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ON INDECOMPOSABILITY IN CHAOTIC ATTRACTORS

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ABSTRACT. We exhibit a Li-Yorke chaotic interval map F such that the inverse limit $X_F = \varprojlim\{F, [0, 1]\}$ does not contain an indecomposable subcontinuum. Our result contrasts with the known property of interval maps: if φ has positive entropy then X_φ contains an indecomposable subcontinuum. Each subcontinuum of X_F is homeomorphic to one of the following: an arc, or X_F, or a topological ray limiting to X_F. Through our research, we found that it follows that X_F is a chaotic attractor of a planar homeomorphism. In addition, F can be modified to give a cofrontier that is a chaotic attractor of a planar homeomorphism but contains no indecomposable subcontinuum. Finally, F can be modified, without removing or introducing new periods, to give a chaotic zero entropy interval map, such that the corresponding inverse limit contains the pseudoarc.

1. Introduction

The strong connection between dynamics of an interval map $\varphi\colon [0, 1] \to [0, 1]$ and topology of the inverse limit $X_\varphi = \varprojlim \{\varphi, [0, 1]\}$ has been well documented in the last 30 years. An extensive study of this and related subjects was triggered by a series of papers by Marcy Barge and his collaborators. Among many results, Barge and Martin [3] showed that for an interval map with a periodic point of period that is not a power of 2, the inverse limit space X_φ must contain an indecomposable subcontinuum. Barge and Martin [4] also showed that for any interval map φ such inverse limit can be realized as an attractor of a planar homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that restricted to X_φ agrees with the shift homeomorphism σ_φ. Since then there has been a lot of attention given to the problem of relating the dynamics of a map to the topological structure of the corresponding inverse limit, and the principle that complicated dynamics induces complicated topology has become well-known and often referred to. The purpose of this article is to show that one must be careful applying this principle, as a chaotic interval map can produce a connected attractor without indecomposable subcontinua. It seems that ours is the first such example presented explicitly. This is despite the fact that for a positive entropy map φ the inverse limit space X_φ must contain an indecomposable subcontinuum [30].

Theorem 1. *There is a map $F\colon [0, 1] \to [0, 1]$ such that the inverse limit $X_F = \varprojlim \{F, [0, 1]\}$ contains no indecomposable subcontinuum (in particular, X_F is decomposable) and the induced shift homeomorphism σ_F on X_F is Li-Yorke chaotic.*

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The map F in the above theorem can be modified to a circle map with the same properties, which by the result of Barge and Martin leads to the following theorem.

Theorem 2. *There are planar homeomorphisms h_1 and h_2, an arc-like continuum Λ_1 and cofrontier Λ_2 such that Λ_i is a Li-Yorke chaotic attractor of h_i, and neither Λ_i contains an indecomposable subcontinuum.*

Before we progress, let us first briefly present definitions of some notions used above. The notion of chaos we use here comes from a paper by Li and Yorke [19]. A continuous map $\varphi\colon X \to X$ acting on a compact metric space (X, ρ) is *Li-Yorke chaotic* if there is an uncountable set $S \subset X$ such that $\liminf_{n\to\infty} \rho(\varphi^n(x), \varphi^n(y)) = 0$ and $\limsup_{n\to\infty} \rho(\varphi^n(x), \varphi^n(y)) > 0$ for any distinct points $x, y \in S$. It is known that there exist maps on the unit interval with zero topological entropy but is Li-Yorke chaotic. These are some among the maps of type 2^∞, i.e. maps with points of period 2^n for every n and no other periods.

A *continuum* is a nondegenerate connected and compact space. A continuum A is a *Li-Yorke chaotic attractor* of a planar homeomorphism h if A is an attractor and $h|A$ is Li-Yorke chaotic. An *arc-like* (also snakelike, or chainable) continuum is a space that can be obtained as the inverse limit of arcs, with continuous bonding maps. Arc-like continua do not separate the plane. A *cofrontier* is a continuum that irreducibly separates the plane into exactly two components and is the boundary of each. A continuum is *decomposable* if it can be written as the union of two proper subcontinua. It is *hereditarily decomposable* if every subcontinuum is decomposable.

