PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 8, August 2015, Pages 3429–3443
http://dx.doi.org/10.1090/proc/12614
Article electronically published on April 21, 2015

NOTE ON THE DERIVATION OF MULTI-COMPONENT FLOW SYSTEMS

D. BRESCH AND M. HILLAIRET

(Communicated by Walter Craig)

ABSTRACT. In this note, we justify rigorously the formal method proposed in [M. HILLAIRET, *J. Math. Fluid Mech.* 2007] to derive viscous and compressible multi-component flow equations. We present here a simpler proof than in [D. BRESCH, X. HUANG, *Arch. Ration. Mech. Anal.* 2011] to show that the homogenized system may be reduced to a viscous and compressible multi-component flow system (with one velocity-field) getting rid of the no-crossing assumption on the partial densities. We also discuss formally why our multi-component system may be seen as a physically-relevant relaxed system for the well-known bi-fluid system with algebraic closure (pressure equilibrium) in the isothermal case.

1. INTRODUCTION

Multi-fluid models are encountered in many environmental problems: modeling of internal waves, violent aerated flows or granular materials, to mention a few. Depending on the context, the models used in simulations greatly differ. However averaged models share the same structure. We distinguish two kinds of such models: models with algebraic closures and models with PDE closures. In both cases, one velocity-field only is sometimes used. For bi-fluid mixture, for instance, we have:
I– *Model with an algebraic closure (common pressure):*

$$\begin{aligned} \alpha^+ + \alpha^- &= 1, \\ \partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+ u) &= 0, \\ \partial_t\rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P &= 0, \\ P = P_-(\rho^-) = P_+(\rho^+), & \end{aligned}$$

where $\rho = \alpha^+\rho^+ + \alpha^-\rho^-$ with $0 \le \alpha^\pm \le 1$. Readers interested in related models are referred to [6], [15], [4] and the references cited therein.

Received by the editors August 12, 2013.
2010 *Mathematics Subject Classification.* Primary 35Q30.

II– *Relaxed model with a PDE closure:*

$$\begin{aligned}
\alpha^+ + \alpha^- &= 1,\\
\partial_t \alpha^+ + u\cdot\nabla\alpha^+ &= \frac{1}{\lambda_P}(P^+ - P^-),\\
\partial_t(\alpha^+\rho^+) + \mathrm{div}(\alpha^+\rho^+ u) &= 0,\\
\partial_t\rho + \mathrm{div}(\rho u) &= 0,\\
\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla P &= 0,
\end{aligned}$$

with $0 \le \alpha^\pm \le 1$, $\rho = \alpha^+\rho^+ + \alpha^-\rho^-$, $P = \alpha^+ P^+ + \alpha^- P^-$. Readers interested in modeling and simulation in the non-isothermal case or related systems are referred for instance to [1] and the references therein. We remark that when λ_P tends to 0, System II formally converges to System I (in the isothermal case under consideration herein).

In this paper, we investigate the case of a barotropic, compressible and viscous mixture. We show how a multi-component model with PDE closure similar to II may be mathematically derived, with a relaxation parameter λ_P related to the common viscosities μ and λ of the phases in the mixture. Therefore our system can be seen as a physically-relevant relaxed system for system I.

To complete the description of the mixture, we assume velocity-fields and stress tensors are continuous at the interfaces between phases. Under the assumption that all phases share the same barotropic law, we obtain that the extended density ρ, velocity-field u and pressure p satisfy globally the barotropic compressible Navier Stokes system:

$$\partial_t\rho + \mathrm{div}(\rho\, u) = 0\,, \tag{1}$$

$$\partial_t(\rho\, u) + \mathrm{div}(\rho\, u\otimes u) = \mathrm{div}\,\Sigma\,, \tag{2}$$

where the stress tensor Σ is given by Newton law:

$$\Sigma := \Sigma(u,p) = \mu\nabla u + [(\lambda+\mu)\mathrm{div} u - p]\,\mathrm{I}$$

with constant viscosities λ and μ. In the isentropic case, the pressure p is computed with respect to ρ via a given power-law function q:

$$q(r) = a r^\gamma\,, \tag{3}$$

for a given $a \in (0,\infty)$ and $\gamma \in (1,\infty)$. A more general pressure law with related open problems is discussed at the conclusion of the paper. Herein, this system is considered on the torus $\mathbb{T}^3$ and completed with periodic boundary conditions and initial conditions:

$$\rho|_{t=0} = \rho^0, \qquad u|_{t=0} = u_0. \tag{4}$$

The set of equations (1)–(3) is the classical system for a barotropic compressible and viscous mono-component flow. In this simplified case, studying a multi-component flow reduces to assuming that the density of the equivalent mono-component flow oscillates between a finite set of values representing the densities of the phases in the multi-component flow. Eventually, the multi-component model is derived by describing the propagation of oscillations in (1)–(3).

This method is introduced in [10, 11], when the fluid is contained in a bounded container Ω and (1)–(3) is completed with homogeneous boundary conditions.

The following multi-component system is obtained:

$$\partial_t \alpha_i + u \cdot \nabla \alpha_i = \frac{\alpha_i(a\rho_i^\gamma - \pi)}{\lambda + 2\mu}, \tag{5}$$

$$\partial_t \rho_i + \mathrm{div}(\rho_i u) = \frac{\rho_i(\pi - a\rho_i^\gamma)}{\lambda + 2\mu}, \tag{6}$$

$$\partial_t(\rho\, u) + \mathrm{div}(\rho\, u \otimes u) + \nabla \pi = \mu \Delta u + (\lambda + \mu)\nabla \mathrm{div} u, \tag{7}$$

where (ρ, π) are given by:

$$\rho = \sum_{i=1}^{k} \alpha_i \rho_i\,, \tag{8}$$

$$\pi = \sum_{i=1}^{k} \alpha_i q(\rho_i)\,. \tag{9}$$

Note that, using the equations satified by ρ_i and α_i, the total density ρ satisfies

$$\partial_t \rho + \mathrm{div}(\rho u) = 0.$$

Here k is the number of phases, the ρ_i's are the partial densities and α_i the volume fractions of the phases. The remaining unknowns u and π stand for the velocity-field and pressure in the mixture. This system is completed with initial conditions

$$u|_{t=0} = u_0, \tag{10}$$

$$\rho_i|_{t=0} = \rho_i^0, \qquad \alpha_i|_{t=0} = \alpha_i^0, \quad \forall\, i = 1, \dots, k, \tag{11}$$

where the initial data satisfy compatibility conditions:

$$\alpha_i^0 \geq 0\,, \quad \forall\, i = 1, \dots, k, \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i^0 = 1\,. \tag{12}$$

