

NEW STRUCTURE FOR ORTHOGONAL QUANTUM GROUP INVARIANTS

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ABSTRACT. Based on the orthogonal Labastida-Mariño-Ooguri-Vafa conjecture made by L. Chen and Q. Chen (2012), we derive an infinite product formula for Chern-Simons partition functions, which generalizes Liu and Peng's recent results to the orthogonal case. Symmetry property of this new infinite product structure is also discussed.

1. INTRODUCTION

In 1984, Jones [9, 10] discovered a polynomial invariant of oriented knots. Later, using this result, the two-variable HOMFLY-PT [7, 22] and one-variable Kauffman polynomials [11] of oriented and unoriented links were also discovered in 1985 and 1987 respectively. HOMFLY-PT polynomial generalizes both the Alexander and Jones polynomial. In 1990, the two-variable Kauffman polynomial [12] was also introduced, which only generalizes the Jones polynomial.

By using the Chern-Simons path integral method, Witten [29] gave a quantum field theory interpretation of the Jones polynomial. Witten [29] also predicted the existence of 3-manifold quantum invariants. Following this track, Reshetikhin and Turaev [26, 27] gave a construction of 3-manifold invariants by using quantum universal enveloping algebra (quantum group), $U_q(sl_2)$ at roots of unity, which led to the colored version of classical HOMFLY-PT and Kauffman polynomial invariants. Their work actually gave a unified understanding of the quantum group invariants of links. Here color means the representation of quantum groups. It turns out that a colored HOMFLY-PT invariant is a special linear quantum group invariant, i.e., the quantum group of A_n type; while colored Kauffman invariant is a quantum group invariant of B_n , C_n and D_n type.

In a series of papers, Labastida, Mariño, Ooguri and Vafa [13, 14, 21] proposed a conjectural description of a deep relationship between reformulated invariants of colored HOMFLY-PT links in the early 2000s. This conjecture was proved by K. Liu and P. Peng in [17, 18].

In some sense the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture can be expressed purely by using mathematical language, i.e., irreducible representation of quantum groups. The physics background of this conjecture can be dated back to 't Hooft's seminal work on large N expansion of $U(N)$ gauge field theories in 1974. Gopakumar and Vafa [8] described the exact theory that closed topological

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string theory on the resolved conifold is dual to the $U(N)$ Chern-Simons theory on S^3 . The Gromov-Witten theory of the resolved conifold actually corresponds to the Chern-Simons theory of an unknot. The LMOV conjecture considers the more general case when the link or knot is nontrivial and the corresponding Wilson loop expectation values, i.e., a colored HOMFLY-PT polynomial of the link. So the LMOV conjecture could be viewed as a counterpart of the invariant Gopakumar-Vafa conjecture.

Previously, people thought that only HOMFLY-PT polynomial could be expressed as a series in $q - q^{-1}$ and $t^{\pm 1}$ with integer coefficients; while the colored HOMFLY-PT invariant was just a Laurent polynomial of $q^{\pm 1}$ and $t^{\pm 1}$ with rational coefficients. The LMOV conjecture predicts an intrinsic symmetry of $q - q^{-1}$ about reformulated invariants of the colored HOMFLY-PT invariant and hidden integrality encoded in the colored HOMFLY-PT invariant.

In 2009, the orthogonal LMOV conjecture was also formulated by Lin Chen and Qingtao Chen [4–6]. and Marcos Mariño [20]. Chen and Chen's formulation put an emphasis solely on the colored Kauffman; while Mariño's put an emphasis on the relation between the composite colored HOMFLY-PT invariants and the colored Kauffman invariants.

More recently, Kefeng Liu and Pan Peng [19] obtained a new structure of the colored HOMFLY-PT polynomial; namely, that the Chern-Simons partition function appearing in the original LMOV conjecture can be expressed as an infinite product, which indicates some potential modularity of the Chern-Simons partition function [19].

In this paper, an infinite product expression for the orthogonal Chern-Simons partition function appearing in the orthogonal LMOV type conjecture [4–6] is established and the case of an unknot is presented in an explicit formula.

