangle\rangle$ be the corresponding Pfister form. Then q splits and neutralizes α.*

Proof. By the Milnor Conjectures, we have an isomorphism $e_n : I^n(F)/I^{n+1}(F) \to k_n(F)$ sending q to α. If q is isotropic, then since it is a Pfister form, it must be hyperbolic and in particular represent the 0 class in $I^n(F)/I^{n+1}(F)$. It then follows that $\alpha = e_n(q)$ is zero as well.

To see that q neutralizes α, suppose that $a = q(v)$, and consider $\alpha \cup (a) = e_{n+1}(q \otimes \langle 1, -a \rangle)$. The form $q \otimes \langle 1, -a \rangle = q \perp -q(v)q$ is isotropic. This is because q, being a Pfister form, represents 1, say $q(w) = 1$, and then we may write

$$(q \perp -q(v)q)(v, w) = q(v) - q(v)q(w) = 0.$$

But as before, it follows that since $q \otimes \langle 1, -a \rangle$ is a Pfister form, it must in fact be hyperbolic implying that $\alpha \cup (a) = e_{n+1}(q \otimes \langle 1, -a \rangle) = 0$ as desired. ☐

Let us now record some applications of these results; the first few of which are well-known consequences of previous results in the area.

Theorem 1.11. *Suppose that F is C_n, and let $p = 2$. Then every element of $k_n(F) = K_n^M(F)/2$ is a symbol and $k_{n+1}(F) = 0$. In particular, $\mathrm{cd}_2(F) \leq n$.*

Proof. For the first claim, suppose that α and β are symbols in $k_n(F)$. It suffices to show that we may find a presentation $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$, so that $a_i = b_i$ for $i = 1, \ldots, n-1$. We do this by induction on the number of slots that a given presentation of α and β share. Suppose that we may write $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$, so that $a_i = b_i$ for $i = 1, \ldots, m$ (or with no constraints on the entries if $m = 0$). By Proposition 1.10, we may find forms H, G of degree 2 on 2^{n-1}-dimensional vector spaces V, W which split and neutralize $(a_1, \ldots, a_{n-1})$ and $(b_1, \ldots, b_{n-1})$ respectively. Let ϕ be the one dimensional form $t^2 a_{n-1} \in F[t]$. By the C_n property, the $2^n + 1$-dimensional form $\phi \oplus H \oplus G$ has a nontrivial zero, say $a_{n-1} t_0^2 + H(v) a_n - G(w) b_n = 0$ for some $t_0 \in F$, $v \in V$ and $w \in W$. If either $H(v)$ or $G(w)$ is zero, then one of the forms α or β is trivial (since these forms split α and β), and we are done. Hence we may assume that $H(v), G(w) \neq 0$. But now, we have

$$(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1}, H(v)a_n) = (a_1, \ldots, t_0^2 a_{n-1}, H(v)a_n), \tag{1}$$

the first equality following from the fact that H neutralizes $(a_1, \ldots, a_{n-1})$, and the second from the assumption $p = 2$. If $t_0^2 a_{n-1} + H(v)a_n = 0$, it would follow by the definition of Milnor K-theory that $(t_0^2 a_{n-1}, H(v)a_n) = 0$ and hence $\alpha = 0$. Hence we may assume that this is not the case. By Lemma 1.8(5), we may therefore write

$$\begin{aligned}(t_0^2 a_{n-1}, H(v)a_n) &= (-t_0^2 a_{n-1}(H(v)a_n)^{-1}, t_0^2 a_{n-1} + H(v)a_n) \\ &= (-t_0^2 a_{n-1}(H(v)a_n)^{-1}, G(w)b_n).\end{aligned}$$

Combining this with equation (1), it follows that we may write

$$\alpha = (a_1, \ldots, a_n) = (a_1, \ldots, -t_0^2 a_{n-1}(H(v)a_n)^{-1}, G(w)b_n),$$

and writing $\beta = (b_1, \ldots, b_n) = (b_1, \ldots, b_{n-1}, G(w)b_n)$, we see that our two new presentations now have $m+1$ slots in common. The claim follows.

