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SOME IMPROVEMENTS OF THE KATZNELSON-TZAFRIRI THEOREM ON HILBERT SPACE

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Abstract. This paper extends two recent improvements in the Hilbert space setting of the well-known Katznelson-Tzafriri theorem by establishing both a version of the result valid for bounded representations of a large class of abelian semigroups and a quantified version for contractive representations. The paper concludes with an outline of an improved version of the Katznelson-Tzafriri theorem for individual orbits, whose validity extends even to certain unbounded representations.

1. Introduction

In [19], Katznelson and Tzafriri proved that, given a power-bounded operator T on a complex Banach space X, $\|T^n(I-T)\| \to 0$ as $n \to \infty$ if (and only if) $\sigma(T) \cap \mathbb{T} \subset \{1\}$, where $\sigma(T)$ denotes the spectrum of T and $\mathbb{T}$ is the unit circle. Given a sequence $a \in \ell^1(\mathbb{Z}_+)$, define $\widehat{a}(\lambda) := \sum_{n=0}^{\infty} a(n)\lambda^n$ ($|\lambda| \leq 1$) and $\widehat{a}(T) := \sum_{n=0}^{\infty} a(n)T^n$, which is a bounded linear operator on X. Katznelson and Tzafriri also showed, in the same paper, that

$$\lim_{n\to\infty} \|T^n \widehat{a}(T)\| = 0 \tag{1.1}$$

provided there exists a sequence (a_n) in $\ell^1(\mathbb{Z}_+)$ such that each $\widehat{a_n}$ vanishes on an open neighbourhood of $\sigma(T) \cap \mathbb{T}$ and $\|a - a_n\|_1 \to 0$ as $n \to \infty$. This result has itself subsequently been extended, first to the case of C_0-semigroups (see [15, Théorème III.4] and [29, Theorem 3.2]) and later to more general semigroup representations (see [6, Theorem 4.3] and [29]).

These results are optimal in various senses (see [11, Section 5]), but improvements are possible when X is assumed to be a Hilbert space. It is shown in [14, Corollary 2.12], for instance, that the weaker (and necessary) condition that $\widehat{a}$ vanish on $\sigma(T) \cap \mathbb{T}$ is sufficient for (1.1) to hold, at least when T is a contraction; see [20, Proposition 1.6] for a slightly more general result. This result has in turn been improved upon in two recent papers. In [21], Léka has extended this result to power-bounded operators on Hilbert space, and Zarrabi in [31] has shown that for contractions, and likewise for pairs of commuting contractions and for contractive C_0-semigroups, the limit appearing in (1.1) is given, more generally, by $\sup\{|\widehat{a}(\lambda)| : \lambda \in \sigma(T) \cap \mathbb{T}\}$ or the appropriate analogue; for related results see [2], [3], [9], [10] and [23].

The purpose of this paper is to improve both Léka's and Zarrabi's versions of the Katznelson-Tzafriri theorem by extending them to representations of a significantly

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larger class of semigroups. The main result, Theorem 3.1, is a Katznelson-Tzafriri type theorem which holds for bounded (as opposed to contractive) representations and thus includes both [21, Theorem 2.1] and various results contained as special cases in [31]. The main implication of Theorem 3.1 is proved first via a certain ergodic condition as in [21] and then by a more direct argument. The second method does not involve the ergodic condition and, as is shown in Theorem 4.2, leads naturally to an extension of Zarrabi's quantified results for contractive representations. Section 5, finally, contains a brief exposition of an improved version of the Katznelson-Tzafriri theorem for individual orbits. First of all, though, Section 2 sets out the necessary preliminary material.