This paper is organized as follows. In Section 2, we introduce the basic setups for the quantum group invariant of links. In Section 3, we describe the original and the orthogonal LMOV conjecture as well as notation. In Section 4, we derive the orthogonal Chern-Simons partition function as an infinite product and illustrate an example of the unknot. In Section 5, we discuss the symmetric properties associated to this infinite product structure.

2. QUANTUM INVARIANTS OF LINKS

Let \mathfrak{g} be a finite dimensional complex semi-simple Lie algebra of rank N with Cartan matrix (C_{ij}) . Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra of \mathfrak{g} . Let V be a vector space over a field k . A linear automorphism c of $V \otimes V$ is said to be an R -matrix if it is a solution of the following Yang-Baxter equation:

$$(2.1) \quad (c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

that holds in the automorphism group of $V \otimes V \otimes V$.

It is well known that the solution of the Yang-Baxter equation provides a representation of the braid group. The solution we used is the following so-called universal R -matrix:

$$(2.2) \quad R = q^{\sum_{i,j} C_{ij}^{-1} H_i \otimes H_j} \prod_{\beta} \exp_q[(1 - q^{-2}) X_{\beta}^{+} \otimes X_{\beta}^{-}],$$

where β runs over positive roots of $sl(N, \mathbf{C})$, (C_{ij}) is the Cartan matrix, and q -exponential is given by

$$(2.3) \quad \exp_q[x] = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!},$$

where

$$[k]_q! = [k]_q \cdot [k-1]_q \cdots [1]_q, [k]_q = \frac{[k]}{[1]}$$

and $[n] = q^n - q^{-n}$.

Given a link \mathcal{L} with L components, it is well known that \mathcal{L} can be represented by an element in some braid group B_m with m strands. For each component, we associate to it an irreducible representation A^α of quantized universal enveloping algebra $U_q(sl(N, \mathbf{C}))$. A^α is labeled by the highest weight Λ_α . As usual, we associate them with the Young diagrams. Without loss of generality, one can assume the first m_1 strands correspond to the first component, the second m_2 strands correspond to the second component, and so on. Let

$$(2.4) \quad \widehat{V} = \bigotimes_{\alpha=1}^L V_{\Lambda_\alpha}^{m_\alpha}$$

and write the braiding $\check{R} = P_{12}R: V \otimes W \rightarrow W \otimes V$, where $P_{12}(s \otimes t) = t \otimes s$.

For a generator of the braid group B_m , σ_i , define

$$(2.5) \quad \pi(\sigma_i^{\pm 1}) = Id_{V_1} \otimes Id_{V_2} \otimes \cdots \otimes Id_{V_{i-1}} \otimes \check{R}^{\pm 1} \otimes \cdots \otimes Id_{V_m}.$$

The quantum group invariants of the link \mathcal{L} is defined as follows:

$$(2.6) \quad W_{A^1, \dots, A^L}^{\mathfrak{g}}(\mathcal{L}) = q^{d(\mathcal{L})} Tr_{\widehat{V}}(\mu^m \cdot \pi(\mathcal{L})),$$

where $\mu = q^{\rho^*}$, ρ^* is the element in $\mathfrak{h} \subset U_q(\mathfrak{h})$ corresponding to the Weyl vector (i.e. the sum of fundamental weights) under the natural isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ and $d(\mathcal{L})$ and is given by the following formula:

$$(2.7) \quad d(\mathcal{L}) = - \sum_{\alpha=1}^L \omega(\mathcal{K}_\alpha)(\Lambda_\alpha, \Lambda_\alpha + 2\rho) + \frac{2}{N} \sum_{\alpha < \beta}^L lk(\mathcal{K}_\alpha, \mathcal{K}_\beta) l_\alpha l_\beta.$$

Special case 1. For the unknot \bigcirc , $W_A(\bigcirc)$ is the quantum dimension $\dim_q(V_A)$ of the corresponding representation space V_A .