It was a long-standing conjecture of Barge that no hereditarily decomposable arc-like continuum admits homeomorphisms with positive entropy. The special case of Barge's conjecture was proved by Ye in 1995 [30] for homeomorphisms induced by square commuting diagrams on inverse limits of arcs. Ingram [14] and Ye independently also showed that homeomorphisms of hereditarily decomposable continua admit only 2^n-periodic orbits, so their dynamics are relatively simple. Barge's conjecture has been recently proved by Mouron [26], and consequently, hereditarily decomposable arc-like continua admit only zero entropy homeomorphisms. However, our result shows that chaotic homeomorphisms on such continua actually do exist.

The starting point of our construction is a simple, zero entropy interval map f of type 2^∞. In Section 2, using a theorem of Bennett and Ingram [15], we are able to show that X_f contains a countable family of decomposable continua, each of which is homeomorphic to X_f. Furthermore, each subcontinuum of X_f is a member of this family, or a topological ray limiting to such a continuum, or an arc. Next, in Section 3, we modify f by a Denjoy-like construction to produce a Li-Yorke chaotic zero entropy map F of type 2^∞. We show that this modification results in a topologically monotone factor map $\Pi\colon X_F \to X_f$, which guarantees that X_F is hereditarily decomposable. We then modify f to a Li-Yorke-chaotic circle map G such that X_G is hereditarily decomposable. The last section contains additional comments and questions related to our construction.

2. A map of type 2^∞ and its inverse limit

In this section we construct a particular example of a map of type 2^∞. While there are numerous methods of construction of such a map (see e.g. [2,12,24]), even of type C^∞, a map f considered in this section has an additional property, that its inverse limit can be easily investigated. It is the main feature demanded by us.

Define a map $f\colon [0,1] \to [0,1]$ determined by the following (see Figure 1)

- $f(0) = \frac{2}{3}, f(1) = 0$,
- $f(1 - \frac{2}{3^n}) = \frac{1}{3^{n-1}}$, and $f(1 - \frac{1}{3^n}) = \frac{2}{3^{n+1}}$ for all $n \geq 1$,
- f is linear between the above points.

This example was developed by Delahaye in [10] who proved that the map is of type 2^∞ (see also [28]).

For the remainder of this section, denote by σ_f the shift homeomorphism induced by f to $X_f = \varprojlim \{f, [0,1]\}$. For convenience, we sometimes denote $\varprojlim \{f|_Y, Y\}$ simply by $\varprojlim \{f, Y\}$. The projection of X onto n-th coordinate is denoted by $\pi_n\colon X \ni x \mapsto x_i \in [0,1]$. Let $I_0^n = [0, 1/3^n]$ for $n = 1, 2, \ldots$. These are intervals

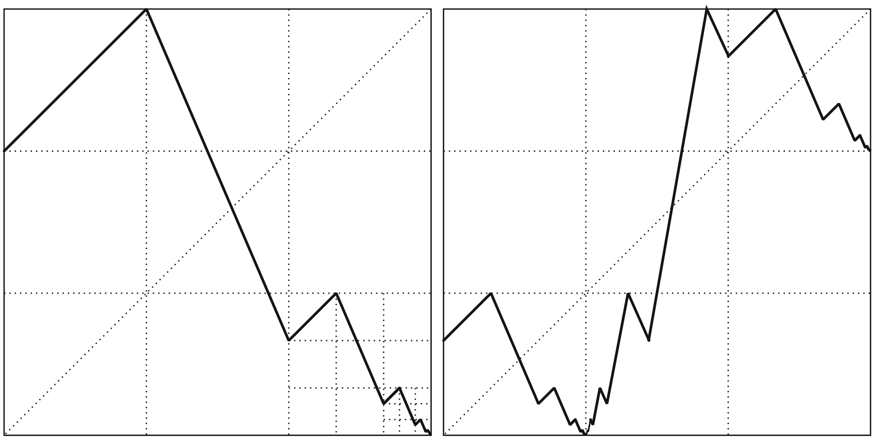

FIGURE 1. Graph of f and f^2

for cycles of length 2^n, i.e. $f^{2^n}(I_0^n) = I_0^n$. Denote $I_j^n = f^j(I_0^n)$ for $j = 0, 1, \ldots, 2^n$ (we keep $I_{2^n}^n = I_0^n$ for simplicity of the notation). It can be proved that if $x \in [0,1]$ and $n > 0$, then either there is $k > 0$ such that $f^k(x) \in I_0^n$ or there is $s > 0$ such that x is a periodic point of period 2^s. It can also be proved that f is not Li-Yorke chaotic.