2. Preliminaries

The setting throughout, even when not stated explicitly, will be that of an abelian semigroup S contained in a locally compact abelian group G satisfying $G = S - S$. The Haar measure on G is denoted by μ, and it is assumed that S is Haar-measurable and hence itself becomes a measure space with respect to the restriction of μ. Assume furthermore that the interior S° of S (in the topology induced by G) is non-empty. The semigroup S becomes a directed set under the relation $\succeq$, where $s \succeq t$ for $s, t \in S$ whenever $s - t \in S \cup \{0\}$; this makes it possible to speak of limits as $s \to \infty$ through S. The dual group of G, consisting of all continuous bounded characters $\chi : G \to \mathbb{C}$, is denoted by Γ, the set of continuous bounded characters on S by S^*. It follows from the assumption that S spans G that the subset of S^* of characters taking values in the unit circle $\mathbb{T}$ can (and will throughout) be identified with Γ. Two important examples of the above are the (semi)groups $\mathbb{Z}_{(+)}$ with counting measure and $\mathbb{R}_{(+)}$ with Lebesgue measure. Here S^* can be identified in a natural way with $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$, respectively, and the dual group Γ is $\mathbb{T}$ and $\mathrm{i}\mathbb{R}$ in each case.

For $\Omega = G$ or S, let $L^1(\Omega)$ denote the algebra (under convolution) of functions $a : \Omega \to \mathbb{C}$ that are integrable with respect to (the restriction of) Haar measure and, given $a \in L^1(\Omega)$, define its Fourier transform by

$$\widehat{a}(\chi) := \int_\Omega a(s)\chi(s)\,\mathrm{d}\mu(s),$$

where χ is an element of Γ or S^*, as appropriate. Given a closed subset Λ of Γ, define $J_\Lambda := \{a \in L^1(G) : \widehat{a} \equiv 0 \text{ in a neighbourhood of } \chi \in \Lambda\}$ and $K_\Lambda := \{a \in L^1(G) : \widehat{a}(\chi) = 0 \text{ for all } \chi \in \Lambda\}$. An element of $L^1(G)$ is said to be of *spectral synthesis* with respect to Λ if it lies in the closure of J_Λ. Since K_Λ is closed, any such function must be an element of K_Λ. If K_Λ coincides with the closure of J_Λ, the set Λ is said to be of *spectral synthesis*.

For any closed subset Λ of Γ, the map $W_\Lambda : a \mapsto \widehat{a}|_\Lambda$ is a well-defined contractive algebra homomorphism from $L^1(G)$ into $C_0(\Lambda)$ whose kernel is K_Λ and whose range, by the Stone-Weierstrass theorem, is dense in $C_0(\Lambda)$; see [27, Theorem 1.2.4]. Since $K_\Lambda(G)$ is a closed ideal of $L^1(G)$, W_Λ induces a well-defined injective algebra homomorphism $U_\Lambda : L^1(G)/K_\Lambda(G) \to C_0(\Lambda)$ which, by the Inverse Mapping Theorem, is an isomorphism precisely when it is surjective. Since its range is dense in $C_0(\Lambda)$, this is the case if and only if the map is an isomorphic embedding (which in turn is equivalent to the dual operator $U'_\Lambda : M(\Lambda) \to K_\Lambda^\perp$ being either a surjection or an isomorphic embedding, where $M(\Lambda)$ denotes the set of complex-valued regular measures on Λ which have finite total variation; see for

instance [27, Appendix C11]). When these conditions are satisfied, Λ is said to be a *Helson set*, and the quantity $\alpha(\Lambda) := \|U_\Lambda^{-1}\|$ is known as its *Helson constant*. Since U_Λ is contractive, $\alpha(\Lambda) \geq 1$ for any Helson set Λ. For some examples of Helson sets, see [31, Remark 5.6].

In what follows, it will be assumed that the set $\{\widehat{a} : a \in L^1(S)\}$ separates points both from each other and from zero and, furthermore, that the interior S° is dense in S. For further details and discussion of these conditions, see for instance [6]. These assumptions ensure, in particular, that there exists a net (Ω_α), known as a *Følner net*, of compact, measurable, non-null subsets of S satisfying

$$\lim_{\alpha\to\infty} \frac{\mu\big(\Omega_\alpha \triangle (\Omega_\alpha + s)\big)}{\mu(\Omega_\alpha)} = 0,$$

uniformly for s in compact subsets of S.