Special case 2. If $\mathfrak{g} = sl_N$ and $A^1 = A^2 = \cdots = A^L = (1)$, the quantum group invariant of links is equal to the HOMFLY polynomial at $t = q^N$ up to a universal factor $\frac{t-t^{-1}}{q-q^{-1}}$.

Special case 3. If $\mathfrak{g} = so_{2N+1}$ and $A^1 = A^2 = \cdots = A^L = (1)$, the quantum group invariant of links is equal to the Kauffman polynomial at $t = q^{2N}$ up to a universal factor $1 + \frac{t-t^{-1}}{q-q^{-1}}$ and some t power of the linking numbers.

Thus the quantum group invariants associated to $\mathfrak{g} = sl_N$ and $\mathfrak{g} = so_{2N+1}$ are called the colored HOMFLY invariants and the colored Kauffman invariants respectively.

Actually the irreducible representation of the quantum groups of special linear and orthogonal cases can be labeled by the Young Tableau. Now we would like

to introduce some basic notation of the partition and the corresponding Young Tableau.

A partition of n is a tuple of positive integers $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ such that $|\mu| \triangleq \sum_{i=1}^k \mu_i = n$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$, where $|\mu|$ is called the degree of μ and k is called the length of μ , denoted by $\ell(\mu)$. A partition can be represented by a Young diagram, for example, partition $(5, 4, 2, 1)$ can be identified as the following Young diagram.

Denote by \mathcal{P} the set of all Young diagrams. Let χ_A be the character of irreducible representation of symmetric group, labeled by partition A . Given a partition μ , define $m_j = \#(\mu_k = j; k \geq 1)$. The order of the conjugate class of type μ is given by:

$$(2.8) \quad \mathfrak{z}_\mu = \prod_{j \geq 1} j^{m_j} m_j!.$$

The theory of symmetric functions has a close relationship with the representations of a symmetric group. The symmetric power functions of a given set of variables $x = \{x_j\}_{j \geq 1}$ are defined as the direct limit of the Newton polynomials:

$$(2.9) \quad p_n(x) = \sum_{j=1}^{\infty} x_j^n, \quad p_\mu(x) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(x).$$

We will consistently denote by \mathcal{L} a link and by L the number of components in \mathcal{L} .

The irreducible $U_q(\mathfrak{g})$ modules associated to \mathcal{L} will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form $\vec{A} = (A^1, \dots, A^L)$.

Let $\vec{x} = (x^1, \dots, x^L)$ be L sets of variables, each of which is associated to a component of \mathcal{L} , and $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{P}^L$ be a tuple of L partitions. Define:

$$\begin{aligned} [\mu] &= \prod_{i=1}^{\ell(\mu)} [\mu_i], & [\vec{\mu}] &= \prod_{\alpha=1}^L [\mu^\alpha], & \mathfrak{z}_{\vec{\mu}} &= \prod_{\alpha=1}^L \mathfrak{z}_{\mu^\alpha}, \\ \chi_{\vec{A}}(C_{\vec{\mu}}) &= \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}), & s_{\vec{A}}(\vec{x}) &= \prod_{\alpha=1}^L s_{A^\alpha}(x^\alpha), & p_{\vec{\mu}}(\vec{x}) &= \prod_{\alpha=1}^L p_{\mu^\alpha}(x^\alpha). \end{aligned}$$

When we consider the orthogonal quantum group invariants, we need to study the Brauer algebra Br_n which contains the group algebra $\mathbb{C}[S_n]$ as a direct summand. Thus all the irreducible representations of S_n are also irreducible representations of Br_n , labeled by partitions of the integer n . Indeed, the set of irreducible representations of Br_n are bijective to the set of partitions of the integers $n - 2k$, where $k = 0, 1, \dots, [\frac{n}{2}]$ [24, 28]. Thus the semi-simple algebra Br_n can be decomposed into the direct sum of simple algebras

$$(2.10) \quad Br_n \cong \bigoplus_{k=0}^{[\frac{n}{2}]} \bigoplus_{\lambda \vdash n-2k} M_{d_\lambda \times d_\lambda}(\mathbb{C}).$$