In [10, 11], the system (5)–(11) is obtained formally under the supplementary assumptions:

$$\rho_i \geq c > 0\,, \quad \forall\, i = 1, \dots, k\,, \tag{13}$$

which means that the solution remains far from vacuum, and

$$\rho_i(t,x) \neq \rho_j(t,x)\,, \quad \forall i \neq j \quad \forall (t,x) \in (0,T) \times \Omega, \tag{14}$$

which we call the *no-crossing* assumption. Note that for densities in $L^\infty(Q_T)$ with $Q_T = (0,T) \times \Omega$ and far from vacuum, denoting

$$L(f) = Q_T \times [\inf_{z \in Q_T} f(z), \sup_{z \in Q_T} f(z)],$$

the *no-crossing* assumption in [2] reads

$$L(\rho_i) \cap L(\rho_j) = \emptyset\,, \quad \forall i \neq j. \tag{15}$$

The previous system is derived by considering a bounded sequence (in the energy norm) of weak solutions (ρ^n, u^n) to (1)-(2) (as constructed by P.–L. Lions [12] or by E. Feireisl and his co-authors [9]) and then computing which system is satisfied by weak cluster points (ρ, u) of this sequence. Classically the key-point is that, as densities converge a priori only in a weak sense, the weak limit π of the sequence of pressures $p^n := q(\rho^n)$ does not satisfy $\pi = q(\rho)$. Hence, following previous studies of W. E [7], D. Serre [16] and P.–L. Lions [12], a family of probability measures, also known as Young measures, $(\nu_{t,x})$, is introduced in order to describe the oscillations of the family of densities (ρ^n). Readers interested in an introduction to Young measures are referred to the book by A. Majda and A.L. Bertozzi [13] and the references cited therein. This permits us to link the weak limit ρ of densities with the weak limit π of pressures. In [11], it is shown that the Young measures satisfy the following advection equation:

$$\partial_t \langle \nu, b \rangle + \mathrm{div}(\langle \nu, b \rangle u) + \langle \nu, (\imath b' - b) \rangle \mathrm{div} u = \frac{\langle \nu, (\imath b' - b) \rangle \pi - \langle \nu, (\imath b' - b) q \rangle}{\lambda + 2\mu}, \tag{16}$$

for all $b \in C(\mathbb{R}^+)$, smooth, with compact support. In this last equation, the symbol $\imath : \mathbb{R}^+ \to \mathbb{R}^+$ stands for the identity function. The density ρ and pressure π are recovered by the identity:

$$\rho = \langle \nu, \imath \rangle \,, \tag{17}$$

$$\pi = \langle \nu, q \rangle \,. \tag{18}$$

The system (7)-(10)-(16)-(17)-(18) is completed with initial conditions:

$$\nu_{0,x} = \nu^0(x) \,, \quad \text{a.e.} \tag{19}$$

We emphasize that, when $\pi \in L^2(0,T;L^{6/5}(\mathbb{T}^3))$, a family of Young measures ν solution to (16) satisfies

$$\int_{\mathbb{T}^3} \langle \nu_{t,x}, b \rangle \varphi(x) \mathrm{d}x \in C([0,T]) \,,$$

whatever the value of $b \in C(\mathbb{R}^+)$, smooth, with compact support and $\varphi \in H^1(\mathbb{T}^3)$. Hence, we might derive (19) from the equalities:

$$\langle \nu, b \rangle_{|t=0} = \langle \nu^0, b \rangle \,,$$

which we might require (in $L^2(\mathbb{T}^3)$) for all $b \in C(\mathbb{R}^+)$ smooth, with compact support.

In [11], the system (5)-(11) is obtained by plugging the ansatz

$$\nu = \sum_{i=1}^{k} \alpha_i \delta_{\rho_i}$$

into (7)-(10)-(16)-(17)-(18)-(19) and assuming the *no-crossing* assumption (14). In this reference, existence of a classical solution to (5)-(11) is also obtained for sufficiently smooth initial data.

The construction in [11] remains formal as, for weak solutions, the Young measures ν, density ρ, velocity-field u and pressure π do not enjoy sufficient regularity to justify rigorously that ν remains a convex combination of Dirac measures. In order to make the full derivation rigorous, the first author, in collaboration with X. Huang [2], apply the method described above to solutions (ρ^n, u^n) with little more regular initial data $(\rho^{n,0}, u^{n,0}) \in L^\infty(\mathbb{T}^3) \times H^1(\mathbb{T}^3)$. For this kind of "smooth" initial data, B. Desjardins proved in 1997 local-in-time existence of a weak solution [3] having more regularity (note that the constraint related to the pressure law may be relaxed). Analyzing the density oscillations of such a sequence of solutions to (1)–(4) with $\rho^{n,0}$ bounded in $L^\infty(\mathbb{T}^3)$ and $u^{n,0} \in H^1(\mathbb{T}^3)$, D. Bresch and X. Huang show the local-in-time existence of a solution (ν, u) to (7)–(16)–(17)–(18) satisfying the further regularity:

[P1] $\operatorname{div} u \in L^1(0,T;L^\infty(\mathbb{T}^3))$,

[P2] there exists $M > 0$ such that

$$\operatorname{Supp} \nu_{t,x} \subset \left[\frac{1}{M}, M\right] \qquad \text{for a.a. } (t,x) \in (0,T) \times \mathbb{T}^3.$$

We note that **[P2]** implies in particular that $\pi \in L^\infty((0,T) \times \mathbb{T}^3)$. This result is based on a weak-strong uniqueness argument for the system (5)–(11). D. Bresch and X. Huang show that, when initial conditions are sufficiently smooth, the classical solution constructed in [11] is unique in the class of solutions to (5)–(11) for which

- α_i and ρ_i are bounded for all $i = 1, \dots, k$,
- $u \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$ satisfies **[P1]**.