Observe that $f^{2^n}|_{I_s^n}\colon I_s^n \to I_s^n$ is an onto map. Denote by X_0^n the inverse limit $X_0^n = \varprojlim \left\{g_i, I^n_{-i \pmod{2^n}}\right\}$ where $g_i = f|_{I^n_{-i \pmod{2^n}}}$ for $i = 1, 2, \ldots$. Denote $X_i^n = \sigma_f^i(X_n^0)$. Clearly X_0^n is periodic under σ_f and $X_{2^n}^n = X_0^n$ and furthermore, $X_0^{n+1} \cup X_{2^n}^{n+1} \subset X_0^n$.

A homeomorphic image of $[0, +\infty)$ is a *topological ray* and homeomorphic image of $(-\infty, +\infty)$ is a *topological line*.

The following useful result is attributed to Ralph Bennett. A proof (with a historical remark) can be found in [15].

Theorem 3 (Bennett)**.** *Suppose that $g\colon [a,b] \to [a,b]$ is continuous and $a < d < b$ is such that $g([d,b]) \subset [d,b]$, $g|_{[a,d]}$ is monotone, and there is $n > 0$ such that $g^n([a,d]) = [a,b]$. Then continuum $K = \varprojlim \{g, [a,b]\}$ is the union of a topological ray R and a continuum $C = \varprojlim \{g, [d,b]\}$ such that $\overline{R} \setminus R = C$.*

Lemma 4. *Each continuum X^i_j is homeomorphic to X_f.*

Proof. By induction, it is easy to see that the graph of f^{2^n} on I^n_0 is the same as $f^{2^{n-1}}$ on I^{n-1}_0, that is, these maps are conjugate, or in other words, continua X^n_0 and X^{n-1}_0 are homeomorphic. The theorem follows for $j \neq 0$ by the fact that for a fixed i, $X^i_j = \sigma^j_f(X^i_0)$ and σ_f is a homeomorphism. $\square$

Lemma 5. *The continuum X_f is the union of two continua K_1 and K_2 such that*

(1) *K_1 is homeomorphic to K_2,*
(2) *K_1 is the union of a topological ray R_1 and X^1_0 that compactifies R_1; i.e. $\overline{R_1} \setminus R_1 = X^1_0$,*
(3) *K_2 is the union of a topological ray R_2 and X^1_1 that compactifies R_2; i.e. $\overline{R_2} \setminus R_2 = X^1_1$, and*
(4) *$K_1 \cap K_2 = R_1 \cap R_2 = \{\hat{p}\}$, where $\hat{p}$ is the fixed point of σ_f.*

Proof. Let p be the fixed point of f. Set $g = f^2$ and let $K_1 = \varprojlim\{g, [p,1]\}$. Note that $g([13/21,1]) \subseteq [13/21,1]$, $g|_{[p,13/21]}$ is monotone, and $g([p,13/21]) = [p,1]$. Therefore, by Theorem 3, we obtain that K_1 is the union of a topological ray R_1 and the continuum $C_1 = \varprojlim\{g,[13/21,1]\}$ that compactifies R_1. Clearly

$$C_1 = \varprojlim\{g,[13/21,1]\} = \varprojlim\{g,[2/3,1]\} = X^1_0,$$

and $\hat{p} = (p,p,p,\ldots)$ is the end point of R_1. Setting $K_2 = \varprojlim\{g,[0,p]\}$ the theorem follows by the fact that $\sigma_f(K_1) = K_2$. $\square$

Corollary 6. *Each X^i_j is the union of a topological line L and the continua X^{i+1}_t and $X^{i+1}_{t'}$ such that $\overline{L} \setminus L = X^{i+1}_t \cup X^{i+1}_{t'}$, for some t and t'.*

Proof. This follows from the previous two lemmas. $\square$

Theorem 7. *Continuum X_f is hereditarily decomposable.*

Proof. Since by Lemma 5 continuum X_f is decomposable, we need to show that so is each subcontinuum of X_f. Let K be a subcontinuum of X_f. Recall that X_f is the union of a topological line L limiting with one end to X^1_0 and with the other to X^1_1. Using the previous lemmas we will keep partitioning X_f (if necessary) to find where K is located and realize that K must be an arc, or homeomorphic to K_1 from Lemma 5, or homeomorphic to X_f. By Lemma 4 we can view each X^n_i as X_f, in particular we can apply partitioning provided by Lemma 5 to it. We will use this fact without any further reference in the proof.

