

OPEN ASPHERICAL MANIFOLDS NOT COVERED BY THE EUCLIDEAN SPACE

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ABSTRACT. We show that any open aspherical manifold of dimension $n \geq 4$ is tangentially homotopy equivalent to an n -manifold whose universal cover is not homeomorphic to \mathbb{R}^n .

1. INTRODUCTION

Davis famously constructed, for every $n \geq 4$, a closed aspherical n -manifold whose universal cover is not homeomorphic to \mathbb{R}^n [Dav83]. We prove:

Theorem 1.1. *If the universal cover of an open smooth manifold V is diffeomorphic to \mathbb{R}^n with $n \geq 4$, then the tangential homotopy type of V contains a continuum of open smooth n -manifolds whose universal covers are not homeomorphic to \mathbb{R}^n .*

Our proof also works in PL and TOP categories. Recall that a homotopy equivalence $f: V \rightarrow W$ of n -manifolds is *tangential* if f^*TX and TV are stably isomorphic, where TX denotes the tangent bundle if X is a smooth manifold and the tangent microbundle in the PL or TOP cases. By [Sie69, Theorem 2.3] this is equivalent to requiring that $f \times \text{id}(\mathbb{R}^s)$ is homotopic to a CAT-homeomorphism for some s (where CAT equals DIFF, PL or TOP).

In the simply-connected case Theorem 1.1 is due to Curtis-Kwun [CK65] for $n \geq 5$ and to Glaser [Gla67] for $n = 4$. The proof combines three ingredients:

(1) A result of Curtis-Kwun [CK65] that for a boundary connected sum S of a countable family of compact n -manifolds, the homeomorphism type of $\text{Int}(S)$ determines the isomorphism class of $\pi_1(\partial S)$.

(2) A recent result of Calcut-King-Siebenmann [CKS12] that any countable collection of CAT properly embedded \mathbb{R}^{n-1} 's in \mathbb{R}^n is CAT unknotted, which generalizes classical results of Cantrell and Stallings.

(3) The existence of infinitely many smooth compact contractible n -manifolds whose boundary homology spheres have freely indecomposable fundamental groups. (Such examples can be found in [CK65, Gla67], and more examples are now known; see e.g. Casson-Harer [CH81] for aspherical homology 3-spheres that bound smooth contractible 4-manifolds, while Kervaire [Ker69] showed that the fundamental group of any homology 3-sphere appears as the fundamental group of the boundary of a smooth contractible n -manifold for any $n \geq 5$.)

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Proof. Since V is open, it contains a CAT properly embedded ray whose CAT regular neighborhood is an embedded closed halfspace; see e.g. [CKS12, Section 3]. Hence V is CAT isomorphic to the interior of a noncompact manifold N whose boundary is an open disk. By the strong version of the Cantrell-Stalling hyperplane unknotting theorem proved in [CKS12, Corollary 9.3], the universal cover of N can be compactified to D^n , the n -disk, in which the preimage of ∂N becomes the union of a countable collection of round open disks with pairwise disjoint closures.

Let $\{C_i\}_{i \in \mathbb{N}}$ be an infinite sequence of compact contractible n -manifolds, such that $\pi_1(\partial C_i)$ are pairwise nonisomorphic and freely indecomposable. Given a subset $\alpha \subseteq \mathbb{N}$, let C_α be a boundary connected sum of C_i 's with indices in α . (For our purposes the choices involved in defining boundary connected sums will always be irrelevant.) Fix a closed $(n-1)$ -disk $\Delta \subset \partial C_\alpha$, and let N_α be a boundary connected sum of N and C_α obtained by identifying Δ with a closed disk in ∂N . A deformation retraction $C_\alpha \rightarrow \Delta$ extends to a deformation retraction of $N_\alpha \rightarrow N$, so $V_\alpha := \text{Int}(N_\alpha)$ is tangentially homotopy equivalent to V .

If Q_α denotes a boundary connected sum of countably many copies of C_α , then the interior of the universal cover of N_α is homeomorphic to the interior of a boundary connected sum of D^n and Q_α , which is homeomorphic to $\text{Int}(Q_\alpha)$. By [CK65, Theorem 4.1] if Q_α, Q_β have homeomorphic interiors, then $\partial Q_\alpha, \partial Q_\beta$ have isomorphic fundamental groups. Now $\pi_1(\partial Q_\alpha)$ is a free product in which each factor $\pi_1(\partial C_{i_k}), i_k \in \alpha$ appears countably many times. Each $\pi_1(\partial C_{i_k})$ is freely indecomposable, so $\alpha = \beta$ by Grushko's theorem. Thus the universal covers of $\text{Int}(N_\alpha)$ lie in a continuum of homeomorphism types. \square

Remark 1.2. The proof of [CK65, Theorem 4.1] is quite technical, which may be due to the fact that the tools of Siebenmann's thesis were not yet available at the time, so we summarize it in a modern language: Given $\alpha = \{i_1, \dots, i_k, \dots\}$ it is easy to construct a cofinal family $\{U_k\}_{k \geq 1}$ of neighborhoods of infinity in Q_α such that in the corresponding inverse sequence of fundamental groups, the group $\{\pi_1(U_k)\}$ is the free product of $\pi_1(C_{i_1}) * \dots * \pi_1(C_{i_k})$, and the map $\pi_1(U_k) \leftarrow \pi_1(U_{k+1})$ is a retraction onto the first k factors, and, in particular, is surjective. Hence the inverse sequence of groups is Mittag-Leffler, and therefore its pro-equivalence class depends only on the homeomorphism type of Q_α . Now if Q_α, Q_β have homeomorphic interiors, then a simple diagram chase in the commutative diagram from the definition of pro-equivalence shows that each free factor of $\pi_1(Q_\alpha)$ occurs as a free factor of $\pi_1(Q_\beta)$, so $\alpha = \beta$.

Remark 1.3. Theorem 1.1 should hold for $n = 3$, but our proof fails. One could try substituting the boundary connected sum of N and Q_α by the end sum of V with a suitable Whitehead manifold, but the multiple hyperplane unknotting is no longer true, due to the existence of an exotic $[0, 1] \times \mathbb{R}^2$ [ST89]. This makes analyzing the fundamental group of infinity more delicate. For the same reason we do not attempt to prove Theorem 1.1 for V whose universal cover is not \mathbb{R}^n .

Remark 1.4. By the Cartan-Hadamard theorem any (complete Riemannian) n -manifold of nonpositive curvature is covered by \mathbb{R}^n . So given an open n -manifold of nonpositive curvature with $n \geq 4$, Theorem 1.1 yields a continuum of n -manifolds in the same tangential homotopy type that admit no metric of nonpositive curvature. In [Bel] the author studied obstructions to nonpositive curvature on open manifolds that go beyond the Cartan-Hadamard theorem.

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