

SOME BALL QUOTIENTS WITH A CALABI–YAU MODEL

EBERHARD FREITAG AND RICCARDO SALVATI MANNI

(Communicated by Kathrin Bringmann)

ABSTRACT. Recently we determined explicitly a Picard modular variety of general type. On the regular locus of this variety there are holomorphic three forms which have been constructed as Borchers products. Resolutions of quotients of this variety, such that the zero divisors are in the branch locus, are candidates for Calabi–Yau manifolds. Here we treat one distinguished example for this. In fact we shall recover a known variety given by the equations

$$X_0X_1X_2 = X_3X_4X_5, \quad X_0^3 + X_1^3 + X_2^3 = X_3^3 + X_4^3 + X_5^3.$$

as a Picard modular variety. This variety has a projective small resolution which is a rigid Calabi–Yau manifold ($h^{1,2} = 0$) with Euler number 72.

1. INTRODUCTION

In a sequence of papers, cf. [FS1], [FS2], [FS3] and [CFS], we studied Siegel threefolds which admit a Calabi–Yau model. We obtained a long list of projective Calabi–Yau manifolds. Some of them were new. In this paper we start to look to other kinds of modular examples. A natural class are the ball quotients which belong to the unitary group $U(1,3)$. Its arithmetic subgroups are called Picard modular groups and the corresponding varieties are called Picard modular varieties. In [FS4] we determined explicitly a Picard modular variety of general type. On the regular locus of this variety there are many holomorphic three forms with known zero divisors which have been constructed as Borchers products. Resolutions of quotients of this variety, such that the zero divisors are in the branch locus, are candidates for Calabi–Yau manifolds. The aim of this note is to treat one distinguished example of this. In fact we shall recover a known variety ([Me1, Chapter 5, Section 6]; see also [Me2]) given by the equations

$$X_0X_1X_2 = X_3X_4X_5, \quad X_0^3 + X_1^3 + X_2^3 = X_3^3 + X_4^3 + X_5^3,$$

as a Picard modular variety with respect to a certain Picard modular group G' . As it has been explained in [Me1], [Me2], this variety has a projective small resolution which is a rigid Calabi–Yau manifold ($h^{1,2} = 0$) with Euler number 72. Moreover, a certain group acts on this variety, fixing the holomorphic differential 3-form. We will show that this group is modular in the sense that it is given by a subgroup of the normalizer of G' in the full modular group.

Received by the editors May 24, 2012 and, in revised form, July 28, 2012.
2010 *Mathematics Subject Classification*. Primary 11F46; Secondary 14J32.

2. DIFFERENTIAL FORMS

Let V be a finite dimensional complex vector space which is equipped with a hermitian form $\langle \cdot, \cdot \rangle$ of signature $(1, n)$. A line (=one dimensional sub-vector-space of V) is called positive if it is represented by an element of positive norm ($\langle z, z \rangle > 0$). We denote by $\mathcal{B} = \mathcal{B}(V) \subset P(V)$ the set of all positive lines. (As usual $P(V)$ denotes the projective space of V , i.e. the set of all lines in V .) The unitary group $U(V) \cong U(1, n)$ acts on \mathcal{B} as a group of biholomorphic automorphisms.

We choose a sub-vector-space $W \in V$ of codimension one which contains no vector of positive norm. We also consider a vector $e \in V$ such that

$$V = \mathbb{C}e \oplus W.$$

If $v = Ce + w$, $w \in W$, is a vector of positive norm, then $C \neq 0$. Hence we can normalize the line $\mathbb{C}v$ such that $C = 1$. This gives an embedding of \mathcal{B} into W . The linear action of $U(V)$ on $\mathcal{B}(V)$ gives a rational action $z \mapsto g\langle z \rangle$ on the image of \mathcal{B} in W . If we write $g(e + z) = Ce + w$, then $g\langle z \rangle = w/C$. We have

$$g(e + z) = j(g, z)(e + g\langle z \rangle) \quad \text{where} \quad j(g, z) := C.$$

The function $j(g, z)$ is an automorphy factor. We call it the canonical automorphy factor. Another automorphy factor is the Jacobian.