This result is a more rigorous justification of (5)–(11) for the study of a multi-component flow. Yet, it still requires the assumption (15) that partial densities do not cross. *In this note, we give a rigorous justification of the derivation of* (5)–(11) *without the no-crossing assumption* (15) and without the defect measure introduced in [10] and strongly used in [2].

For simplicity, in what follows we denote (HCNS), for Homogenized Compressible Navier Stokes System, the set of equations (7),(10),(16),(19),(17),(18), while we denote (PHCNS), for Positive Homogenized Compressible Navier Stokes System, the set of equations (6)–(11). We distinguish this way between the system obtained by homogenizing the isentropic compressible Navier Stokes system in terms of Young measures (HCNS) and the same system in terms of partial densities and volume fractions (PHCNS). As we explained above, homogenization of the isentropic compressible Navier Stokes system (whether starting from solutions in the weak sense as constructed by P.–L. Lions or starting from solutions in a semi-strong sense as it was constructed by B. Desjardins), yield solutions to (HCNS). From now on, we consider only solutions obtained by homogenizing B. Desjardins' solution to the isentropic Navier Stokes system. As is proved in [2], these solutions to (HCNS) satisfy the further regularity statements **[P1]** and **[P2]**.

Our main result reads:

Theorem 1. *Let* $u \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$ *and* ν *a family of Young measures solution to* (HCNS). *We assume:*

- u *satisfies* **[P1]** *and* ν *satisfies* **[P2]**,
- *there exists* $(\alpha_i^0, \rho_i^0) \in [L^\infty(\mathbb{T}^3)]^2$ *(for* $i = 1, \dots, k$*), satisfying* (12)-(13)*, so that*

$$\nu_{0,x} = \sum_{i=1}^k \alpha_i^0(x) \delta_{\rho_i^0(x)}\,, \qquad \textit{for a.a. } x \in \mathbb{T}^3.$$

Then, there exists $0 < T_0 \le T$ *and a solution* $((\alpha_i, \rho_i)_{i=1,\dots,k}, u)$ *to* (PHCNS) *with initial data* $((\alpha_i^0, \rho_i^0)_{i=1,\dots,k}, u^0)$ *on* $(0, T_0)$*, such that*

$$\nu_{t,x} = \sum_{i=1}^k \alpha_i(t,x) \delta_{\rho_i(t,x)}\,, \quad \textit{for a.a. } (t,x) \in (0,T_0) \times \mathbb{T}^3.$$

The proof of this theorem contains several difficulties. First, one needs to consider velocity-field u for which uniqueness of solution to transport equation (16) is guaranteed. This justifies the construction of homogenized solutions to (HCNS) in the sense of B. Desjardins (for which $\operatorname{div} u \in L^1(0,T;L^\infty(\mathbb{T}^3))$). Even so, setting $b = q$ in (16) for instance, we remark that proving uniqueness of the subsequent equation requires us to compute an equation for $\langle \nu, q^2 \rangle$. We obtain such an equation by setting $b = q^2$ in (16). But to reach uniqueness for this second equation, one needs to compute an equation for $\langle \nu, q^3 \rangle$ and so on. We obtain a cascade of equations that it seems difficult to handle simultaneously. One way to resolve this difficulty is to restrict Young measures to the class of convex combination of a given number of Dirac measures. Fortunately, such Young measures are preserved by (16) justifying the previous computations in [2,11]. Unfortunately, in this case, one needs to assume a no-crossing assumption in order to transform (16) into (5)-(6).

To get rid of this supplementary assumption, we first consider that (u, π) are given with the regularity of homogenized solutions to (HCNS) in the sense of B. Desjardins. For this pair, we construct a solution $(\alpha_i, \rho_i)_{i=1,\dots,k}$ to (5)-(6). This construction seems to be standard but requires us to handle non-linearities with care. This is the content of the following lemma:

Lemma 2. *Let $T > 0$. Assume that*

- $u \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$ *satisfy* **[P1]**,
- $\pi \in L^\infty((0,T) \times \mathbb{T}^3)$,
- $(\alpha_i^0, \rho_i^0) \in [L^\infty(\mathbb{T}^3)]^2$, $i = 1, \dots, k$, *satisfies* (12)-(13).

Then, there exists $T_0 < T$ and $(\alpha_i, \rho_i) \in L^\infty((0,T_0) \times \mathbb{T}^3)$, $i = 1, \dots, k$, solution to (5)-(6)-(11). *Furthermore, there exists $M > 0$ such that*

$$\frac{1}{M} \leq \rho_i(t,x) \leq M \quad \text{a.e. on } (0,T_0) \times \mathbb{T}^3. \tag{20}$$

Then, we prove uniqueness of solutions to (5)–(11) in the regularity class of solutions to (HCNS) in the sense of B. Desjardins:

Lemma 3. *Let $T_0 > 0$. Assume that*

- $u \in L^\infty(0,T_0;L^2(\mathbb{T}^3)) \cap L^2(0,T_0;H^1(\mathbb{T}^3))$ *satisy* **[P1]**,
- $\pi \in L^\infty((0,T_0) \times \mathbb{T}^3)$,
- $(\alpha_i^0, \rho_i^0) \in [L^\infty(\mathbb{T}^3)]^2$, $i = 1, \dots, k$, *satisfy* (12)-(13),
- $(\alpha_i, \rho_i) \in L^\infty((0,T_0) \times \mathbb{T}^3)$, $i = 1, \dots, k$, *is a solution to* (5)-(6)-(11) *satisfying* (20).

Then, any family of Young measures ν that satisfies **[P2]** *and that is a solution to* (16)-(19) *on $(0,T_0)$ with:*

$$\nu_x^0 = \sum_{i=1}^k \alpha_i^0(x) \delta_{\rho_i^0(x)}, \qquad \text{for a.a. } x \in \mathbb{T}^3.$$

verifies:

$$\nu_{t,x} = \sum_{i=1}^k \alpha_i(t,x) \delta_{\rho_i(t,x)}, \quad \text{for a.a. } (t,x) \in (0,T_0) \times \mathbb{T}^3.$$

Eventually, given a solution (u, ν) to (HCNS) satisfying **[P1]** and **[P2]**, applying Lemma 2 we construct a solution to (5)-(6). Then, with Lemma 3, we prove that ν is almost everywhere equal to the convex combination of Dirac measures associated with the solution to (5)-(6). This concludes the proof of Theorem 1. The next section is devoted to the proofs of the two lemmas. In the last section, we discuss the extension of the method to non-monotone pressure laws and the link with other models for multi-component flows that have been mentioned at the beginning of this introduction.