For the second assertion, it suffices to show that every symbol $\alpha = (a_1, \ldots, a_{n+1})$ is trivial in $k_{n+1}(F)$. But by the veracity of the Milnor Conjectures, we may identify α as the e_{n+1} invariant of the $(n+1)$-fold Pfister form $\langle\langle a_1, \ldots, a_{n+1} \rangle\rangle$, which has dimension 2^{n+1}. Since F is C_n, it follows that this Pfister form is isotropic and hence hyperbolic. Consequently α, its e_{n+1}-invariant, must vanish. $\square$

We next recall a consequence of the work of Merkur′ev and Suslin:

Theorem 1.12. *Let F be C_2. Then $k_3(F) = K_3^M(F)/p = 0$, and so $\mathrm{cd}(F) \leq 2$.*

Proof. Let $\alpha = (a, b, c) \in k_3(F)$. Let $D = D_{(a,b)}$ be the symbol algebra associated to (a, b), and let N_D be its reduced norm. By Example 1.6, N_D splits and neutralizes (a, b). Therefore, by Proposition 1.9, the form $N = N_D \oplus -cN_D$ splits and neutralizes α. But note that this form has dimension $2p^2 > p^2$ and degree p. Since F is C_2, it follows that it is isotropic and therefore $\alpha = 0$ as desired. $\square$

The next result is a known consequence of the theory of Albert algebras.

Theorem 1.13. *Let F be C_3. Then $K_4^M(F)/3 = 0$, and so $\mathrm{cd}_3(F) \leq 3$.*

Proof. Let $(a, b, c, d) \in k_4(F)$. If A is the Albert algebra for (a, b, c), then $N_A \oplus -dN_A$ is a $2 \cdot 3^3 = 54$ dimensional form splitting and neutralizing (a, b, c, d). Since $54 > 27$ we are done. $\square$

The following theorems are new:

Theorem 1.14. *Let F be C_4. Then $K_5^M(F)/3 = 0$, and so $\mathrm{cd}_4(F) \leq 4$.*

Proof. Let $(a, b, c, d, e) \in k_5(F)$. If A is the Albert algebra for (a, b, c), then $N = N_A \oplus -dN_A$ is a $2 \cdot 3^3 = 54$ dimensional form splitting and neutralizing (a, b, c, d), and $N' = N \oplus -eN$ is a 108 dimensional form splitting and neutralizing (a, b, c, d, e). Since $108 > 81 = 3^4$ we are done. $\square$

Theorem 1.15. *Let F be C_n. Then*

(1) $\mathrm{cd}_2(F) \leq n$,

(2) $\mathrm{cd}_3(F) \leq \begin{cases} n & \text{if } n \leq 4, \\ \lceil (n-3)(\log_2(3)) + 3 \rceil & \text{otherwise,} \end{cases}$

(3) $\mathrm{cd}_p(F) \leq \begin{cases} n & \text{if } n \leq 2, \\ \lceil (n-2)(\log_2(p)) + 1 \rceil & \text{otherwise.} \end{cases}$

For example, if F is C_3, then $\mathrm{cd}_5(F), \mathrm{cd}_7(F) \leq 4$. Note that, in particular, the p-cohomological dimension is bounded linearly with respect to the Diophantine dimension, with slope $\log_2(p)$.

Proof. The first part, corresponding to cd_2 follows from Theorem 1.11.