Given a Banach space X and $\Omega = G$ or S for S and G as above, a *representation* of Ω on X is a strongly continuous homomorphism $T : \Omega \to \mathcal{B}(X)$ which, if $0 \in \Omega$, satisfies $T(0) = I$. The representation is said to be *bounded* if $\sup\{\|T(s)\| : s \in \Omega\} < \infty$ and in this case, given $a \in L^1(\Omega)$, the operator $\widehat{a}(T) \in \mathcal{B}(X)$ is defined, for each $x \in X$, by

$$\widehat{a}(T)x := \int_\Omega a(s)T(s)x \,\mathrm{d}\mu(s).$$

Given a bounded representation T of G on a Banach space X and a closed subspace Λ of the dual group Γ, the corresponding *spectral subspace* $M_T(\Lambda)$ is defined as

$$M_T(\Lambda) := \bigcap\nolimits_{a\in J_\Lambda} \operatorname{Ker} \widehat{a}(T),$$

the *(Arveson) spectrum* $\operatorname{Sp}(T)$ of T as

$$\operatorname{Sp}(T) := \bigcap \big\{\Lambda \subset \Gamma : \Lambda \text{ is closed and } M_T(\Lambda) = X\big\}.$$

Thus the spectrum is a closed subset of Γ, and it is shown in [25, Theorems 8.1.4 and 8.1.12], respectively, that $\operatorname{Sp}(T)$ is non-empty whenever X is non-trivial and that it is compact if and only if T is continuous with respect to the norm topology on $\mathcal{B}(X)$. If T is a representation by isometries, this notion of spectrum coincides with the *finite L-spectrum* of [22, Section 5.2]. By [25, Proposition 8.1.9], furthermore, $\operatorname{Sp}(T)$ has the alternative description

$$\operatorname{Sp}(T) = \big\{\chi \in \Gamma : |\widehat{a}(\chi)| \leq \|\widehat{a}(T)\| \text{ for all } a \in L^1(G)\big\}.$$

Accordingly, given a bounded representation T of a semigroup S on a Banach space X, the *spectrum* $\operatorname{Sp}(T)$ of T is defined as

$$\operatorname{Sp}(T) := \big\{\chi \in S^* : |\widehat{a}(\chi)| \leq \|\widehat{a}(T)\| \text{ for all } a \in L^1(S)\big\},$$

and the *unitary spectrum* of T is given by $\operatorname{Sp}_{\mathrm{u}}(T) := \operatorname{Sp}(T) \cap \Gamma$; see [6] for details. In the examples mentioned above, bounded semigroup representations correspond to a single power-bounded operator $T \in \mathcal{B}(X)$ if $S = \mathbb{Z}_+$ and to a bounded C_0-semigroup if $S = \mathbb{R}_+$. The spectrum is given by $\sigma(T)$ and $\sigma(A)$, respectively, where A denotes the generator of the semigroup.

3. A general Katznelson-Tzafriri type result

The aim of this section is to prove the following generalisation of [21, Theorem 2.1].

Theorem 3.1. *Let T be a bounded representation of a semigroup S on a Hilbert space X, and suppose that $a \in L^1(S)$. Then the following are equivalent:*

(i) $\widehat{a}(\chi) = 0$ *for every* $\chi \in \mathrm{Sp_u}(T)$;

(ii) Given any Følner net (Ω_α) *for* S *and any* $\chi \in \mathrm{Sp_u}(T)$,

$$\lim_{\alpha\to\infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s)\widehat{a}(T)\,\mathrm{d}\mu(s) \right\| = 0; \tag{3.1}$$

(iii) $\|T(s)\widehat{a}(T)\| \to 0$ *as* $s \to \infty$.

