

A NOTE ON BI-LINEAR MULTIPLIERS

SAURABH SHRIVASTAVA

(Communicated by Alexander Iosevich)

ABSTRACT. In this paper we prove that if $\chi_E(\xi - \eta)$ – the indicator function of a measurable set $E \subseteq \mathbb{R}^d$ – is a bi-linear multiplier symbol for exponents p, q, r satisfying the Hölder's condition $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and exactly one of p, q , or $r' = \frac{r}{r-1}$ is less than 2, then E is equivalent to an open subset of \mathbb{R}^d .

1. INTRODUCTION AND STATEMENT OF RESULTS

The remarkable work of M. Lacey and C. Thiele [4], [5] on boundedness of the bi-linear Hilbert transform motivated a lot of research in the area of Euclidean harmonic analysis. For f, g in $\mathcal{S}(\mathbb{R})$ – the Schwartz class on \mathbb{R} – the bi-linear Hilbert transform is defined by

$$H(f, g)(x) = p.v. \int_{\mathbb{R}} f(x - y)g(x + y) \frac{dy}{y},$$

or equivalently,

$$H(f, g)(x) = -i \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\eta) \operatorname{sgn}(\xi - \eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta,$$

where $\hat{\cdot}$ denotes the Fourier transform and

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi = 0, \\ -1, & \xi < 0. \end{cases}$$

We would like to remark that the bi-linear Hilbert transform is invariant under the operations of simultaneous translation, dilation, and modulation. The modulation invariance is a subtle property shared by the bi-linear Hilbert transform and poses additional difficulties while proving suitable L^p –estimates for the operator. It is not difficult to convince ourselves that the classical approach of Littlewood-Paley decomposition, which is a useful technique to handle singular integral operators, is not quite helpful to deal with operators having modulation in-variance property. In papers [4] and [5] M. Lacey and C. Thiele systematically developed very powerful techniques to handle such operators. These techniques are commonly referred to as the time-frequency techniques. In seminal papers they proved the following L^p –estimates for the bi-linear Hilbert transform.

Received by the editors March 18, 2014.

2010 *Mathematics Subject Classification.* Primary 42A45, 42B15; Secondary 42B25.

Key words and phrases. Fourier multipliers, Littlewood-Paley operators, bi-linear multipliers, transference methods.

Theorem 1.1 ([4, 5]). *Let $1 < p, q \leq \infty$ and $\frac{2}{3} < r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for all functions $f, g \in \mathcal{S}(\mathbb{R})$, there exists a constant $C > 0$ such that*

$$\|H(f, g)\|_{L^r(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

In this paper we are interested in studying bi-linear multiplier operators having modulation in-variance property. The bi-linear multiplier operators in general are defined as follows: Let $m(\xi - \eta)$ be a bounded measurable function on \mathbb{R}^d and (p, q, r) , $0 < p, q, r \leq \infty$, be a triplet of exponents. Consider the bi-linear operator T_m initially defined for functions f and g in a suitable dense class by

$$(1.1) \quad T_m(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\eta) m(\xi - \eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We say that T_m is a bi-linear multiplier operator for the triplet (p, q, r) if T_m extends to a bounded operator from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$, i.e., there exists a constant $C > 0$, independent of functions f and g , such that

$$\|T_m(f, g)\|_{L^r(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

The bounded function m is said to be a bi-linear multiplier symbol for the triplet (p, q, r) if the corresponding operator T_m is a bi-linear multiplier operator for (p, q, r) .

We denote by $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$ the space of all bi-linear multiplier symbols for the triplet (p, q, r) . Further, the norm of $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ is defined to be the norm of the corresponding bi-linear multiplier operator T_m from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$, i.e.,

$$\|m\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} = \|T_m\|_{L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d)}.$$

The bi-linear multiplier symbols on the torus group \mathbb{T}^d and discrete group \mathbb{Z}^d are defined similarly. The space of bi-linear multiplier symbols on \mathbb{T}^d and \mathbb{Z}^d will be denoted by $\mathcal{M}_{p,q}^r(\mathbb{T}^d)$ and $\mathcal{M}_{p,q}^r(\mathbb{Z}^d)$, respectively.

Remark 1.2. Unless specified otherwise, we shall always assume that exponents p, q, r satisfy $0 < p, q, r \leq \infty$ and the Hölder’s condition $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Now we describe some important properties of bi-linear multipliers. Some of these properties will be used later in the paper.