Lemma 2.1. *The Jacobian of the transformation $z \mapsto g\langle z \rangle$ for $g \in U(V)$ is*

$$\det(g)j(g, z)^{-(n+1)}.$$

Proof. The proof can be given for reflections by a somewhat tedious computation. \square

Let M be a lattice in V , i.e. a discrete subgroup with compact quotient. We assume that M admits complex multiplication. Then the set of all complex numbers $a \in \mathbb{C}$ such that $aM \subset M$ is an order in an imaginary number field. The Picard modular group $U(M)$ is the subgroup of $U(V)$ that preserves M .

We recall the notion of a modular form of integral weight k . For this we consider the inverse image $\tilde{\mathcal{B}}$ of \mathcal{B} in $V - \{0\}$. It consists of all $z \in V$ with $\langle z, z \rangle > 0$. This is a connected open subset.

Let $G \subset U(M)$ be a subgroup of finite index and let $v : G \rightarrow \mathbb{C}^\bullet$ be a character.

A modular form of weight k on G with respect to a v is a holomorphic function $f : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$ with the following properties:

- 1) $f(tz) = t^{-k}f(z)$ for all $t \in \mathbb{C}^\bullet$,
- 2) $f(\gamma z) = v(\gamma)f(z)$ for all $\gamma \in G$,
- 3) f is regular at the cusps.

We denote this space of modular forms by $[G, k, v]$ or simply by $[G, k]$ when v is trivial. The subspace of cusp forms is denoted by $[G, k]_0$. The ring of modular forms

$$A(G) = \bigoplus [G, k]$$

is a finitely generated algebra. From the theory of Baily and Borel it follows that $A(G)$ is a finitely generated algebra whose associated projective variety can be identified with the compactification $X(G)$ of \mathcal{B}/G by (finitely many) cusps,

$$\text{proj}A(G) \cong X(G).$$

From Lemma 2.1 we obtain the following result.

Lemma 2.2. *There is a one-to-one correspondence between G -invariant differential forms of top degree n on \mathcal{B} and modular forms of weight $n+1$ and multiplier system $v(\gamma) = \det(\gamma)^{-1}$.*

We mention a well-known general fact. A G -invariant differential form of top degree n extends holomorphically to a smooth compactification of \mathcal{B}/G if and only if the associated modular form is cuspidal (compare [Fr, Satz III.2.6]).

3. A SPECIAL GROUP

We equip $V = \mathbb{C}^4$ with the hermitian form

$$\langle a, b \rangle = \bar{a}_1 b_2 + \bar{a}_2 b_1 - \bar{a}_3 b_3 - \bar{a}_4 b_4.$$

Let \mathcal{E} be the ring of Eisenstein numbers. We consider the lattice $M = \mathcal{E}^4$. We use the notation

$$G_3 = U(M).$$

We also consider the principal congruence subgroups

$$G_3[a] = \text{kernel}(G_3 \longrightarrow \text{GL}(4, \mathcal{E}/a)) \quad (a \in \mathcal{E}, a \neq 0).$$

In the paper [FS1] the ring of modular forms of $G_3[3]$ has been determined. We recall some of the main results. There are 15 forms B_1, \dots, B_{15} of weight 1 and 10 cusp forms C_1, \dots, C_{10} of weight 2. These 25 forms generate the ring of modular forms $A(G_3[3])$. The relations between the generators have been determined:

$$\dim[G_3[3], k] = \begin{cases} 0 & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ 15 & \text{for } k = 1, \\ 130 & \text{for } k = 2, \\ 750 & \text{for } k = 3, \\ 3115 & \text{for } k = 4, \\ -1377 + (8019/2)k - 2187tk^2 + (729/2)k^3 & \text{for } k > 4, \end{cases}$$

and

$$\dim[G_3[3], k] - \dim[G_3[3], k]_0 = \begin{cases} 15 & \text{for } k = 1, \\ 120 & \text{for } k = 2, \\ 405 & \text{for } k = 3, \\ 765 & \text{for } k = 4, \\ 810 & \text{for } k > 4. \end{cases}$$

Basic elements of G_3 are reflections along vectors b of norm $\langle b, b \rangle = -1$. They are defined by

$$a \longmapsto \frac{a - (1 - \eta)\langle b, a \rangle}{\langle b, b \rangle b}$$

where η is a 6th root of unity (hence a power of $-\zeta$). They transform b to ηb and act as an identity on the orthogonal complement of b . Their order equals the order of η . We call them biflections, triflections or hexfections corresponding to the order of η .