For cd_3, the cases with $n \leq 4$ follow from Theorems 1.13 and 1.14 respectively. In general, to check whether or not $\mathrm{cd}_3(F) \leq m$, write a degree $m+1$ symbol in $K_{m+1}^M(F)/3$ as $\alpha = (a, b, c) \cup (d_1, \dots, d_{m-2})$. We wish to find a criterion on m which will guarantee that $\alpha = 0$. This works as follows: if N_A is the norm form for the first Tits process Albert algebra A defined by (a, b, c), then inductively applying Proposition 1.9 we obtain a form N in $27 \cdot 2^{m-2}$ variables of degree 3 which splits α. In particular, by the C_n property, we find that this form is isotropic, and hence α is split when $27 \cdot 2^{m-2} > 3^n$ or $2^{m-2} > 3^{n-3}$. But this translates to $m > (n-3)(\log_2(3)) + 3$, since $\log_2 3$ is irrational. This is equivalent to saying $m \geq \lceil (n - 3(\log_2(3)) + 3 \rceil$.

The general case of cd_p follows much like the case of cd_3 above, considering a symbol $\alpha = (a, b) \cup (c_1, \dots, c_{m-1})$, starting with Theorem 1.12 and applying Proposition 1.9 to obtain a form N of degree p and dimension $p^2 \cdot 2^{m-1}$ splitting α. Again, in this case, the C_n property guarantees that α is split in case $2^{m-1} > p^{n-2}$. The result follows. $\square$

Remark 1.16. Although it is a weaker result, it is interesting to note that one may associate an "obvious form" which splits a given symbol. In particular if $\alpha = (a_1, \dots, a_n) \in k_n(F)$, then the form $a_1 t_1^p + \cdots + a_n t_n^p$ splits α.

Proof. We induct on n, the case $n = 1$ being trivial and the case $n = 2$ following from the fact that $N_{D_{(a,b)}}$ when restricted to the subspace $Fx + Fy$ is exactly the form $at^p + bs^p$. For the general induction step, suppose that $\sum a_i r_i^p = 0$ for some choice of $r_i \in F$, and set $u = \sum_{i=0}^{n-2} a_i r_i^p$. Then $-u = a_{n-1} r_{n-1}^p + a_n r_n^p$. By Lemma 1.8(5), we may write $(a_{n-1}, b_{n-1}) = (-u, v)$ for some $v \in F^*$. But therefore we may write $\alpha = (a_1, \dots, a_{n-2}, -u, v)$. Considering the form $(a_1, \dots, a_{n-2}, -u)$ we find that by $\sum_{i=1}^{n-2} a_i r_i^p + (-u)1 = 0$, the induction hypothesis implies $(a_1, \dots, a_{n-2}, -u) = 0$, and so $\alpha = (a_1, \dots, a_{n-2}, -u) \cup (v) = 0$ as well. $\square$

2. Appendix: Norm forms on power associative algebras

Recall that an algebra A is called *power associative* if all associators of the form $\{a^i, a^j, a^k\}$ vanish, or equivalently, if for every $a \in A$, the subalgebra $F[a]$ generated by a is associative and commutative. We say that A is *strictly power associative* if for every field extension L/F, the algebra $A_L = A \otimes_F L$ is power associative. Note that in case F has characteristic not equal to 2, by linearizing associator relations such as $\{x, x, y\}$, one sees that every power associative algebra is automatically strictly power associative.

Let A be a unital finite dimensional strictly power associative F-algebra. We now recall the definition of the reduced norm function on A. Set $R = F[A^*]$ to be the ring of polynomial functions on A. Consider the generic element $x \in A_R$, defined as follows: if $a_1, \ldots a_n$ is a basis for A and $f_1, \ldots, f_n \in A^* \subset R = F[f_1, \ldots, f_n]$ is the corresponding dual basis, then $x = \sum a_i f_i$. In a coordinate free way, we may also write x as the identity map, thought of as an element of $Hom_F(A, A) = A \otimes_F A^* \subset A \otimes_F R = A_R$. We may now consider the subalgebra $R[x] \subset A_R$. Since A_R is a finitely generated module and R is Noetherian, it follows that $R[x]$ is finitely generated R-module and hence an integral extension. Let $m(T) = m_0 + m_1 T + \cdots + m_{r-1} T^{r-1} + T^r \in R[T]$ be the minimal polynomial of x over R. Since $m(x)$ is the zero polynomial function on A, it follows that for any $a \in A$, a is a root of the polynomial