Proposition 1.3 ([3]). *Let p, q, r be exponents satisfying the Hölder’s condition. Then bi-linear multiplier symbols satisfy the following properties:*

- (1) *If $c \in \mathbb{C}$ and m, m_1 , and m_2 are in $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$, then so are cm and $m_1 + m_2$. Moreover, $\|cm\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} = |c| \|m\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)}$ and $\|m_1 + m_2\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \leq C(\|m_1\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} + \|m_2\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)})$.*
- (2) *If $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ and $\eta \in \mathbb{R}^d$, then $\tau_\eta m(\cdot) = m(\cdot - \eta)$ is in $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$ with the same norm as m .*
- (3) *If $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ and $\lambda > 0$, then $m_\lambda(\cdot) = m(\lambda \cdot)$ is in $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$ with the same norm as m .*
- (4) *If $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ and $A = (a_{i,j})_{d \times d}$ is an orthogonal matrix acting on \mathbb{R}^d , then $m(A \cdot)$ is in $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$ with the same norm as m .*
- (5) *If $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d)$, then the convolution $m * h \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$, provided $r \geq 1$. Moreover, we have $\|m * h\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \leq \|h\|_{L^1(\mathbb{R}^d)} \|m\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)}$.*

- (6) If m_j is a sequence of bi-linear multiplier symbols in $\mathcal{M}_{p,q}^r(\mathbb{R}^d)$ such that the norm $\|m_j\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \leq C$ for some fixed constant $C > 0$ uniformly for all $j = 1, 2, \dots$, and if we further assume that m_j are uniformly bounded by a bounded measurable function on \mathbb{R}^d , then if $m_j \rightarrow m$ a.e. as $j \rightarrow \infty$, we have that $m \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ and $\|m\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \leq C$.

Remark 1.4. We would like to remark that corresponding properties hold true for bi-linear multiplier symbols on \mathbb{T}^d and \mathbb{Z}^d .

1.1. Statement of the result. In this paper we study some particular type of bi-linear multiplier symbols, namely those which are indicator functions of measurable sets. In general, there is no effective method to decide that the indicator function of a measurable set is a bi-linear multiplier symbol for some exponents. An important example in this direction is given by $\chi_I(\xi - \eta)$, for an interval $I \subset \mathbb{R}$. This is a consequence of boundedness of the bi-linear Hilbert transform that $\chi_I(\xi - \eta)$ is a bi-linear multiplier symbol for all exponents satisfying the conditions of Theorem 1.1.

In the current paper, we study structural properties (in the sense of measure theory) of sets whose indicator functions give rise to bi-linear multiplier symbols. The motivation for this paper comes from the beautiful work of V. Lebedev and A. Olevskii [6] on the classical Fourier multipliers. We will prove an analogue of their result in the context of bi-linear multiplier operators. In order to describe V. Lebedev and A. Olevskii’s result, we need the following definition.

Definition 1.5. We say that measurable sets E and E' are equivalent if the symmetric difference $E\Delta E'$ has Lebesgue measure zero.

Theorem 1.6 ([6]). *Let $E \subseteq \mathbb{R}^d$ be a measurable set and $p \neq 2$. If χ_E – the indicator function of E – is an L^p –multiplier, i.e., the linear operator $f \rightarrow (\chi_E \tilde{f})$, $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, extends boundedly from $L^p(\mathbb{R}^d)$ into itself, then E is equivalent to an open set in \mathbb{R}^d .*

This theorem tells us that the structure of the set whose indicator function is an L^p –multiplier, $p \neq 2$, cannot be very complicated in the sense of measure theory. As an immediate consequence of this, we see that the indicator function of a nowhere dense set of positive Lebesgue measure is never an L^p –multiplier for $p \neq 2$.