2. Proof of Theorem 1

The proof of Lemma 2 is standard. We recall the main ingredients for completeness. The arguments for the second lemma are less standard and will be more detailed.

2.1. **Proof of Lemma 2.** Following the assumptions of this lemma, we let $T > 0$ and (u, π) with the given regularity. We also let initial data $(\alpha_i^0, \rho_i^0) \in L^\infty(\mathbb{T}^3)$. We split the proof into three steps. First we introduce and study properties of two mappings: the first one solves (5)-(6) assuming that right-hand sides and initial data are given; the second one computes non-linear right-hand sides of (5)-(6). In the second step, we perform a fixed-point argument to compute solutions to (5)-(6). In the last step, we prove the bound below (20).

Step 1. We first solve (5)-(6) with given right-hand sides and initial data, i.e., we consider the system, on $(0, \tilde{T}) \times \mathbb{T}^3$:

$$\partial_t \alpha_i + u \cdot \nabla \alpha_i = \alpha_i f_i, \tag{21}$$

$$\partial_t \rho_i + \mathrm{div}(\rho_i u) = \rho_i g_i, \tag{22}$$

for $i = 1, \ldots, k$, with initial conditions:

$$\alpha_i(0, x) = \alpha_i^0(x), \quad \rho_i(0, x) = \rho_i^0(x), \quad \text{on } \mathbb{T}^3, \tag{23}$$

for $i = 1, \ldots, k$. We assume here that $(f_i, g_i)_{i=1,\ldots,k}$ are bounded functions. All the equations in this system are examples of transport equations as considered in [5]. Applying [5, Corollary II.1], we get existence and uniqueness of a solution (21)-(22). This yields a family of mappings (indexed by the existence time $\tilde{T} \leq T$):

$$\begin{array}{rccl} \mathcal{L}_{\tilde{T}}: & L^\infty((0,\tilde{T}) \times \mathbb{T}^3) & \longrightarrow & L^\infty((0,\tilde{T}) \times \mathbb{T}^3)\,, \\ & (f_i, g_i)_{i=1,\ldots,k} & \longmapsto & (\alpha_i, \rho_i) \text{ solution to (21)–(23)}\,. \end{array}$$

This family of mappings satisfies the following proposition:

Proposition 4. *There exists $C_0 < \infty$ depending only on:*

$$\sup_{i=1,\ldots,k} \left(\|\alpha_i^0\|_{L^\infty(\mathbb{T}^3)} + \|\rho_i^0\|_{L^\infty(\mathbb{T}^3)} \right) \quad \textit{and} \quad \|\mathrm{div}\, u\|_{L^1(0,T;L^\infty(\mathbb{T}^3))}$$

such that, for arbitrary $\tilde{T} < T$,

- *given $M > 0$, there holds:*

$$\mathcal{L}_{\tilde{T}}(B(0, M)) \subset B(0, C_0 \exp(\tilde{T} M))\,,$$

where $B(\cdot, \cdot)$ denotes the ball in $L^\infty((0, \tilde{T}) \times \mathbb{T}^3)$,

- *if* $\tilde{T} < 1$ *the restriction of* $\mathcal{L}_{\tilde{T}}$ *to* $B(0,M)$*, endowed with the* $L^\infty(0,\tilde{T}; L^1(\mathbb{T}^3))$ *norm, is lipschitz with lipschitz constant*

$$K_{\mathcal{L}}(\tilde{T},M) = \tilde{T}C_0 \exp(2M + C_0). \tag{24}$$

Proof. The first item is obtained applying classical regularizing arguments (see [5, Theorem II.1]) and classical multiplier arguments. We leave the details to the reader.

Concerning the second item, we consider two source terms $(\tilde{f}_i, \tilde{g}_i)_{i=1,\dots,k}$ and $(\hat{f}_i, \hat{g}_i)_{i=1,\dots,k}$ associated with their respective images $(\tilde{\alpha}_i, \tilde{\rho}_i)$ and $(\hat{\alpha}_i, \hat{\rho}_i)$. The differences $\alpha_i = \tilde{\alpha}_i - \hat{\alpha}_i$ and $\rho_i = \tilde{\rho}_i - \hat{\rho}_i$ satisfy (keeping convention for differences: $f_i = \tilde{f}_i - \hat{f}_i$ and $g_i = \tilde{g}_i - \hat{g}_i$):

$$\partial_t \alpha_i + u \cdot \nabla \alpha_i = \alpha_i \tilde{f}_i + \hat{\alpha}_i f_i,$$
$$\partial_t \rho_i + \mathrm{div}(\rho_i u) = \rho_i \tilde{g}_i + \hat{\rho}_i g_i,$$

for $i = 1, \dots, k$, with zero initial conditions. Concerning α_i for instance, up to regularizing arguments we skip for conciseness, we multiply the first equation with "sign(α_i)". After time-integration, this yields, for all $t \leq \tilde{T}$:

$$\begin{aligned}
&\int_{\mathbb{T}^3} |\alpha_i(t,x)| \mathrm{d}x \\
&\leq \int_0^t \exp\left(\int_s^t \|\tilde{f}_i\|_{L^\infty(\mathbb{T}^3)} + \|\mathrm{div} u\|_{L^\infty(\mathbb{T}^3)} \mathrm{d}\sigma \right) \|\hat{\alpha}_i\|_{L^\infty(\mathbb{T}^3)} \|f_i\|_{L^1(\mathbb{T}^3)} \mathrm{d}s \\
&\leq \tilde{T} \exp(\tilde{T}(M + C_0)) C_0 \exp(\tilde{T}M) \sup_{s \in (0,\tilde{T})} \|f_i(s,\cdot)\|_{L^1(\mathbb{T}^3)} \\
&\leq \tilde{T} C_0 \exp(2M + C_0) \sup_{s \in (0,\tilde{T})} \|f_i(s,\cdot)\|_{L^1(\mathbb{T}^3)}\,.
\end{aligned}$$