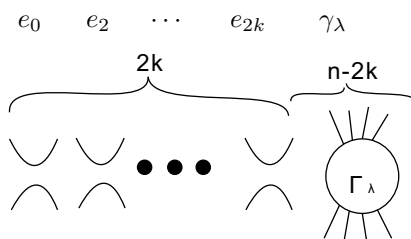
The work of Beliakova and Blanchet [1] constructed an explicit basis of the above decomposition. An up and down tableau $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a tube of n Young diagrams such that $\lambda_1 = (1)$ and each λ_i is obtained by adding or removing one

box from λ_{i-1} . Let λ be a partition of $n - 2k$. Denote by $|\Lambda| = \lambda$ if $\lambda_n = \lambda$, and we say an up and down tableau Λ is of shape λ . There is a minimal path idempotent $p_\Lambda \in Br_n$ associated to each Λ . Then the minimal central idempotent π_λ of Br_n corresponding to the irreducible representation labeled by λ is given by

$$(2.11) \quad \pi_\lambda = \sum_{|\Lambda|=\lambda} p_\Lambda.$$

In particular, the dimension of the irreducible representations d_λ is the number of up and down tableau of shape λ . More details can be found in [1, 28].

The table of the characters and orthogonal relations can be found in [23–25]. The values of a character of Br_n is completely determined by its values on the set of elements $e^k \otimes \gamma_\lambda$, where e is the conjugacy class of e_1, \dots, e_{n-1} and γ_λ is the conjugacy class in S_{n-2k} labeled by the partition λ of $n - 2k$. The notion $e^k \otimes \gamma_\lambda$ stands for the tangle in the following diagram:



where Γ_λ is a diagram in the conjugacy class of S_{n-2k} labeled by a partition λ of $n - 2k$.

Denote by χ_A the character of the irreducible representation of Br_n labeled by a partition $A \vdash n - 2k$ for some k , and denote by $\chi_B^{S_n}$ the character of the irreducible representation of S_n labeled by a partition $B \vdash n$. It is known that when A is a partition of n , then $\chi_A(e^m \otimes \gamma_\lambda) = 0$ for all $m > 0$ and partition $\lambda \vdash n - 2m$, and $\chi_A(\gamma_\mu) = \chi_A^{S_n}(\gamma_\mu)$ for partition $\mu \vdash n$ coincide with the characters of the permutation group S_n [24].

3. LABASTIDA-MARIÑO-OOGURI-VAFA TYPE CONJECTURE FOR THE COLORED KAUFFMAN INVARIANT

Let's quickly review the original LMOV conjecture first.

For each link \mathcal{L} , the type- A Chern-Simons partition function of \mathcal{L} is defined by

$$(3.1) \quad Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{x}) = \sum_{\vec{A} \in \mathcal{P}^L} W_{\vec{A}}^{SL}(\mathcal{L}; q, t) s_{\vec{A}}(\vec{x}) = 1 + \sum_{\vec{\mu} \neq \vec{0}} Z_{\vec{\mu}}^{SL} p_{\vec{\mu}}(\vec{x}),$$

where $s_{\vec{A}}(\vec{x})$ are the Schur polynomials.

The original LMOV conjecture describes a very subtle structure of $Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{x})$, which was proved by Kefeng Liu and Pan Peng [17, 18], based on the cabling technique and a careful degree analysis of the cut-join equation. As an application, the LMOV conjecture gives highly nontrivial relations between colored HOMFLY invariants. The first such relation is the classical Lickorish-Millett theorem [15].

The study of the colored Kauffman invariants is more difficult. For instance, the definition of the Chern-Simons partition function for the orthogonal quantum groups involves the representations of Brauer centralizer algebras, which admit a

more complicated orthogonal relation [23–25]. In a joint work with Lin Chen, we [5] rigorously formulate the orthogonal quantum group version of LMOV conjecture by using the representation of the Brauer centralizer algebra.

Now we set $\mathfrak{g} = \mathfrak{so}_{2N+1}$.