(1) suppose that $K \cap L \neq \emptyset$. If $L \subseteq K$ then $K = X_f$ and we are done. Otherwise, if $L \setminus K \neq \emptyset$, then K is an arc (this is when $K \subseteq L$), or it is the union of a topological ray limiting to either X^1_0 or X^1_1, and we are done as well.
(2) suppose that $K \cap L = \emptyset$. Then either $K \subseteq X^1_0$ or $K \subseteq X^1_1$. Without loss of generality assume $K \subseteq X^1_0$.
(3) let L_1 be the topological line whose union with the continua X^2_0 and X^2_2, that compactify L_1, is X^1_0. In other words $\overline{L_1} \setminus L_1 = X^2_0 \cup X^2_1$ and $\overline{L_1} = X^1_0$. If $K \cap L_1 \neq \emptyset$ then we are done by the same reasoning as in (1).

(4) if $K \cap L_1 = \emptyset$ then, as in (2), we deduce that $K \subseteq X_0^2$.
(5) from the fact that $\lim_{i\to\infty} \operatorname{diam}(X_0^i) = 0$ it follows that after finitely many steps we will be able to deduce that K is an arc, or the union of a topological ray limiting to some X_j^n or $K = X_j^n$ for some integers n, j. Namely, for X_0^n such that $\operatorname{diam}(X_0^n) < \operatorname{diam}(K)$ we cannot have $K \subseteq X_0^n$ so the above procedure terminates.

The proof is complete. □

A continuum that contains exactly n topologically distinct subcontinua is called n-equivalent. As we exhibited in the above proof, X_f is 3-equivalent. It is worth emphasizing, that an interesting example of 2-equivalent continuum was recently constructed by Islas [16], who proved that his example was hereditarily decomposable but without investigating the dynamical properties of it. In fact, Islas is using a sequence of bonding maps, so there is no easy way to induce a homeomorphism on the resulting continuum.

3. Chaos in the sense of Li and Yorke

The aim of this section is to prove Theorems 1 and 2. A starting point is the map constructed in Section 2 (recall that its graph is on Figure 1) which we consequently denote f.

We will perform a construction similar to that of a Denjoy map [11, Example 14.9]. First note that for all but countably many points $c \in (0,1)$ there is an open set $U \ni c$ such that f is injective on U.

Denote by Q the ω-limit set of 0 under f (i.e. $Q = \omega(0, f)$) and observe that for every $c \in Q$ and every n there is j such that $c \in I_j^n$ and hence orbit of c visits each interval I_i^n with period 2^n. But $\operatorname{diam} I_j^n = 3^{-n}$ hence the family of iterates of $f|_Q$ is equicontinuous. Note that $f|_Q$ is a homeomorphism, since every transitive map that has equicontinuous iterates is a homeomorphism (see [1]). It is also not hard to see that if $c \in [0,1]$ then $\omega(c, f)$ is periodic orbit (i.e. c is eventually periodic) or $Q = \omega(c, f)$. Namely, if $\omega(c, f)$ in not periodic orbit then for every n the orbit of c has to eventually intersect the interval I_0^n.

Choose a point $z \in Q$, denote $D_0 = \{z, f(z)\} \cup f^{-1}(\{z\})$ and inductively $D_{n+1} = f(D_n) \cup f^{-1}(D_n)$. Finally put

$$D_z = \bigcup_{n=1}^{\infty} D_n. \tag{1}$$

Since f is a homeomorphism on Q, for points z from different orbits, sets D_z are disjoint. But Q is uncountable and each point has finite preimage under f, hence we can find z such that for every $c \in D_z$ there is an open set $U \ni c$ such that f is an injection on U. Note that there at most countably many points $q \in Q$ such that $(q, q+\varepsilon) \cap Q = \emptyset$ or $(q-\varepsilon, 1) \cap Q = \emptyset$ for some $\varepsilon > 0$. Hence we may also assume that for every $\varepsilon > 0$ and for every $c \in D_z$ we have $(c-\varepsilon, c) \cap Q \neq \emptyset$ and $(c, c+\varepsilon) \cap Q \neq \emptyset$.