$$\chi_a(T) = m_0(a) + m_1(a)T + \cdots + m_{r-1}(a)T^{r-1} + T^r,$$

which we call the reduced characteristic polynomial of a. We define the reduced norm of a, $N_A(a)$, to be $m_0(a)$. The reduced norm function is the function $N_A = m_0 \in R$. We define the *adjunct* of a, defined by $\mathrm{adj}_A(a)$ to be $m_1(a) + m_2(a)a + \cdots m_{r-1}(a)a^{r-2} + a^{r-1}$. We then have $a(\mathrm{adj}_A(a)) = N_A(a)$. Note that since these coincide with the standard characteristic and minimal polynomials of the R-algebra $R[x]$ acting on itself, we have the familiar identity $\chi_a(T) = N_A(T - a)$.

We say that a strictly power associative finite dimension F-algebra A has degree d if χ_A (and hence also N_A) is a degree d polynomial function.

Recall that a nonassociative algebra A is called division in case left and right multiplication by every nonzero element $a \in A$ are both bijective. We say that A is *principally division*, if for every $a \in A$, the subalgebra $F[a]$ generated by a is division. If A is power associative, this is equivalent to saying that $F[a]$ is a field. We say that $a \in A$ is invertible if there exists b such that $ab = 1 = ba$.

Lemma 2.1. *Suppose that A is a strictly power associative F-algebra of prime degree p. Then A is principally division if and only if N_A is anisotropic.*

Proof. Suppose N_A is anisotropic and let $a \in A$. We must show that $F[a]$ is division. Let $b \in F[a] \subset A$ with $b \neq 0$. Since $N_A(b) \neq 0$, we may write $b(N_A(b))^{-1}(\mathrm{adj}_A(b)) = 1$, and since $\mathrm{adj}_A(b) \in F[b] \subset F[a]$, it follows that b is invertible in $F[a]$. Therefore $F[a]$ is division.

Conversely, if A is principally division, we wish to show that for all $a \in A \setminus 0$, $N_A(a) \neq 0$. Let $m_a(t)$ be the minimal polynomial of a. It follows from [Jac59], that $m_a(t)$ and $\chi_a(t)$ have the same prime factors. Since $F[a] = F[t]/(m_a(t))$ is a division algebra, it follows that $m_a(t)$ is irreducible, and in particular, not divisible by t. But therefore t cannot divide $\chi_a(t)$ as well. It therefore follows that $N_A(a)$, being (up to a sign) the constant coefficient of $\chi_a(t)$ is also nonzero, as desired. $\square$

Lemma 2.2. *Suppose A is a finite dimensional alternative algebra. Then A is principally division if and only if it is division.*

Proof. Certainly if A is division, it is also a domain. Since each subalgebra $F[a]$ is then a finite dimensional commutative associative domain, they are fields, and hence A is principally division.

For the converse, suppose that A is principally division, and let $a \in A$. We wish to show that left multiplication by a gives a bijection from A to itself. The proof that right multiplication also has this property follows from the same argument.

To start, since A is principally division and power associative, it follows that $F[a]$ is a field. Using the fact that A is left alternative, it follows that left multiplication gives A the structure of a vector space over $F[a]$. But scalar multiplication by a nonzero scalar is automatically an isomorphism of A with inverse given by a^{-1}. □

Lemma 2.3. *Let A be a first Tits process Albert algebra over F. Then A is division, if and only if it is principally division.*

Proof. Clearly if A is division it is principally division. For the converse, suppose that A is not division. By [Jac68, Chapter 9, Theorem 20], A is split, and hence must contain the algebra $M_n(F)^+$. But it follows that A is therefore not principally division, since, for example, it contains nilpotent elements. □

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