By a short mirror in $\tilde{\mathcal{B}}$ we understand the fixed point set of a triflection. The short mirrors decompose under $G_3[\sqrt{-3}]$ into 15 orbits:

1	2	3	4	5
(0, 0, 1, 0)	(0, 0, 0, 1)	(1, 0, 1, 0)	(1, 0, -1, 0)	(1, 0, 0, 1)
6	7	8	9	10
(1, 0, 0, -1)	(0, 1, 1, 0)	(0, 1, -1, 0)	(0, 1, 0, 1)	(0, 1, 0, -1)
11	12	13	14	15
(ζ , -1, 1, 1)	(ζ , -1, 1, -1)	(ζ , -1, -1, 1)	(ζ , -1, -1, -1)	(ζ , 1, 0, 0).

We recall the basic properties of the forms B_1, \dots, B_{15} . Their zero divisor (considered mod $G_3[3]$) consists of three short mirrors which are described in the following list:

(1,2,15), (2,4,8), (2,3,7), (1,6,10), (1,5,9), (12,13,15), (11,14,15), (4,6,11), (4,5,12), (8,10,14), (8,9,13), (3,6,13), (3,5,14), (7,10,12), (7,9,11).

We also recall the definition of the cusp forms C_i :

$$\begin{aligned}
 C_1 &= B_2 B_4 B_{15} / B_8, \\
 C_2 &= B_2 B_{13} B_{15} / B_3, \\
 C_3 &= B_3 B_6 B_{10} / B_{14}, \\
 C_4 &= B_3 B_5 B_8 / B_{15}, \\
 C_5 &= B_8 B_{13} B_{14} / B_9, \\
 C_6 &= B_5 B_7 B_{14} / B_{15}, \\
 C_7 &= B_2 B_6 B_{15} / B_{11}, \\
 C_8 &= B_1 B_8 B_{11} / B_2, \\
 C_9 &= B_6 B_{13} B_{15} / B_7, \\
 C_{10} &= B_2 B_4 B_6 / B_1.
 \end{aligned}$$

Each of them vanishes along six short mirrors (counted mod $G_3[3]$). For example C_1^2 vanishes along the mirrors with the digits 1, 2, 7, 8, 9, 10. We denote by G' the group which is generated by $G_3[3]$ and the six triflections corresponding to the above six short mirrors.

From Lemma 2.2 it follows that C_1^2 defines a G' -invariant differential form. Moreover, its zeros (of order two) disappear in the quotient \mathcal{B}/G' due to the ramification. Hence there is a chance that a suitable desingularization gives a Calabi–Yau manifold. We will now show that this is actually the case.

Lemma 3.1. *The index of the group $G_3[3]$ in G' is 486. It contains the negative of the identity. The covering degree of $X(G_3[3]) \rightarrow X(G')$ is 243.*

Proof. The proof can be given by means of a direct computation with the help of a computer. We used the computer algebra system MAGMA which allows us to define subgroups of $GL(n, R)$ for a finite R . The subgroups can be defined by means of a finite system of generators (in our case $R = \mathcal{E}/3\mathcal{E}$). □

The forms $B_6, B_7, B_8, B_9, B_{12}, B_{13}$ do not vanish along one of the six short mirrors. Hence they have a trivial multiplier with respect to G' .

Theorem 3.2. *The six modular forms above generate the ring $A(G')$. Defining relations are*

$$B_7B_9B_{12} = B_6B_8B_{13}, \quad B_6^3 + B_7^3 - B_8^3 + B_9^3 - B_{12}^3 - B_{13}^3 = 0.$$

The associated variety \mathcal{X} has 108 singularities which are nodes.