In particular, D_z is countable and so we can enumerate its elements: assume that $D = \{y_i \;:\; i \in \mathbb{Z}\}$ where $y_i \neq y_j$ for $i \neq j$. Furthermore observe that if $f^n(y_i) = y_j$ for some $n > 0$ then $i \neq j$ and $y_i \notin \operatorname{Orb}^+(y_j, f)$, as otherwise z would be an eventually periodic point. Just by the definition, both sets D_z and $[0,1] \setminus D_z$ are invariant, i.e. $f(D_z) = D_z$ and $f([0,1] \setminus D_z) = [0,1] \setminus D_z$. There is also a function $\phi\colon \mathbb{Z} \to \mathbb{Z}$ so that $f(y_i) = y_{\phi(i)}$.

As the final step of our construction we remove all the points y_i from $[0,1]$ and fill each obtained hole with an interval I_i of length $2^{-|i|}$. This way a new continuous map F is defined on the extended space in such a manner that:

(1) each interval I_i is mapped homeomorphicaly onto $I_{\phi(i)}$,
(2) if all intervals I_i are collapsed back to single points then F reverts back to the map f.

Condition (1) can be satisfied because the preimage $f^{-1}(y_i)$ of every y_i is finite and, by the choice of z, the map f is injective on some small neighborhood of every $y \in f^{-1}(y_i)$.

As the domain of F is isometric to $[0,4]$ we can assume that $F\colon [0,4] \to [0,4]$. In this way every interval I_i becomes some interval $[a_i, b_i] \subset (0,4)$ and there is a quotient map $\pi\colon [0,4] \to [0,1]$ that does not increase distance, collapses every interval $[a_i, b_i]$ into a single point (i.e. $\pi([a_i, b_i]) = \{y_i\}$), and has the property that $f \circ \pi = \pi \circ F$. If we fix indices $i, j \in \mathbb{Z}$, such that $y_i \notin \mathrm{Orb}^+(y_j)$ then $F^n((a_j, b_j)) \cap (a_i, b_i) = \emptyset$ for all $n > 0$. This implies that there is one-to-one correspondence between periodic points of f and F, which implies that F is also of type 2^∞, in particular has zero topological entropy. Simply, by Misiurewicz theorem, on the interval positive entropy is equivalent to the existence of a horseshoe for some power of the map [23], which easily implies existence of a periodic point with period which is not a power of 2.

In [29] Smítal characterized Li-Yorke chaos in terms of separable orbits in ω-limit sets. We will use this result here. Let $\varphi\colon [0,1] \to [0,1]$ be continuous and fix two points $x_0, x_1 \in [0,1]$. If there are two disjoint intervals J_0, J_1 and two integers $k_0, k_1 > 0$ such that for $i = 0, 1$ we have $x_i \in J_i$, $\varphi^{k_i}(J_i) = J_i$ and $\varphi^j(J_i)$ are pairwise disjoint for $j = 0, 1, \ldots, k_i - 1$ then we say that x_0, x_1 are *φ-separable.*

It was proved in [29, Theorem 2.2] that a map $\varphi\colon [0,1] \to [0,1]$ is Li-Yorke chaotic if and only if there is an infinite ω-limit set containing two points which are not φ-separable. Note that if we fix $q \in Q \setminus D_z$ then for every $c \in D_z$ and every $\varepsilon > 0$ we have $k, s > 0$ such that $f^k(q) \in Q \cap (c - \varepsilon, c)$ and $f^s(q) \in Q \cap (c, c + \varepsilon)$. If we denote the unique point $v \in \pi^{-1}(q)$ then it is clear that $\pi^{-1}(Q \setminus D_z)$ is contained in the ω-limit set of v under F, i.e.

$$v \in \omega(v, F) \supset \overline{\pi^{-1}(Q \setminus D_z)} \supset \bigcup_{i \in \mathbb{Z}} \{a_i, b_i\}.$$

Since diameters of intervals $\lim_{i\to\infty} \operatorname{diam} I_i = 0$, there is an asymptotic (hence not F-separable) pair for F in $\omega(v, F)$, e.g. pair a_0, b_0. This shows that F is Li-Yorke chaotic.