The corresponding computations for the ρ_i's are left to the reader. □

Next, we construct the non-linear mapping corresponding to the computations of right-hand sides in (5)-(6). Hence, given $\tilde{T} < T$ we introduce:

$$\begin{array}{cccc}
\mathcal{NL}_{\tilde{T}} : & L^\infty((0,\tilde{T}) \times \mathbb{T}^3) & \to & L^\infty((0,\tilde{T}) \times \mathbb{T}^3) \\
& (\alpha_i, \rho_i)_{i=1,\dots,k} & \longmapsto & (f_i, g_i)_{i=1,\dots,k}\,,
\end{array}$$

where, for all $i = 1, \dots, k$ we set:

$$f_i = \frac{a\rho_i^\gamma - \pi}{\lambda + 2\mu}\,, \quad g_i = \frac{\pi - a\rho_i^\gamma}{\lambda + 2\mu}\,, \quad \text{on } (0,\tilde{T}) \times \mathbb{T}^3.$$

Straightforward computations yield the following proposition:

Proposition 5. *There exists a constant* K_0 *depending only on parameters* a, λ, μ, $\|\pi\|_{L^\infty((0,T)\times\mathbb{T}^3)}$ *for which, given* $\tilde{T} < T$ *and* $R > 0$*, the following statements hold true:*

- $\mathcal{NL}_{\tilde{T}}$ *maps* $B(0,R)$ *into* $B(0, K_0(R^\gamma + 1))$*,*
- *the restriction of* $\mathcal{NL}_{\tilde{T}}$ *to* $B(0,R)$*, endowed with the* $L^\infty(0,\tilde{T}; L^1(\mathbb{T}^3))$ *norm, is lipschitz with lipschitz constant*

$$K_{\mathcal{NL}}(R) := \frac{a\gamma}{\lambda + \mu} R^{\gamma - 1}\,. \tag{25}$$

The proof of these statements is left to the reader.

Step 2. It is clear that, given $0 < \tilde{T} < T$, if $(\alpha_i, \rho_i) \in L^\infty((0,\tilde{T}) \times \mathbb{T}^3)$ is a fixed-point of $\Phi_{\tilde{T}} := \mathcal{L}_{\tilde{T}} \circ \mathcal{NL}_{\tilde{T}}$, then it is a solution to (5)-(6)-(11) on $(0,\tilde{T})$. To prove the existence of such a pair, we apply a fixed-point argument.

First, we prove that $\Phi_{\tilde{T}}$ fixes a ball of sufficiently large radius for a sufficiently small time. Indeed, with the notation of the previous propositions, let

$$R_0 := 2C_0 \quad T_0 := \inf\left(\frac{\ln 2}{K_0((2C_0)^\gamma + 1)}, 1\right).$$

Then, applying Proposition 5 we get that

$$\mathcal{NL}_{T_0}(B(0,R_0)) \subset B(0, K_0(R_0^\gamma + 1))$$

and, applying Proposition 4 yields:

$$\mathcal{L}_{T_0}(B(0,K_0(R_0^\gamma+1))) \subset B(0, C_0 \exp(T_0 K_0(R_0^\gamma + 1))).$$

Due to our choices, there holds

$$C_0 \exp(T_0 K_0 (R_0^\gamma + 1)) \leq 2C_0 = R_0\,,$$

so that we obtain finally:

$$\Phi_{T_0}(B(0,R_0)) \subset B(0,R_0)\,.$$

Second, we prove that, restricting the size of T_0, the mapping Φ_{T_0} realizes a contraction on $B(0,R_0)$ endowed with the $L^\infty((0,T_0);L^1(\mathbb{T}^3))$. Indeed, skipping indices for legibility, we have, for any two $(\tilde{\alpha}_i,\tilde{\rho}_i)$ and $(\hat{\alpha}_i,\hat{\rho}_i)$ in $B(0,R_0)$ that, as $T_0 < 1$:

$$\|\Phi_{T_0}((\tilde{\alpha}_i,\tilde{\rho}_i)) - \Phi_{T_0}((\hat{\alpha}_i,\hat{\rho}_i))\| \leq K_{\mathcal{L}}(T_0, K_0(R_0^\gamma + 1)) K_{\mathcal{NL}}(R_0) \|(\tilde{\alpha}_i,\tilde{\rho}_i) - (\hat{\alpha}_i,\hat{\rho}_i)\|,$$

where $K_{\mathcal{L}}$ is given in (24) and $K_{\mathcal{NL}}$ in (25). In particular, as R_0 is fixed by initial data, there exists $\tilde{C}_0$, depending only on initial data and physical parameters a, λ, μ, γ, such that:

$$K_{\mathcal{L}}(T_0, K_0(R_0^{\gamma-1} + 1)) K_{\mathcal{NL}}(R_0) \leq T_0 \tilde{C}_0\,.$$

Restricting T_0 in order that $T_0\tilde{C}_0 \leq 1/2$, we obtain then that Φ_{T_0} is a contraction.

Consequently, we consider the sequence $(\alpha_i^n, \rho_i^n)_{n \in \mathbb{N}\cup\{0\}}$ defined by

$$(\alpha_i^{n+1}, \rho_i^{n+1})_{i=1,\dots,k} = \Phi_{T_0}((\alpha_i^n,\rho_i^n)_{i=1,\dots,k})\,, \quad \forall\, n \in \mathbb{N}\cup\{0\}\,,$$

where we use the convention that the value for $n = 0$ is the extension to $[0,T_0]$ of the initial given value $(\alpha_i^0,\rho_i^0)_{i=1,\dots,k}$. According to the previous arguments, this sequence satisfies:

- $(\alpha_i^n,\rho_i^n)_{i=1,\dots,k} \in B(0,R_0)$, for all $n \in \mathbb{N}\cup\{0\}$,
- $((\alpha_i^n,\rho_i^n)_{i=1,\dots,k})_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence for the $L^\infty(0,T_0;L^1(\mathbb{T}^3))$ norm.

Consequently, this sequence admits a limit $(\alpha_i,\rho_i)_{i=1,\dots,k}$ which satisfies:

- $(\alpha_i,\rho_i)_{i=1,\dots,k} \in L^\infty((0,T_0)\times\mathbb{T}^3)$,
- $(\alpha_i,\rho_i)_{i=1,\dots,k}$ is a fixed-point for Φ_{T_0}.

We obtain existence of a solution to (5)-(6)-(11).