Remark 3.2. In [21, Theorem 2.1], conditions (ii) and (iii) above are presented in a slightly more general form, with the operator $\widehat{a}(T)$ replaced by an arbitrary $Q \in \mathcal{B}(X)$ that commutes with the representation. The presentation here is restricted to the case $Q = \widehat{a}(T)$ purely for simplicity.

The proof of this result will be broken up into a number of separate steps, all of which correspond to some part of the proof of [21, Theorem 2.1] given by Léka but typically with some modifications to accommodate the more general setting in which the representation need not be norm continuous. The following lemma constitutes the main step towards proving that (i) $\Longrightarrow$ (ii); it corresponds to [21, Lemma 2.2]. Note that the Hilbert space assumption is not required for this part of the argument.

Lemma 3.3. *Let T be a bounded representation of a semigroup S on a Banach space X, and let $a \in L^1(S)$. Then, for all $\chi \in \Gamma$,*

$$\lim_{\alpha\to\infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s)\big(\widehat{a}(T) - \widehat{a}(\chi)\big)\,\mathrm{d}\mu(s) \right\| = 0,$$

where (Ω_α) is any Følner net for S and the integral is taken in the strong sense.

Proof. With $a \in L^1(S)$ and $\chi \in \Gamma$ fixed, let $x \in X$ have unit norm and set

$$I_x(\alpha) := \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s)\big(\widehat{a}(T)x - \widehat{a}(\chi)x\big)\,\mathrm{d}\mu(s) \right\|.$$

Then, by a simple application of Fubini's theorem,

$$I_x(\alpha) \le M \int_S \frac{\mu\big(\Omega_\alpha \triangle (\Omega_\alpha + s)\big)}{\mu(\Omega_\alpha)} |a(s)|\,\mathrm{d}\mu(s),$$

where $M := \sup\{\|T(s)\| : s \in S\}$. Let $\varepsilon > 0$. Since $a \in L^1(S)$, there exists a compact subset K of S such that $\int_{S\setminus K} |a(s)|\,\mathrm{d}\mu(s) < \varepsilon/4M$. Defining

$$\xi_K(\alpha) := \sup\left\{ \frac{\mu(\Omega_\alpha \triangle (\Omega_\alpha + s))}{\mu(\Omega_\alpha)} : s \in K \right\},$$

it follows from the definition of a Følner net that $\xi_K(\alpha) \to 0$ as $\alpha \to \infty$. Since

$$I_x(\alpha) \le M\|a\|_1 \xi_K(\alpha) + 2M \int_{S\setminus K} |a(s)|\,\mathrm{d}\mu(s),$$

$I_x(\alpha) < \varepsilon$ for all sufficiently large α, and the result follows. $\square$

Corollary 3.4. *Let T be a bounded representation of a semigroup S on a Banach space X, and let $\chi \in \Gamma$. Suppose that $a \in L^1(S)$ is such that $\widehat{a}(\chi) = 0$. Then* (3.1) *holds for any Følner net (Ω_α) for S.*

The next result is an important step towards establishing the implication (ii) $\Longrightarrow$ (iii) in Theorem 3.1 and should be compared with [21, Lemma 2.4].

Proposition 3.5. *Let S be a semigroup and T a representation of a group $G = S - S$ by unitary operators on a Hilbert space X. Suppose that $a \in L^1(G)$ and that, for each $\chi \in \mathrm{Sp}(T)$,*

$$\lim_{\alpha\to\infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s)\widehat{a}(T)\,\mathrm{d}\mu(s) \right\| = 0,$$

where (Ω_α) is any Følner net for S. Then $\widehat{a}(T) = 0$.