Let \mathcal{L} be a link with L components and

$$(3.2) \quad pb_n(z) = \sum_{j=-\infty}^{\infty} z_j^n, \quad pb_{\mu}(z) = \prod_{i=1}^{\ell(\mu)} pb_{\mu_i}(z), \quad pb_{\vec{\mu}}(\vec{z}) = \prod_{\alpha=1}^L pb_{\mu^{\alpha}}(z^{\alpha}).$$

Let $Z_{CS}^{SO}(\mathcal{L}, q, t)$ be the orthogonal Chern-Simons partition function defined by

$$(3.3) \quad Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \sum_{\vec{\mu} \in \mathcal{P}^L} \frac{pb_{\vec{\mu}}(\vec{z})}{\mathfrak{z}_{\vec{\mu}}} \sum_{\vec{A} \in \widehat{Br}_{|\vec{\mu}|}} \chi_{\vec{A}}(\gamma_{\vec{\mu}}) W_{\vec{A}}^{SO}(\mathcal{L}; q, t),$$

where $\mathfrak{z}_{\vec{\mu}} = \frac{||\vec{\mu}||!}{|C_{\vec{\mu}}|}$, $|\vec{\mu}| = (d^1, \dots, d^L)$, $\widehat{Br}_{|\vec{\mu}|}$ denotes the set $\widehat{Br}_{d^1} \times \dots \times \widehat{Br}_{d^L}$ (every element is a representation of the Brauer algebra), $\vec{\mu} = (\mu^1, \dots, \mu^L)$ for partitions μ^i of $d^i \in \mathbb{Z}$ and $\chi_{\vec{A}}(\gamma_{\vec{\mu}}) = \prod_{i=1}^L \chi_{A^i}(\gamma_{\mu^i})$ for the character χ_{A^i} of Br_{d^i} labeled by A^i .

Expend the free energy

$$(3.4) \quad F_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \log Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{z}) = \sum_{\vec{\mu} \neq \vec{0}} F_{\vec{\mu}}^{SO}(\mathcal{L}; q, t) pb_{\vec{\mu}}(\vec{z}).$$

Then the reformulated invariants are defined by

$$(3.5) \quad g_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{k|\vec{\mu}} \frac{\mu(k)}{k} F_{\vec{\mu}/k}(\mathcal{L}; q^k, t^k).$$

The orthogonal LMOV conjecture was formulated by L. Chen and Q. Chen [4–6] as follows.

Conjecture 3.1 (Orthogonal LMOV, Chen-Chen [4–6]). *Let*

$$\frac{\mathfrak{z}_{\vec{\mu}}[1]^2 \cdot [g_{\vec{\mu}}(\mathcal{L}; q, t) - g_{\vec{\mu}}(\mathcal{L}; q, -t)]}{2[\vec{\mu}]} = \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu}, g, \beta} (q - q^{-1})^g t^{\beta},$$

where $N_{\vec{\mu}, g, \beta}$ are the integer coefficients and vanish for sufficiently large g and $|\beta|$.

This conjecture is a rigorous mathematical formulation of the LMOV type conjecture about the colored Kauffman invariant; while in [2, 20], their conjecture emphasizes the relationship between colored HOMFLY-PT and colored Kauffman invariants. The integer coefficients $N_{\vec{\mu}, g, \beta}$ are closely related to the BPS numbers.

4. INFINITE PRODUCT FORMULA FOR ORTHOGONAL CHERN-SIMONS PARTITION FUNCTIONS

To derive an infinite product formula, we will state the result for a knot first, since the notation in the computation for a knot is relatively simpler.

4.1. **The case of a knot.** Given $z = \{z_i\}_{-\infty < i < \infty}$, $x = \{x_j\}_{j \geq 1}$, define

$$x * y = \{x_i \cdot y_j\}_{-\infty < i < \infty, j \geq 1}.$$

We also define $z^d = \{z_i^d\}_{-\infty < i < \infty}$. The d -th Adam operation of a type-B Schur function is given by $sb_A(z^d)$.