Step 3. We end the proof by obtaining (20). To this end, we remark that, for each $i \in \{1,\dots,k\}$, we have that ρ_i is a solution to:

$$\partial_t \rho_i + u\cdot\nabla\rho_i = f_i \qquad \text{with } f_i = -\rho_i \mathrm{div} u + \rho_i \frac{(a\rho_i^\gamma - \pi)}{\lambda+2\mu}\,.$$

We then let c_i be the unique solution to

$$\partial_t c_i + u\cdot\nabla c_i = -\|f_i\|_{L^\infty(\mathbb{T}^3)}\,.$$

As $\|f_i\|_{L^\infty(\mathbb{T}^3)} \in L^1(0,T_0)$, there exists $\tilde{T}_0 \le T_0$ such that

$$\int_0^{\tilde{T}_0} \|f_i\|_{L^\infty(\mathbb{T}^3)} \le \frac{c}{2}\,.$$

Consequently, $c_i \ge c/2$ on $(0,\tilde{T}_0)\times\mathbb{T}^3$ for all $i\in\{1,\dots,k\}$. We conclude by a comparison argument. We remark that $\tilde{\rho}_i := \rho_i - c_i$ satisfies:

$$\begin{aligned} \tilde{\rho}_i(0,\cdot) &\ \ge\ 0\,,\\ \partial_t\tilde{\rho}_i + u\cdot\nabla\tilde{\rho}_i &\ \ge\ 0\,. \end{aligned}$$

Consequently, $\tilde{\rho}_i \ge 0$ on $(0,\tilde{T}_0)$ and we obtain $\rho_i \ge c/2$ on $(0,\tilde{T}_0)\times\mathbb{T}^3$. This ends the proof.

2.2. **Proof of Lemma 3.** With the notation of the lemma, we introduce the Young measures

$$\bar{\nu} := \sum_{i=1}^k \alpha_i \delta_{\rho_i}\,.$$

Our aim is to show that $\bar{\nu} = \nu$ almost everywhere. By assumption, this identity already holds true for $t = 0$. To extend it to positive times, we proceed in three steps.

First, we show that $\bar{\nu}$ and ν are solutions to similar transport equations. Indeed, by construction ν satisfies (16), i.e.,

$$\partial_t\langle\,\nu, b\,\rangle + \mathrm{div}(\langle\,\nu,b\,\rangle u) + \langle\,\nu,(\imath b'-b)\,\rangle \mathrm{div} u = \frac{\langle\,\nu,(\imath b'-b)\,\rangle\pi - \langle\,\nu,(\imath b'-b)q\,\rangle}{\lambda+2\mu},$$

for all $b\in C(\mathbb{R}^+)$, smooth, with compact support. We note that, as ν has compact support in $(0,\infty)$ we may extend the above identity to all smooth $b \in C((0,\infty))$. Up to a regularizing argument in the spirit of [5, Proof of Corollary II.2], we combine (5) and (6) to get that $\bar{\nu}$ satisfies:

$$\partial_t\langle\,\bar{\nu}, b\,\rangle + \mathrm{div}(\langle\,\bar{\nu},b\,\rangle u) + \langle\,\bar{\nu},(\imath b'-b)\,\rangle \mathrm{div} u = \frac{\langle\,\bar{\nu},(\imath b'-b)\,\rangle\pi - \langle\,\bar{\nu},(\imath b'-b)q\,\rangle}{\lambda+2\mu},$$

for all smooth $b\in C((0,\infty))$.

Second, we prove by an induction argument that

$$\langle \nu, \rho^{1-k\gamma}\rangle = \langle\bar{\nu},\rho^{1-k\gamma}\rangle\,,\quad \forall\, k\in\mathbb{N}\cup\{0\}\,,\quad \text{a.e.} \tag{26}$$

Indeed, for $k=0$, we have that $\langle\nu,\imath\rangle$ and $\langle\bar{\nu},\imath\rangle$ are both bounded solutions to

$$\begin{aligned} \partial_t\beta + \mathrm{div}\beta u &= 0\\ \beta(0,\cdot) &= \sum_{i=1}^k \alpha_i^0\rho_i^0\,. \end{aligned}$$

Applying the uniqueness part in [5, Corollary II.1], we get that

$$\langle \nu, 1\rangle = \langle \bar{\nu}, 1\rangle\,, \quad \text{a.e.}$$

Assuming then that, for a given $k \geq 0$, there holds $\langle \nu, \rho^{1-k\gamma}\rangle = \langle \bar{\nu}, \rho^{1-k\gamma}\rangle$ we have that $\langle \nu, \rho^{1-(k+1)\gamma}\rangle$ and $\langle \bar{\nu}, \rho^{1-(k+1)\gamma}\rangle$ are both solutions to

$$\begin{aligned}
\partial_t \beta + \operatorname{div}(\beta u) - (k+1)\gamma\beta \operatorname{div} u &= -(k+1)\gamma \frac{a\langle \nu, \rho^{1-k\gamma}\rangle - \beta\pi}{\lambda + 2\mu} \\
\beta(0,\cdot) &= \sum_{i=1}^{k} \alpha_i^0 |\rho_i^0|^{1-(k+1)\gamma}\,.
\end{aligned}$$

Uniqueness in [5, Corollary II.1] leads again to

$$\langle \nu, \rho^{1-(k+1)\gamma}\rangle = \langle \bar{\nu}, \rho^{1-(k+1)\gamma}\rangle\,, \qquad \text{a.e.}$$

A standard argument on zero-measure sets then yields (26).

To conclude, we restrict to the set on which (26) holds true. For all Young measures on this set, we obtain by combination of (26) that there holds:

$$\langle \nu, b\rangle = \langle \bar{\nu}, b\rangle\,, \tag{27}$$

for all b of the form

$$b(s) = k + \beta(s^{-\gamma}) \tag{28}$$

where $k \in \mathbb{R}$ and β is a polynomial function. As ν and $\bar{\nu}$ have compact supports in $(0,\infty)$, we might apply a density argument to extend the identity to any $\beta \in C((0,\infty))$.

Then, we remark that $s \mapsto s^{\gamma}$ realizes a homeomorphism of $(0,\infty)$ so that (27) is actually true for all b of the form (28) with $\beta \in C((0,\infty))$. In particular, (27) holds true for all polynomial functions. Again, as ν and $\bar{\nu}$ have compact support in $(0,\infty)$, a density argument yields that (27) holds true for all $b \in C((0,\infty))$. This concludes the proof.