As mentioned previously, in this paper our aim is to prove an analogue of Theorem 1.6 in the context of bi-linear multipliers. In particular, we prove the following result:

Theorem 1.7. *Let E be a non-empty measurable subset of \mathbb{R}^d , and let p, q, r be exponents such that $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and exactly one of p, q , or $r' = \frac{r}{r-1}$ is less than 2. Suppose that $\chi_E(\xi - \eta)$ is a bi-linear multiplier symbol for the triplet (p, q, r) . Then E is equivalent to an open subset of \mathbb{R}^d .*

Remark 1.8. In the case of classical Fourier multipliers, $p = 2$ plays a special role by virtue of the Plancherel theorem. In a sharp contrast to this, in the theory of bi-linear mutipliers there is absolutely no easy way by which one can test that a given bounded function $m(\xi - \eta)$ is a bi-linear multiplier symbol for some triplet of exponents. The range of exponents covered by Theorem 1.7 falls in the complement of what is commonly known as local L^2 –range of exponents. The local L^2 –range consists of exponents p, q , and r satisfying $2 \leq p, q \leq \infty$ and $1 \leq r \leq 2$. Generally, it is believed that in this range of exponents most of the bi-linear multiplier operators are well behaved as far as the boundedness is concerned.

2. BASIC RESULTS AND PROOF OF THEOREM 1.7

In this section first we provide some basic definitions and results, which would be required to complete the proof of Theorem 1.7.

Definition 2.1. Let E be a measurable set of \mathbb{R}^d . Then we say that a point $x \in \mathbb{R}^d$ is a density point for E if

$$(2.1) \quad \lim_{t \rightarrow 0} \frac{|B(x, t) \cap E|}{|B(x, t)|} = 1,$$

where $B(x, t)$ denotes the Euclidean ball of radius $t > 0$ centered at $x \in \mathbb{R}^n$ and $|\cdot|$ denotes the Lebesgue measure of a set.

The set of all density points for the set E is denoted by E^d . The set $\partial_e E = \overline{E^d} \cap \overline{(E^c)^d}$ is referred to as the essential boundary of E , where E^c denotes the complement of E .

Lemma 2.2. *If $E \subseteq \mathbb{R}^d$ is a measurable set, then E and E^c are both equivalent to open sets if and only if $\partial_e E$ – the essential boundary of E – has Lebesgue measure zero.*

This lemma is easy to verify and hence its proof is not included here. See V. Lebedev and A. Oleviskiĭ [6] for more details. Next, we describe an important lemma proved in the seminal paper by V. Lebedev and A. Oleviskiĭ [6].

Lemma 2.3 ([6]). *Let $E \subseteq \mathbb{R}^d$ be a measurable set such that $|\partial_e E| > 0$. Then for every $N \in \mathbb{N}$ and for every subset $A \subseteq A_N = \{1, 2, \dots, N\}$, there exist $x_0, h \in \mathbb{R}^d$ such that the arithmetic progression*

$$x_n = x_0 + nh, \quad n \in A_N$$

satisfies the conditions

$$(2.2) \quad x_n \in E^d, \text{ if } n \in A, \text{ and } x_n \in (E^c)^d, \text{ if } n \in A_N \setminus A.$$

This lemma plays a crucial role in the proof of Theorem 1.6. Since, we have exploited the methodology of [6], in order to prove the main result of this paper, Lemma 2.3 is a key tool for the current paper as well.

Finally, we would require de Leeuw's type transference result for bi-linear multipliers. This would allow us to restrict bi-linear multiplier symbols defined on \mathbb{R}^d to the discrete group \mathbb{Z}^d . As a consequence of this, we will have to deal with some bi-linear multiplier operators on \mathbb{T}^d . In the context of this paper, it turns out that working with operators on \mathbb{T}^d is more convenient compared to working with the original operator defined on \mathbb{R}^d . This approach helps us in proving some good estimates on bi-linear operators under consideration and eventually helps us completing the proof of our main result.

We first recall the definition of a regulated function as it is needed to state the de Leeuw's type transference result for bi-linear multipliers, as mentioned previously.

Definition 2.4. A bounded function m on \mathbb{R}^n is called regulated at $x_0 \in \mathbb{R}^n$ if

$$\lim_{t \rightarrow 0} \frac{1}{t^n} \int_{|x| < t} (m(x_0 - x) - m(x_0)) dx = 0.$$

We say that the function m is regulated if it is regulated at every point $x \in \mathbb{R}^n$.

The following is a bi-linear analogue of the celebrated transference result proved by de Leeuw [2] for the classical Fourier multipliers.

Theorem 2.5 ([1]). *Let $m(\xi - \eta) \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ be a regulated function. Then $m \upharpoonright_{\mathbb{Z}^d}$ belongs to $\mathcal{M}_{p,q}^r(\mathbb{T}^d)$ with norm bounded by a constant multiple of $\|m\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)}$.*

We are now in a position to prove the main result of this paper.