Proof. Writing $B(\Gamma)$ for the set of Borel subsets of the dual group Γ, let $E : B(\Gamma) \to \mathcal{B}(X)$ denote the spectral measure associated with T (see [25, Theorem 8.3.2]) and, for $s \in G$ and $\Lambda \in B(\Gamma)$, let $T_\Lambda(s) := T(s)E(\Lambda)$. Then

$$T_\Lambda(s) := \int_\Lambda \chi(s)\,\mathrm{d}E(\chi),$$

the integral being taken in the weak sense, and, by Fubini's theorem,

$$\widehat{b}(T_\Lambda) = \int_\Lambda \widehat{b}(\chi)\,\mathrm{d}E(\chi) \tag{3.2}$$

for all $b \in L^1(G)$. Thus if $\Lambda \in B(\Gamma)$ is closed and $b \in J_\Lambda(G)$, then $\widehat{b}(T_\Lambda) = 0$, and it follows that $M_{T_\Lambda}(\Lambda) = X$, so that $\mathrm{Sp}(T_\Lambda) \subset \Lambda$. Choosing $\Lambda \in B(\Gamma)$ to be compact ensures that the representation T_Λ of G on X is norm continuous.

Set $Q := \widehat{a}(T)$ and, for a given compact subset Λ of $\mathrm{Sp}(T)$, define $Q_\Lambda := QE(\Lambda)$, noting that Q_Λ is normal and that $Q_\Lambda \to Q$ in the weak (and indeed the strong) operator topology as Λ approaches $\mathrm{Sp}(T)$ through compact subsets. Furthermore, let $\mathcal{A}_\Lambda$ denote the commutative unital C^*-algebra generated by $\{Q_\Lambda, Q_\Lambda^*\} \cup \{T_\Lambda(s) : s \in G\}$, and let $\Delta(\mathcal{A}_\Lambda)$ denote its character space. Write $\Phi_\Lambda : \mathcal{A}_\Lambda \to C_0(\Delta(\mathcal{A}_\Lambda))$ for the Gelfand transform of $\mathcal{A}_\Lambda$, which is an isometric $*$-isomorphism, and consider the map $\chi_\xi : G \to \mathbb{C}\backslash\{0\}$ given, for $\xi \in \Delta(\mathcal{A}_\Lambda)$ and $s \in G$, by $\chi_\xi(s) := \Phi_\Lambda(T_\Lambda(s))(\xi)$. Since the representation T_Λ is norm continuous, χ_ξ is a continuous group homomorphism, and the fact that each $\xi \in \Delta(\mathcal{A}_\Lambda)$ is a bounded linear functional on $\mathcal{A}_\Lambda$ with $\|\xi\| = |\xi(E(\Lambda))| = 1$ implies that $|\chi_\xi(s)| \le 1$ for all $s \in G$. Hence $\chi_\xi \in \Gamma$. Moreover, if $b \in L^1(G)$, then

$$\left|\widehat{b}(\chi_\xi)\right| = \left|\xi\left(\int_G b(s)T_\Lambda(s)\,\mathrm{d}\mu(s)\right)\right| \le \left\|\widehat{b}(T_\Lambda)\right\|,$$

which is to say that $\chi_\xi \in \mathrm{Sp}(T_\Lambda)$, and hence $\chi_\xi \in \mathrm{Sp}(T)$. Let $g_\Lambda := \Phi_\Lambda(Q_\Lambda)$. Then

$$\begin{aligned}
|g_\Lambda(\xi)| &= \frac{1}{\mu(\Omega_\alpha)} \left| \int_{\Omega_\alpha} |\chi_\xi(s)|^2 g_\Lambda(\xi)\, \mathrm{d}\mu(s) \right| \\
&\leq \frac{1}{\mu(\Omega_\alpha)} \left\| \Phi_\Lambda \left(\int_{\Omega_\alpha} \overline{\chi_\xi(s)} T_\Lambda(s) Q_\Lambda \, \mathrm{d}\mu(s) \right) \right\|_\infty \\
&= \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi_\xi(s)} T_\Lambda(s) Q_\Lambda \, \mathrm{d}\mu(s) \right\| \\
&\leq \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi_\xi(s)} T(s) Q \, \mathrm{d}\mu(s) \right\|,
\end{aligned}$$

for any $\xi \in \Delta(\mathcal{A}_\Lambda)$ and letting $\alpha \to \infty$ shows that $g_\Lambda = 0$. Since Φ_Λ is an isometry, it follows that $Q_\Lambda = 0$, and allowing Λ to approach $\mathrm{Sp}(T)$ through compact subsets gives $Q = 0$, as required. □