Denote $X_F = \varprojlim \{F, [0,4]\}$. Let $\Pi\colon X_F \to X_f$ be given by

$$\Pi(x) = (\pi(x_1), \pi(x_2), \pi(x_3), \ldots).$$

Recall that a map $T\colon X_1 \to X_2$ between two continua X_1 and X_2 is *(topologically) monotone* if $T^{-1}(x)$ is a subcontinuum of X_2 for every $x \in X_1$. Equivalently, T is monotone if $T^{-1}(K)$ is a subcontinuum of X_2 for every subcontinuum K of X_1.

Proposition 8. $\Pi\colon X_F \to X_f$ *is an onto and monotone map.*

Proof. First note that, by definition, $\pi\colon [0,4] \to [0,1]$ is a monotone map. Now let $x \in X_F$. If $\pi_1(x) = y_j$ for some j then $\Pi^{-1}(x)$ is an arc, as it is the inverse limit of I_i's with the homeomorphism F, when restricted to either I_i. If $\pi_1(x) \neq y_j$ for every j then $\Pi^{-1}(x)$ is a point. □

Lemma 9. *Continuum X_F is hereditarily decomposable.*

Proof. Let Z be a nondegenerate subcontinuum of X_F. It is enough to show that Z is decomposable. Note that if $\Pi(Z)$ is a point, then the projection of Z from X_F onto either factor space is contained in I_j, for some j. Consequently Z is homeomorphic to an arc, by definition of F. If $\Pi(Z)$ is a nondegenerate subcontinuum of X_f, then $\Pi(Z) = W_1 \cup W_2$ for two proper subcontinua W_1 and W_2 of X_f. Since Π is monotone we deduce that $\Pi^{-1}(W_1)$ and $\Pi^{-1}(W_2)$ are subcontinua of X_F such that $Z = \Pi^{-1}(W_1) \cup \Pi^{-1}(W_2)$. This completes the proof. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 9, X_F is hereditarily decomposable and by previous discussion F is a continuous onto map of type 2^∞ which is Li-Yorke chaotic. But Li-York chaos is shared by the shift homeomorphism on inverse limits [9], hence the result follows. □

Clearly, not every map of type 2^∞ defines a hereditarily decomposable inverse limit. For example, when constructing the map F we can define $F\colon I_i \to I_{\phi(i)}$ using any map fixing endpoints (e.g. maps presented in Example 4 or Example 5 in [3]), not necessarily linear homeomorphism. While such a modification has no influence on either the type of a map (new periodic points cannot be produced), or Li-Yorke chaos, an indecomposable subcontinuum such as the Knaster buckethandle continuum, or even the pseudoarc can be introduced in X_F.

Remark 10. There is a Li-Yorke chaotic interval map φ of type 2^∞ such that X_φ contains the pseudoarc.

The above observation also explains why we were so careful about the choice of the point z (and the set D_z) for the Denjoy extension. For example $0 \in Q$ however $f^k(0)$ is a singular point (i.e. point in which f is not monotone) for infinitely many values of $k > 0$. But if we insert I_i in a point at which f is not monotone, then F must send both endpoints of I_i into the same endpoint of $I_{\phi(i)}$. This forces us to send an inner point of I_i into the second endpoint of $I_{\phi(i)}$, and could lead to an indecomposable subcontinuum in X_F.

Recall that a continuum X is said to be *Suslinean* if every family of pairwise-disjoint subcontinua of X is countable (finite or not). Note that each Suslinean continuum is hereditarily decomposable. We note that both X_f and X_F are Suslinean.

Proposition 11. *Continuum X_f is Suslinean.*

Proof. We take advantage of the partition of X_f used in the proof of Theorem 7. By contradiction, suppose $\aleph$ is an uncountable cardinal and $\{C_\beta : \beta < \aleph\}$ is a family of pairwise disjoint subcontinua of X_f. Because the topological line limiting to the continua X_0^1 and X_1^1 is Susliniean, uncountably many C_β's must be contained in either X_1^0 or X_1^1. Without loss of generality suppose X_0^1 contains uncountably many C_β's. Since, according to Theorem 7, X_0^1 is a union of a topological line L and two continua X_0^2 and X_3^2 homeomorphic to X_f, and L is Susliniean, either X_0^2 or X_3^2 must contain uncountably many C_β's. Proceeding with the continua X_j^i by induction on i we obtain a contradiction since otherwise for some sequence i_n the set $\bigcap_{n=1}^{\infty} X^n_{i_n}$ must contain at least one continuum C_β while it is a singleton. □