3. Conclusion

In this concluding section, we first envisage the case of non-monotone pressure laws. Indeed, we motivated our paper by the study of flows which are made of different phases. In this case, the unknowns (ρ, u) obtained by extending the different phase-densities and phase-velocities satisfy the compressible Navier Stokes system (1)–(3). However, in applications, it is unlikely that the obtained pressure law is monotone. For instance, in the case of a diphasic fluid, let us assume that the first fluid has density ρ_+ and velocity u_+ such that ρ_+ ranges an interval $I_+ \subset \mathbb{R}$. Let us denote q_+ its pressure law. Let us also denote by the subscript "$-$" the corresponding quantities for the second phase. Assuming that $\overline{I_+} \cap \overline{I_-} = \emptyset$, the extended unknown (ρ, u) satisfy the compressible Navier Stokes system with any extended pressure law q satisfying

$$q(s) = q_+(s)\,, \quad \forall\, s \in I_+\,, \quad q(s) = q_-(s)\,, \quad \forall\, s \in I_-\,.$$

Hence, in most cases q is not necessarily a monotone pressure law.

The derivation of the multi-component flow equations that has been developed in the previous section adapts formally to this case. However, a fully rigorous proof remains an open problem. We recall that our construction is divided into the following steps:

- computation of the system satisfied by any cluster point of a sequence of bounded-energy solutions (ρ^n, u^n) to the compressible Navier Stokes system (introduction of Young measures),
- construction of solutions to the homogenized system in which Young measures are convex combinations of a fixed number of Dirac measures,
- proof of uniqueness of solutions to the system satisfied by Young measures.

In the case of a pressure given by a power law, the first point is computed herein in the "regular solution" framework provided by [3] and compactness arguments are due to P.–L. Lions [12]. These arguments extend to the case of a non-monotone pressure law (see [8] for the compactness arguments, for instance). Construction of solutions to the homogenized system with non-monotone pressure laws, for Young measures made of a convex combination of Dirac measures, is also straightforward since the extended pressure law q is chosen sufficiently smooth. The weakest point of the method is the third argument. Indeed, the uniqueness result proven herein relies heavily on an algebraic property that is satisfied by power functions only, i.e. the unique solution to

$$\left\{ \begin{array}{rcl} \beta'(\rho)\rho - \beta(\rho) & = & \dfrac{1}{p(\rho)}\,, \quad \forall\, \rho > 0, \\ \beta(0) & = & 0 \end{array} \right.$$

satisfies:

$$\rho \mapsto \beta'(\rho)\rho - \beta(\rho) \quad \text{is proportional to } \rho \mapsto \beta(\rho)\,.$$

This enables us to write the induction argument to obtain (26). In the general case, we note that equation (16) introduces two different functions on the right-hand side:

$$\langle \nu, \imath b - b \rangle \qquad \text{and} \qquad \langle \nu, (\imath b - b) q \rangle.$$

Hence, for proving uniqueness of solutions to (16) we face again the problem that we have to handle with an infinite number of equations simultaneously. This seems to lead to an unmanageable difficulty. A deeper problem is that the construction of defect-measures, which enable us to ensure that solutions to (16) remain a convex combination of a fixed number of Dirac measures (see [2]), also rely on "power-law" properties of the pressure-law. Hence, it does not generalize to non-monotone pressure-law. Eventually, it is even not clear that the Young measures remain convex combinations of a fixed number of Dirac measures with time.

Motivation for non-monotone pressure and relaxed system. One motivation for tackling the case of non-monotone pressure-law is that it gives very thorough insight in non-viscous diphasic models (such as the ones that have been presented in the introduction). With the derivation we developed previously, we obtain:

$$\begin{aligned}
\alpha^+ + \alpha^- &= 1, \\
\partial_t \alpha^+ + u \cdot \nabla \alpha^+ &= \alpha^+ \frac{(q_+(\rho^+) - \pi)}{\lambda + 2\mu}, \\
\partial_t(\alpha^+ \rho^+) + \mathrm{div}(\alpha^+ \rho^+ u) &= 0, \\
\partial_t \rho + \mathrm{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \pi &= \mu \Delta u + (\lambda + \mu) \nabla \mathrm{div} u, \\
\alpha^+ \rho^+ + \alpha^- \rho^- &= \rho, \\
\alpha^+ q_+(\rho^+) + \alpha^- q_-(\rho^-) &= \pi\,,
\end{aligned}$$

where we relabeled with $+$ and $-$ the unknowns associated with both phases for simplicity. We also introduced explicitly that $q = q_+$ (resp. q_-) on the domain ranged by ρ^+ (resp. ρ^-). We note that the right-hand side of the second equation can also be written:

$$\alpha^+ \alpha^- \frac{q_+(\rho^+) - q_-(\rho^-)}{\lambda + 2\mu}.$$

As α^+ remains bounded independant of λ and μ, in the limit $\lambda \to 0$ and $\mu \to 0$ we obtain formally:

$$q_+(\rho^+) = q_-(\rho^-)$$

together with:

$$\begin{aligned}
\alpha^+ + \alpha^- &= 1, \\
\partial_t(\alpha^+ \rho^+) + \mathrm{div}(\alpha^+ \rho^+ u) &= 0, \\
\partial_t(\alpha^- \rho^-) + \mathrm{div}(\alpha^- \rho^- u) &= 0, \\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \pi &= 0, \\
\alpha^+ \rho^+ + \alpha^- \rho^- &= \rho, \\
\alpha^+ q_+(\rho^+) + \alpha^- q_-(\rho^-) &= \pi\,,
\end{aligned}$$

This yields the mono-velocity diphasic non-viscous system with algebraic closure law. In this last system, the second equation also reads:

$$\alpha^+ \left(\partial_t \rho^+ + u \cdot \nabla \rho^+ + \rho^+ \mathrm{div} u \right) + \left(\partial_t \alpha^+ + u \cdot \nabla \alpha^+ \right) \rho^+ = 0.$$

Multiplying this last equation by $q_+'(\rho^+)$ we get, introducing $a_+^2 := q_+'(\rho^+)$:

$$\alpha^+ \left(\partial_t q_+(\rho^+) + u \cdot \nabla q_+(\rho^+) + a_+^2 \rho^+ \mathrm{div} u \right) + \left(\partial_t \alpha^+ + u \cdot \nabla \alpha^+ \right) a_+^2 \rho^+ = 0.$$