Introducing variables $q^\rho = \{-q^{2j-1}\}_{j \geq 1}$, we have

$$(4.1) \quad p_n(q^\rho) = \frac{1}{[n]},$$

where we assume $|q| < 1$.

Set $w = z * q^\rho$. Then we have

$$(4.2) \quad pb_n(w) = \frac{pb_n(z)}{[n]}$$

and

$$(4.3) \quad pb_\mu(w) = \frac{pb_\mu(z)}{[\mu]}.$$

Consider the free energy weighted by the $pb_\mu(w)$. The orthogonal LMOV conjecture implies the following reformulation of the free energy:

$$(4.4) \quad F^{SO}(\mathcal{K}; q, t; w) = \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_\mu(\mathcal{K}; q^d, t^d) pb_\mu(w^d)$$

and

$$(4.5) \quad g_\mu(\mathcal{K}; q, t) - g_\mu(\mathcal{K}; q, -t) = \frac{2[\mu]}{\mathfrak{z}_\mu[1]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu}, g, \beta} (q - q^{-1})^g t^\beta.$$

There exists integers $n_{B, g, \beta}$ such that

$$(4.6) \quad \sum_{g=0}^{\infty} N_{B, g, \beta} (q - q^{-1})^g = \sum_{g=0}^{\infty} n_{B, g, \beta} \sum_{k=0}^g q^{g-2k}.$$

By the orthogonal LMOV conjecture, $N_{B, g, \beta}$ vanish for sufficiently large g and $|\beta|$, thus $n_{B, g, \beta}$ vanish for sufficiently large g and $|\beta|$.

We have

$$\begin{aligned}
 & F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\
 = & \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_{\mu}(\mathcal{K}; q^d, t^d) pb_{\mu}(w^d) - \sum_{d=1}^{\infty} \sum_{\mu \neq 0} \frac{1}{d} g_{\mu}(\mathcal{K}; q^d, (-1)^d t^d) pb_{\mu}(w^d) \\
 = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} (g_{\mu}(\mathcal{K}; q^d, t^d) - g_{\mu}(\mathcal{K}; q^d, -t^d)) pb_{d\mu}(w) \\
 = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2[d\mu]}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\mu, g, \beta} (q^d - q^{-d})^g t^{d\beta} pb_{d\mu}(w) \\
 (4.7) \quad = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\mu, g, \beta} (q^d - q^{-d})^g t^{d\beta} pb_{d\mu}(z) \\
 = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}[d]^2} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{k=0}^g n_{\mu, g, \beta} q^{(g-2k)d} t^{d\beta} pb_{d\mu}(z) \\
 = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}} \sum_{m=1}^{\infty} m q^{2md} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{k=0}^g n_{\mu, g, \beta} q^{(g-2k)d} t^{d\beta} pb_{d\mu}(z) \\
 = & \sum_{d \in O\mathbb{Z}_+} \sum_{\mu \neq 0} \frac{1}{d} \frac{2}{\mathfrak{z}_{\mu}} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g m n_{\mu, g, \beta} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z) \\
 (4.8) \quad = & \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} \sum_{m=1}^{\infty} \sum_{k=0}^g \frac{m n_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z),
 \end{aligned}$$

where $O\mathbb{Z}_+ = \{1, 3, 5, \dots\}$ denotes the set of all positive odd integers.

Now we analyze the following computation in detail:

$$\begin{aligned}
 & \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} pb_{d\mu}(z) \\
 = & \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} \prod_{j=1}^{\ell(\mu)} \left(\sum_{i=-\infty}^{\infty} (z_i)^{d\mu_j} \right) \\
 = & \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} \sum_{i_1, \dots, i_{\ell(\mu)}} (z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})^d \\
 = & \sum_{i_1, \dots, i_{\ell(\mu)}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} (q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})^d.
 \end{aligned}$$

Now we compute the series $\sum_{d \in O\mathbb{Z}_+} \frac{2}{d} x^d$ as follows:

$$\begin{aligned} & \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} x^d \\ &= 2 \left(\sum_{d=1}^{\infty} \frac{1}{d} x^d - \sum_{d \in 2\mathbb{Z}_+} \frac{1}{d} x^d \right) \\ &= 2 \left(\sum_{d=1}^{\infty} \frac{1}{d} x^d - \sum_{d=1}^{\infty} \frac{1}{2d} x^{2d} \right) \\ &= 2 \left(-\log(1-x) + \frac{1}{2} \log(1-x^2) \right) \\ &= \log \frac{1+x}{1-x}, \end{aligned}$$

where $2\mathbb{Z}_+ = \{2, 4, 6, \dots\}$ denotes the set of all positive even integers.