Proposition 12. *Continuum X_F is Suslinean.*

Proof. Notice that it follows from the definition of the map F that the continuum X_F is obtained from X_f by blow-up of some of the points to an arc. There are two types of blow-up points in X_f. Specifically, $f|_Q$ is a homeomorphism and there are countably many blow-up points in Q, hence there are also at most countably many points blown up to intervals in $\varprojlim\{f,Q\}$. Now, let $b \in X_f \setminus \varprojlim\{f,Q\}$ be a blow-up point. Denote $I_k = [0, 1/3^k]$ for $k = 0, 1, 2, \ldots$. First of all, since $b \notin \varprojlim\{f,Q\}$ there exists minimal k and $N > 0$ such that $b_j \notin \mathrm{Orb}^+(I_{k+1})$ for all $j \geq N$ and if $b_j \in I_s$ then $b_i \in \mathrm{Orb}^+(I_s)$ for all $i \geq j$. But note that if $b_j \in \mathrm{Orb}^+(I_k)\setminus\mathrm{Orb}^+(I_{k+1})$ for all $j \geq N$, then each b_j is uniquely determined by b_N. It is easy to see that it is true for $\mathrm{Orb}^+(I_0) \setminus \mathrm{Orb}^+(I_1) = (1/3, 2/3)$ and then using mathematical induction and symmetry of the graph of f we obtain (similarly to Lemma 4) that the same holds for all other $k > 0$. This shows that every $b \notin \varprojlim\{f,Q\}$ is unique after dropping a few first positions. But then, since $\#f^{-1}(t) \leq 3$ for every $t \in [0,1]$ and the set D used in the construction of F from f is countable, we obtain that there are at most countably many blown up points in $X_f \setminus \varprojlim\{f,Q\}$ (when we know N, there are at most countably many choices for first N coordinates is each $b \notin \varprojlim\{f,Q\}$ and then the choice for all subsequent coordinates is unique). Indeed, we have countably many blow-up points in X_f.

Next, suppose by the way of contradiction that X_F is not Suslinean. Again, suppose $\aleph$ is an uncountable cardinal and $\{C_\beta : \beta < \aleph\}$ is a family of pairwise disjoint subcontinua of X_f. By Proposition 8 there is a monotone onto map $\Pi\colon X_F \to X_f$. Since this map is continuous the family $\{\Pi(C_\beta) : \beta < \aleph\}$ consists of compact and connected subsets of X_f (some of which may be singletons). If $\Pi(C_\beta)$ is not a singleton for uncountably many β's, then we obtain a contradiction with the fact that X_f is Suslinean. So $\Pi(C_\beta)$ is a singleton for uncountably many β's. But then it follows from the definition of Π that there would be uncountably many blow-up points in X_f, which is a contradiction. □

In [20] in Example 3.1 the authors provided a sequence of bonding maps $f_1, f_2, \ldots$ such that $f_n(0) = 0$ and $f_n(1) = 1$, but the inverse limit

$$X = \varprojlim\{\{f_n\}_{n=0}^{\infty}, [0,1]\}$$

is not Sulinean, while is hereditarily decomposable. Hence, if we take a sequence i_j such that $i_0 = 0$ and iterate backwards, so that $i_k = \phi(i_{k+1})$, then putting $(F\colon I_{k+1} \to I_k) = f_k$ (after appropriate rescaling of domain of f_k) we can embed X as a subcontinuum of X_F creating a non-Suslinean continuum.

Remark 13. There is a Li-Yorke chaotic interval map φ of type 2^∞ such that X_φ is not Suslinean (but is hereditarily decomposable).

Our next objective is to prove Theorem 2.

Lemma 14. *There is a Li-Yorke chaotic circle map $G\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that the inverse limit $X_G = \varprojlim\{G, \mathbb{S}^1\}$ contains no indecomposable subcontinuum.*

Proof. Consider the map $\bar{f}\colon [-1,2] \to [-1,2]$, a modification of the interval map f represented in Figure 2. Since $x = -1$ and $x = 2$ are fixed points of $\bar{f}$, we can identify them to a point to obtain a circle map g. It is easily checked that the inverse limit X_g is hereditarily decomposable and g can be modified again to give a Li-Yorke chaotic circle map G with X_G that contains no indecomposable subcontinuum. □