Proof of Theorem 1.7. Notice that the same property holds for the complement E^c , i.e., the indicator function χ_{E^c} is a bi-linear multiplier symbol for the triplet (p, q, r) . Therefore, in the view of Lemma 2.2, it is enough to show that the essential boundary $\partial_e E$ has Lebesgue measure zero.

The proof is given by contradiction. Suppose on the contrary that $|\partial_e E| > 0$. Let $N \in \mathbb{N}$ be fixed and $\{\epsilon_n\}_{n=1}^N$ be a random sequence of 0 and 1. Then as an application of Lemma 2.3 we know that there exists $x_0, h \in \mathbb{R}^d$ such that

$$x_n = x_0 + nh \in \begin{cases} E^d, & \epsilon_n = 1, \\ (E^c)^d, & \epsilon_n = 0. \end{cases}$$

For $t > 0$, we consider the function

$$\begin{aligned} m_t(x) &= \frac{1}{|B(x, t)|} \int_{B(x, t)} \chi_E(y) dy \\ &= \frac{|B(x, t) \cap E|}{|B(x, t)|}. \end{aligned}$$

Observe that $m_t = \chi_E * A_t$, where $A_t(\cdot) = \frac{1}{|B(0, t)|} \chi_{B(0, t)}(\cdot)$. Also note that, for all t the norm $\|A_t\|_{L^1(\mathbb{R}^d)} = 1$. Since $r \geq 1$, by Proposition 1.3 we obtain that $m_t \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$. Moreover

$$\|m_t\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \leq \|\chi_E\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)}, \quad \forall t > 0.$$

Let $e_1 = (1, 0, \dots, 0)$ be the unit vector of the x_1 -axis and ψ be an affine mapping of \mathbb{R}^d which maps the vector ne_1 to x_n . At this point we invoke Proposition 1.3 once again and obtain that the composition $m_t \circ \psi \in \mathcal{M}_{p,q}^r(\mathbb{R}^d)$ with uniform multiplier norm with respect to the parameter t .

Recall the definition of density points and notice that $m_t(x_n) \rightarrow \epsilon_n$ as $t \rightarrow 0$, which is the same as $m_t \circ \psi(ne_1) \rightarrow \epsilon_n$ as $t \rightarrow 0$.

Now we identify the set $\{ne_1 : n \in \mathbb{Z}\}$ with the discrete group \mathbb{Z} and invoke de Leeuw's type transference theorem for bi-linear multipliers, namely Theorem 2.5 from [1]. Note that we can apply de Leeuw's type transference theorem to $m_t \circ \psi$ as m_t is a continuous function and ψ is an affine mapping. This yields that $m_t \circ \psi \upharpoonright_{\mathbb{Z}}$ – the restriction of $m_t \circ \psi$ to \mathbb{Z} – belongs to $\mathcal{M}_{p,q}^r(\mathbb{T})$ and

$$\|m_t \circ \psi \upharpoonright_{\mathbb{Z}}\|_{\mathcal{M}_{p,q}^r(\mathbb{T})} \leq C \|m_t\|_{\mathcal{M}_{p,q}^r(\mathbb{R}^d)} \quad \forall t > 0.$$

Since the above estimate is uniform with respect to $t > 0$, a standard limiting argument (see Proposition 1.3) allows us to conclude that the sequence $\{\epsilon_n\}_{n=1}^N$ is a bi-linear multiplier symbol for the triplet (p, q, r) , i.e., $\{\epsilon_n\} \in \mathcal{M}_{p,q}^r(\mathbb{T})$. Moreover the norm is independent of $N \in \mathbb{N}$.

We shall show that this leads to a contradiction.