Thus we obtain

$$\begin{aligned} & F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\ &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} q^{(g-2k+2m)d} t^{d\beta} p_{b_{d\mu}}(z) \\ &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{i_1, \dots, i_{l(\mu)}} \sum_{d \in O\mathbb{Z}_+} \frac{2}{d} (q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})^d \\ &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \sum_{i_1, \dots, i_{l(\mu)}} \log \frac{1 + q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}}{1 - q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}} \\ &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \log \prod_{i_1, \dots, i_{l(\mu)}} \frac{1 + q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}}{1 - q^{g-2k+2m} t^{\beta} z_{i_1}^{\mu_1} z_{i_2}^{\mu_2} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}}. \end{aligned}$$

Define the symmetric product as shown in the following formula:

$$\langle 1 \pm \omega z^{\mu} \rangle = \prod_{i_1, \dots, i_{\ell(\mu)}} \left(1 \pm \omega z_{i_1}^{\mu_1} \dots z_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right).$$

Thus we have

$$\begin{aligned} & F^{SO}(\mathcal{K}; q, t; w) - F^{SO}(\mathcal{K}; q, -t; w) \\ &= \sum_{\mu \neq 0} \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}m=1}^{\infty} \sum_{k=0}^g \frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}} \log \frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \\ &= \log \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}m=1}^{\infty} \prod_{k=0}^g \left(\frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu, g, \beta}}{\mathfrak{z}_{\mu}}}. \end{aligned}$$

Now we obtain the infinite product formula for the orthogonal Chern-Simons partition function of knots.

Theorem 4.1 (Orthogonal infinite product formula for knots). *Based on Conjecture 3.1, the Chern-Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite product formula:*

$$(4.9) \quad \frac{Z_{CS}^{SO}(\mathcal{K}; q, t; w)}{Z_{CS}^{SO}(\mathcal{K}; q, -t; w)} = \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left(\frac{\langle 1 + q^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu, g, \beta}}{3\mu}}.$$

4.2. The case of a link. Now we consider the case of a link.

Given a link \mathcal{L} of L components, let $\vec{w} = (w^1, \dots, w^L)$ and $\vec{z} = (z^1, \dots, z^L)$ satisfying $w^i = z^i * q^{\rho}$, for $i = 1, \dots, L$.

We generalize the symmetric product to the case of a link as follows:

$$\begin{aligned} & \langle 1 \pm \omega(z^1)^{\mu^1} \cdots (z^L)^{\mu^L} \rangle \\ &= \prod_{i_1, 1, \dots, i_1, \ell(\mu^1), \dots, i_L, 1, \dots, i_L, \ell(\mu^L)} \left(1 \pm \omega \prod_{\alpha=1}^L \left((z_{i_{\alpha, 1}}^{\alpha})^{\mu_1^{\alpha}} \cdots (z_{i_{\alpha, \ell(\mu^{\alpha})}}^{\alpha})^{\mu_{\ell(\mu^{\alpha})}^{\alpha}} \right) \right). \end{aligned}$$

There exist integers $n_{\vec{B}; g, \beta}$ such that

$$\sum_{g=0}^{\infty} N_{\vec{B}; g, \beta} (q - q^{-1})^g = \sum_{g=0}^{\infty} n_{\vec{B}; g, \beta} \sum_{k=0}^g q^{g-2k}.$$

In a similar way, the infinite product formula for the orthogonal Chern-Simons partition function of a link can be obtained as follows.