Proof. From the description of the zeros one can derive that the six forms have no joint zero. By a result of Hilbert, the ring $A(G')$ is integral over the subring $\mathbb{C}[B_6, B_7, B_8, B_9, B_{12}, B_{13}]$. The relations described in the theorem generate a prime ideal. A dimension argument shows that they are defining relations between the six forms. The Hilbert polynomial of the subring can be computed. The result is $(27t^2 + 9t^3)/3!$. Its highest coefficient is $3/2$. We also know that the highest coefficient of the Hilbert polynomial of $A(G')$ is $729/2$. The quotient of the two highest coefficients, 243, equals the covering degree $[G' : G_3[3]]$ of $X(G') \rightarrow X(G_3[3])$. So we see that $A(G')$ and the subring $\mathbb{C}[B_6, B_7, B_8, B_9, B_{12}, B_{13}]$ have the same field of fractions. So $A(G')$ is the normalization of this subring. One can check that the relations in the theorem define a normal ring. In fact it is a complete intersection and regular in codimension 2. This finishes the proof. \square

We introduce variables X_0, \dots, X_5 and consider the homomorphism

$$\mathbb{C}[X_0, \dots, X_5] \longrightarrow A(G')$$

which is defined by

$$X_0 \longmapsto -B_6, \quad X_1 \longmapsto B_8, \quad X_2 \longmapsto B_{13}, \quad X_3 \longmapsto B_7, \quad X_4 \longmapsto B_9, \quad X_5 \longmapsto -B_{12}.$$

Then the defining relations get

$$X_0X_1X_2 = X_3X_4X_5, \quad X_0^3 + X_1^3 + X_2^3 = X_3^3 + X_4^3 + X_5^3.$$

This is a very well-known variety. We learned from [Me1], [Me2] that it has 108 singularities which are all nodes. They correspond to certain cusps in the ball-model. Moreover, there is a projective small resolution which is a rigid Calabi-Yau manifold ($h^{1,2} = 0$) with Euler number 72. The Picard number is $h^{1,1} = 36$.

Theorem 3.3. *The variety \mathcal{X} defined in $P^5\mathbb{C}$ as the intersection of two cubics*

$$X_0X_1X_2 = X_3X_4X_5, \quad X_0^3 + X_1^3 + X_2^3 = X_3^3 + X_4^3 + X_5^3$$

can be identified with (compactified) ball quotient $X(G')$ with respect to the group G' . This gives an example of a three dimensional ball-quotient which admits a Calabi-Yau model.

There is an obvious group H of automorphisms of order 5832 acting on this variety. First we can consider permutations of the variables X_0, X_1, X_2 and X_3, X_4, X_5 separately and by interchanging these two blocs. This gives a group of order 72. Then we consider transformations $X_\nu \mapsto \zeta_\nu X_\nu$ where ζ_ν are third roots of unity with the property

$$\zeta_0\zeta_1\zeta_2 = \zeta_3\zeta_4\zeta_5.$$

Taking into account that (ζ, \dots, ζ) acts as an identity, this gives us a group of order 81 acting on the projective variety. Both types generate a group of order $72 \cdot 81 = 5832$ as has been stated. In fact, these automorphisms are modular.

Remark 3.4. Let N' be the normalizer of G' in the full modular group. There is a natural surjective homomorphism

$$N' \longrightarrow H$$

which is compatible with the actions of N' on $X(G')$ and of H on \mathcal{X} . The kernel of this homomorphism is generated by G' and by the transformation “multiplication by ζ ”.

Proof. Since we know the action of the full modular group on the generators, the statement can be checked by means of a direct computation. We did it with the help of MAGMA which can handle homomorphisms of finite groups, describing them on generators and computing their kernels. \square

We want to determine the subgroup that leaves the Calabi–Yau form invariant. The Calabi–Yau form (given in the modular picture by the modular form C_1^2) can be described in the model \mathcal{X} using a result of van Geemen that we learned by [GKY], namely

Lemma 3.5. *Let X be a complete intersection variety in P^n of dimension $d =: n - k$ with homogeneous equations f_1, \dots, f_k in the homogeneous variables x_0, \dots, x_n . Then on the open set where $Dx_0 \neq 0$, a holomorphic d -form is given by*

$$\omega = \frac{dx_{k+1} \wedge \dots \wedge dx_n}{D}$$

where

$$D = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i, j \leq k}.$$

Furthermore, when X is a Calabi–Yau variety, then ω extends to a holomorphic d -form on all of X .