First, we will prove that exponent p or q cannot be smaller than 2. Since $\{\epsilon_n\} \in \mathcal{M}_{p,q}^r(\mathbb{T})$ with a uniformly bound on multiplier norms with respect to parameter $N \in \mathbb{N}$, we use Khintchine's inequality to deduce that the bi-linear Littlewood-Paley

operator

$$S : (P, Q) \rightarrow \left(\sum_n |S_n(P, Q)|^2 \right)^{\frac{1}{2}}$$

is bounded from $L^p(\mathbb{T}) \times L^q(\mathbb{T})$ into $L^r(\mathbb{T})$, where P and Q are trigonometric polynomials and S_n is the bi-linear multiplier operator on \mathbb{T} with corresponding bi-linear multiplier symbol $\chi_{\{n\}}$. But, we already know that $p, q \geq 2$ is a necessary condition for the boundedness of the bi-linear Littlewood-Paley operator S (see the paper by P. Mohanty and S. Shrivastava [8] for a proof of this assertion) and hence we get a contradiction if either of p and q is less than 2.

Next, we assume that $r' < 2$, and show that this assumption also gives a contradiction. Let p, q, r be exponents such that $p, q, r > 2$.

Since the sequence $\{\epsilon_n\} \in \mathcal{M}_{p,q}^r(\mathbb{T})$ with multiplier norm uniformly bounded in $N \in \mathbb{N}$, we have that

$$(2.3) \quad \|S_\Gamma(P, Q)\|_{L^r(\mathbb{T})} \leq C \|P\|_{L^p(\mathbb{T})} \|Q\|_{L^q(\mathbb{T})},$$

where S_Γ is the bi-linear multiplier operator on \mathbb{T} with corresponding bi-linear multiplier symbol $\{\epsilon_n\}$.

Take $Q \equiv 1$ on \mathbb{T} and consider

$$\begin{aligned} S_\Gamma(P, Q)(x) &= \sum_{m,l \in \mathbb{Z}} \hat{P}(m) \hat{Q}(l) \epsilon_{l-m} e^{2\pi i(m+l)x} \\ &= \sum_{m \in \mathbb{Z}} \hat{P}(m) \epsilon_{-m} e^{2\pi i m x}. \end{aligned}$$

Therefore, we obtain that the linear operator $T : f \rightarrow \sum_{n \in \mathbb{Z}} \epsilon_n \hat{P}(n) e^{2\pi i n \cdot}$ is bounded from $L^p(\mathbb{T})$ into $L^r(\mathbb{T})$, i.e., we have

$$(2.4) \quad \|TP\|_{L^r(\mathbb{T})} \leq C \|P\|_{L^p(\mathbb{T})}.$$

This leads to a contradiction as we know that the inequality (2.4) does not hold true for all choices of $\{\epsilon_n\}_{n=1}^N$, $N \in \mathbb{N}$ as $r > 2$.

This completes the proof of Theorem 1.7. \square

REFERENCES

- [1] Oscar Blasco, *Bilinear multipliers and transference*, Int. J. Math. Math. Sci. **4** (2005), 545–554, DOI 10.1155/IJMMS.2005.545. MR2172393 (2006g:42013)
- [2] Karel de Leeuw, *On L_p multipliers*, Ann. of Math. (2) **81** (1965), 364–379. MR0174937 (30 #5127)
- [3] Loukas Grafakos and Rodolfo H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164, DOI 10.1006/aima.2001.2028. MR1880324 (2002j:42029)
- [4] Michael Lacey and Christoph Thiele, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$* , Ann. of Math. (2) **146** (1997), no. 3, 693–724, DOI 10.2307/2952458. MR1491450 (99b:42014)
- [5] Michael Lacey and Christoph Thiele, *On Calderón's conjecture*, Ann. of Math. (2) **149** (1999), no. 2, 475–496, DOI 10.2307/120971. MR1689336 (2000d:42003)
- [6] V. Lebedev and A. Olevskiĭ, *Idempotents of Fourier multiplier algebra*, Geom. Funct. Anal. **4** (1994), no. 5, 539–544, DOI 10.1007/BF01896407. MR1296567 (95i:42010)
- [7] Parasar Mohanty and Saurabh Shrivastava, *A note on the bilinear Littlewood-Paley square function*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2095–2098, DOI 10.1090/S0002-9939-10-10233-0. MR2596047 (2011b:42065)

- [8] Parasar Mohanty and Saurabh Shrivastava, *Bilinear Littlewood-Paley for circle and transference*, Publ. Mat. **55** (2011), no. 2, 501–519, DOI 10.5565/PUBLMAT_55211_11. MR2839453 (2012h:42038)

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH
BHOPAL, INDORE BY-PASS ROAD BHAURI, BHOPAL-462066, INDIA

E-mail address: `saurabhk@iiserb.ac.in`