Theorem 4.2 (Orthogonal infinite product formula for links). *Based on Conjecture 3.1, the Chern-Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite product formula:*

$$(4.10) \quad \frac{Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{w})}{Z_{CS}^{SO}(\mathcal{L}; q, -t; \vec{w})} = \prod_{\vec{\mu} \neq \vec{0}} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left(\frac{\langle 1 + q^{g-2k+2m} t^{\beta} (z^1)^{\mu^1} \cdots (z^L)^{\mu^L} \rangle}{\langle 1 - q^{g-2k+2m} t^{\beta} (z^1)^{\mu^1} \cdots (z^L)^{\mu^L} \rangle} \right)^{\frac{mn_{\vec{\mu}, g, \beta}}{3\vec{\mu}}}.$$

4.3. The case of the unknot. In Proposition 10.2 of [5], we have computed the free energy associated to the orthogonal Chern-Simons partition function of the knot as

$$F^{SO}(\bigcirc; q, t; z) = \sum_{k=1}^{\infty} \frac{1}{k} \left(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}} \right) pb_k(z).$$

Thus we have

$$\begin{aligned} & F^{SO}(\bigcirc; q, t; w) - F^{SO}(\bigcirc; q, -t; w) \\ &= \sum_{k \in \mathbb{Z}_+} \frac{2}{k} \frac{t^k - t^{-k}}{q^k - q^{-k}} pb_k(w) \\ &= \sum_{k \in \mathbb{Z}_+} \frac{2}{k} \frac{t^k - t^{-k}}{[k]^2} pb_k(z). \end{aligned}$$

Compared with (4.7), we obtain

$$(4.11) \quad N_{(1), 0, 1} = -N_{(1), 0, -1} = 1.$$

All other coefficients $N_{B;g,Q}$ are zero.

Thus we have

$$(4.12) \quad n_{\mu,g,\beta} = \delta_{\mu,(1)} \delta_{g,0} \text{sign}(\beta)$$

and

$$(4.13) \quad \frac{Z_{CS}^{SO}(\bigcirc; q, t; w)}{Z_{CS}^{SO}(\bigcirc; q, -t; w)} = \prod_{m=1}^{\infty} \prod_{i=-\infty}^{\infty} \left(\frac{(1 + q^{2m} t z_i)(1 - q^{2m} t^{-1} z_i)}{(1 - q^{2m} t z_i)(1 + q^{2m} t^{-1} z_i)} \right)^m.$$

4.4. Symmetry property of $q \rightarrow q^{-1}$ in infinite product structure. In this subsection, we discuss a basic symmetric property of this infinite product structure obtained from the orthogonal LMOV partition function. Here we focus on the knot case only, while the case of links exactly follows from the same analysis. In the derivation of the infinite product formula, we assume $|q| < 1$ for the Taylor expansion of $\frac{1}{[d]^2}$. In the case of $|q| > 1$, the Taylor expansion is given by

$$(4.14) \quad \frac{1}{[d]^2} = \sum_{m=1}^{\infty} m q^{-2md}.$$

Therefore, the infinite product formula will be read as

$$\begin{aligned} & \frac{Z_{CS}^{SO}(\mathcal{K}; q, t; w)}{Z_{CS}^{SO}(\mathcal{K}; q, -t; w)} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \left(\frac{\langle 1 + q^{g-2k-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{g-2k-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \left(\frac{\langle 1 + q^{-g+2(g-k)-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{-g+2(g-k)-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \left(\frac{\langle 1 + q^{-g+2k-2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - q^{-g+2k-2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}} \\ (4.15) \quad &= \prod_{\mu \neq 0}^{\infty} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}}^{\infty} \prod_{m=1}^g \left(\frac{\langle 1 + (q^{-1})^{g-2k+2m} t^{\beta} z^{\mu} \rangle}{\langle 1 - (q^{-1})^{g-2k+2m} t^{\beta} z^{\mu} \rangle} \right)^{\frac{mn_{\mu,g,\beta}}{\delta_{\mu}}}. \end{aligned}$$

This is the symmetry of $q \rightarrow q^{-1}$ for the infinite product formula.

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