Proceeding similarly with $\alpha^- \rho^-$ and substracting the obtained equation multiplied respectively by α^- and α^+, we obtain:

$$\begin{aligned}
\alpha^+ \alpha^- \left(\partial_t \delta_q + u \cdot \nabla \delta_q \right) &+ \alpha^+ \alpha^- (a_+^2 \rho^+ - a_-^2 \rho^-) \mathrm{div} u \\
&+ \left(\partial_t \alpha^+ + u \cdot \nabla \alpha^+ \right) (\alpha^- a_+^2 \rho^+ + \alpha^+ a_-^2 \rho^-) = 0\,,
\end{aligned}$$

where $\delta_q := q_+(\rho^+) - q_-(\rho^-)$. Finally, we have that $q_+(\rho^+) = q_-(\rho^-)$ implies:

$$\partial_t \alpha^+ + u \cdot \nabla \alpha^+ + \frac{\alpha^+ \alpha^- (a_+^2 \rho^+ - a_-^2 \rho^-)}{(\alpha^- a_+^2 \rho^+ + \alpha^+ a_-^2 \rho^-)} \mathrm{div} u = 0\,,$$

which would be the usual second closure law as derived by A. Murrone and H. Guillard in [14]. The converse implication is also true for well-prepared initial data. Finally, in the isentropic case, the two closure equations we gave in the introduction are completely equivalent in the relaxation limit $\lambda_p \to 0$ and appear as inviscid limits of the viscous diphasic model that we derived herein. Our homogenized system may be seen as a physically relevant relaxed system for System I. It would also be interesting to look at the non-isentropic case where the relation $P_+ = P_-$ and

$$\partial_t \alpha^+ + u \cdot \nabla \alpha^+ + \frac{\alpha^+ \alpha^- (a_+^2 \rho^+ - a_-^2 \rho^-)}{(\alpha^- a_+^2 \rho^+ + \alpha^+ a_-^2 \rho^-)} \mathrm{div} u = 0$$

are not equivalent because pressures depend on densities and entropies.

References

[1] Rémi Abgrall and Vincent Perrier, *Asymptotic expansion of a multiscale numerical scheme for compressible multiphase flow*, Multiscale Model. Simul. **5** (2006), no. 1, 84–115 (electronic), DOI 10.1137/050623851. MR2221311 (2007d:76189)

[2] D. Bresch and X. Huang, *A multi-fluid compressible system as the limit of weak solutions of the isentropic compressible Navier-Stokes equations*, Arch. Ration. Mech. Anal. **201** (2011), no. 2, 647–680, DOI 10.1007/s00205-011-0400-8. MR2820360 (2012f:76098)

[3] Benoît Desjardins, *Regularity of weak solutions of the compressible isentropic Navier-Stokes equations*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 977–1008, DOI 10.1080/03605309708821291. MR1452175 (98e:35126)

[4] Frédéric Dias, Denys Dutykh, and Jean-Michel Ghidaglia, *A two-fluid model for violent aerated flows*, Comput. & Fluids **39** (2010), no. 2, 283–293, DOI 10.1016/j.compfluid.2009.09.005. MR2600797 (2011b:76097)

[5] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), no. 3, 511–547, DOI 10.1007/BF01393835. MR1022305 (90j:34004)

[6] Donald A. Drew and Stephen L. Passman, *Theory of multicomponent fluids*, Applied Mathematical Sciences, vol. 135, Springer-Verlag, New York, 1999. MR1654261 (99k:76143)

[7] Weinan E, *Propagation of oscillations in the solutions of 1-D compressible fluid equations*, Comm. Partial Differential Equations **17** (1992), no. 3-4, 347–370, DOI 10.1080/03605309208820846. MR1163429 (93a:35120)

[8] Eduard Feireisl, *Compressible Navier-Stokes equations with a non-monotone pressure law*, J. Differential Equations **184** (2002), no. 1, 97–108, DOI 10.1006/jdeq.2001.4137. MR1929148 (2003h:35209)

[9] Eduard Feireisl, Antonín Novotný, and Hana Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations*, J. Math. Fluid Mech. **3** (2001), no. 4, 358–392, DOI 10.1007/PL00000976. MR1867887 (2002k:35253)

[10] M. Hillairet. *Interactive features in fluid mechanics.* Ph.D. Thesis, Ecole normale supérieure de Lyon, 2005.

[11] M. Hillairet, *Propagation of density-oscillations in solutions to the barotropic compressible Navier-Stokes system*, J. Math. Fluid Mech. **9** (2007), no. 3, 343–376, DOI 10.1007/s00021-005-0202-6. MR2336073 (2009c:76062)

[12] Pierre-Louis Lions, *Mathematical topics in fluid mechanics. Vol. 2*, Oxford Lecture Series in Mathematics and its Applications, vol. 10, The Clarendon Press, Oxford University Press, New York, 1998. Compressible models; Oxford Science Publications. MR1637634 (99m:76001)

[13] Andrew J. Majda and Andrea L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002. MR1867882 (2003a:76002)

[14] Angelo Murrone and Hervé Guillard, *A five equation reduced model for compressible two phase flow problems*, J. Comput. Phys. **202** (2005), no. 2, 664–698, DOI 10.1016/j.jcp.2004.07.019. MR2145395 (2006e:76133)

[15] H. Bruce Stewart and Burton Wendroff, *Two-phase flow: models and methods*, J. Comput. Phys. **56** (1984), no. 3, 363–409, DOI 10.1016/0021-9991(84)90103-7. MR768670 (86a:76040)

[16] Denis Serre, *Variations de grande amplitude pour la densité d'un fluide visqueux compressible* (French, with English summary), Phys. D **48** (1991), no. 1, 113–128, DOI 10.1016/0167-2789(91)90055-E. MR1098658 (92b:76069)

Laboratoire de Mathématiques, UMR 5127 CNRS, Université de Savoie, 73376 Le Bourget du Lac, France

E-mail address: didier.bresch@univ-savoie.fr

Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier, UMR 5149 CNRS, 34095 Montpellier cedex 5, France

E-mail address: matthieu.hillairet@univ-mont2.fr