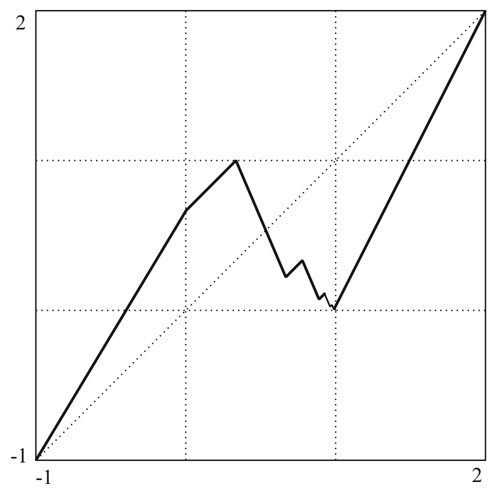

FIGURE 2. The map $\bar{f}$.

Proof of Theorem 2. The homeomorphism h_1 and the arc-like attractor Λ_1 exist by Theorem 1 and [4]. The homeomorphism h_2 and the cofrontier Λ_2 can be constructed according to [5], by the fact that G in Lemma 14 is a degree 1 circle map. □

4. Concluding remarks

Clearly, there exist Li-Yorke chaotic maps of type 2^∞ which are C^∞-smooth [24]. It would be interesting to know if one can improve the differentiability of our example.

Problem 1. Is there $n > 0$ such that φ is a C^n-smooth Li-Yorke chaotic interval map with the X_φ that is hereditarily decomposable? Does X_φ have a "periodic" topological structure similar to X_f or X_F (see Lemmas 4, 5 and Figure 3)?

Also, it is known that there is an arc-like hereditarily decomposable continuum that contains no arc (e.g. see page 29 in [27]). Therefore the following question seems to be of interest.

Problem 2. Is there a Li-Yorke chaotic interval map φ such that X_φ is hereditarily decomposable and contains no arc?

An arc-like hereditarily decomposable continuum that contains no arc should not be confused with a pseudoarc, which is hereditarily indecomposable. Recall that the pseudoarc is the unique homogeneous arc-like continuum [6],[7]. The pseudoarc contains no arc, as all subcontinua of it are indecomposable (in fact it is

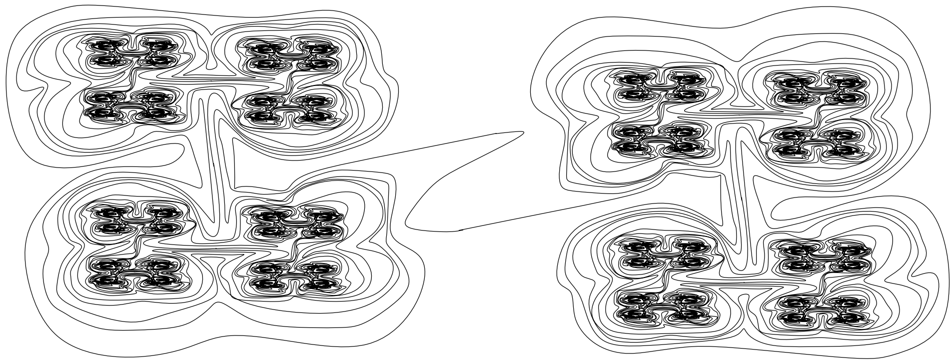

FIGURE 3. A hereditarily decomposable attractor X_F.

homeomorphic to each of its nondegenerate subcontinua). Every interval map is semi-conjugate to a pseudoarc homeomorphism [18] and the pseudoarc admits transitive homeomorphisms [17, 21]. Recently, Mouron showed in [25] that if X_φ is the pseudoarc then the entropy of φ (and the shift map σ_φ) is either 0 or ∞. It is still an open question if there is a homeomorphism, or even a map, of the pseudoarc with positive finite entropy. Note that there is a zero entropy map ψ with very simple dynamics, such that X_ψ is the pseudoarc [13]. Motivated by our examples and the aforementioned results we ask the following.

Problem 3. Is there a Li-Yorke chaotic zero entropy homeomorphism of the pseudoarc?

At this point, it is also worth mentioning that a positive answer to Problem 3 cannot be obtained using the inverse limit approach. It was proved in [8, Theorem F] that if a map $\varphi\colon [0,1] \to [0,1]$ has a periodic point of period 2 or larger, and X_φ is the pseudoarc, then it has a periodic point of odd period other than one. In particular, the inverse limit of a map of type 2^∞ is never the pseudoarc.

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