Hence in this case we have

$$\omega = \frac{dX_3 \wedge dX_4 \wedge dX_5}{3(X_2^3 - X_1^3)}.$$

The group H extends to $P^5\mathbb{C}$ in an obvious way. It acts on ω with a character χ . This character can be computed easily.

Remark 3.6. The character χ of H that describes the action of H on the Calabi–Yau form is defined by the following properties:

- 1) For a permutation of X_0, X_1, X_2 and similarly of X_3, X_4, X_5 it is the sign of the permutation.
- 2) For the permutation $X_0 \leftrightarrow X_3, X_1 \leftrightarrow X_4, X_2 \leftrightarrow X_5$ it is -1 .
- 3) For a transformation $X_\nu \mapsto \zeta_\nu X_\nu$ where ζ_ν are third roots of unity with the property $\zeta_0 \zeta_1 \zeta_2 = \zeta_3 \zeta_4 \zeta_5$ the value of χ is $\zeta_0 \zeta_1 \zeta_2$. The kernel of χ is a subgroup of index 6 in H which has the order 972.

It might be worthwhile to study subgroups of the kernel of χ such that the corresponding quotient of \mathcal{X} admits a Calabi–Yau model.

REFERENCES

- [CFS] S. Cynk, E. Freitag, and R. Salvati Manni, *The geometry and arithmetic of a Calabi-Yau Siegel threefold*, Internat. J. Math. **22** (2011), no. 11, 1585–1602, DOI 10.1142/S0129167X1100732X. MR2863441 (2012m:11061)
- [Fr] E. Freitag, *Siegelsche Modulfunktionen* (German), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 254, Springer-Verlag, Berlin, 1983. MR871067 (88b:11027)
- [FS1] Eberhard Freitag and Riccardo Salvati Manni, *On Siegel three-folds with a projective Calabi-Yau model*, Commun. Number Theory Phys. **5** (2011), no. 3, 713–750, DOI 10.4310/CNTP.2011.v5.n3.a5. MR2864465
- [FS2] Eberhard Freitag and Riccardo Salvati Manni, *Some Siegel threefolds with a Calabi-Yau model II*, Kyungpook Math. J. **53** (2013), no. 2, 149–174, DOI 10.5666/KMJ.2013.53.2.149. MR3078079
- [FS3] Eberhard Freitag and Riccardo Salvati Manni, *On Siegel three-folds with a projective Calabi-Yau model*, Commun. Number Theory Phys. **5** (2011), no. 3, 713–750, DOI 10.4310/CNTP.2011.v5.n3.a5. MR2864465
- [FS4] Eberhard Freitag and Riccardo Salvati Manni, *A three dimensional ball quotient*, Math. Z. **276** (2014), no. 1-2, 345–370, DOI 10.1007/s00209-013-1203-4. MR3150208
- [GKY] Fernando Q. Gouvêa, Ian Kiming, and Noriko Yui, *Quadratic twists of rigid Calabi-Yau threefolds over \mathbb{Q}* , Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, Fields Inst. Commun., vol. 67, Springer, New York, 2013, pp. 517–533, DOI 10.1007/978-1-4614-6403-7_20. MR3156430
- [Me1] C. Meyer, *A dictionary of modular threefolds*, thesis, University of Mainz, Fachbereich Mathematik und Informatik (2005)
- [Me2] Christian Meyer, *Modular Calabi-Yau threefolds*, Fields Institute Monographs, vol. 22, American Mathematical Society, Providence, RI, 2005. MR2176545 (2007f:11070)

MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, D69120 HEIDELBERG, GERMANY
E-mail address: freitag@mathi.uni-heidelberg.de

UNIVERSITÁ DI ROMA, PIAZZALE ALDO MORO, 2, I-00185 ROMA, ITALY
E-mail address: salvati@mat.uniroma1.it