Remark 3.6. The result remains true when $\widehat{a}(T)$ is replaced by any $Q \in \mathcal{B}(X)$ which commutes with T. If Q is normal, this follows from the same argument as above; the general case can then be obtained by considering the operator Q^*Q; see also [21, Lemma 2.4].

Propositions 3.7 and 3.8 below correspond in essence to the two main stages in the proof of [21, Theorem 2.1] and show, via an intermediate result for the strong operator topology, that (ii) $\Longrightarrow$ (iii) in Theorem 3.1.

Proposition 3.7. *Let T be a bounded representation of a semigroup S on a Hilbert space X, and let $a \in L^1(S)$. Suppose that, for some Følner net (Ω_α) for S,* (3.1) *holds for all $\chi \in \mathrm{Sp_u}(T)$. Then, for all $x \in X$, $\|T(s)\widehat{a}(T)x\| \to 0$ as $s \to \infty$.*

Proof. Fix a Banach limit ϕ on $L^\infty(S)$. A construction analogous to [6, Proposition 3.1] and [20, Section 1] shows that there exist a Hilbert space X_ϕ, a representation T_ϕ of S on X_ϕ by isometries with $\mathrm{Sp}(T_\phi) \subset \mathrm{Sp}(T)$ and an operator $\pi_\phi : X \to X_\phi$ with the following properties: π_ϕ is bounded with norm $\|\pi_\phi\| \leq \sup\{\|T(s)\| : s \in S\}$, $\|\pi_\phi(x)\|^2 = \phi(\|T(\cdot)x\|^2)$ for all $x \in X$, $\mathrm{Ran}\,\pi_\phi$ is dense in X_ϕ, $\mathrm{Ker}\,\pi_\phi = \{x \in X : \|T(s)x\| \to 0 \text{ as } s \to \infty\}$ and $\pi_\phi T(s) = T_\phi(s)\pi_\phi$ for all $s \in S$ (so π_ϕ is an *intertwining operator*). In particular, for any operator $Q \in \mathcal{B}(X)$ that commutes with T, the operator $Q_\phi \in \mathcal{B}(X_\phi)$ defined by $\pi_\phi Q = Q_\phi \pi_\phi$ satisfes $\|Q_\phi\| \leq \|Q\|$. By a construction analogous to [4, Proposition 2.1] (see also [6, Proposition 3.2], [8], [13] and [18]), there exist a further Hilbert space Y_ϕ, a representation T_G of the group $G = S - S$ by unitary operators on Y_ϕ with $\mathrm{Sp}(T_G) = \mathrm{Sp_u}(T_\phi)$ and an isometric intertwining operator $\pi_G : X_\phi \to Y_\phi$ such that $\{T_G(-s)\pi_G(x) : s \in S, x \in X_\phi\}$ is dense in Y_ϕ. The latter implies, in particular, that $\|Q_G\| = \|Q_\phi\|$ for all $Q_\phi \in \mathcal{B}(X_\phi)$ and all $Q_G \in \mathcal{B}(X_G)$ which commute with T_G and satisfy $\pi_G Q_\phi = Q_G \pi_G$. Thus it is possible to assume, sacrificing only the density condition on the range of the intertwining operator, that T_ϕ itself is in fact a representation of G by unitary operators on X_ϕ.

Now, given $\chi \in \mathrm{Sp}(T_\phi)$, define operators $Q_\alpha \in \mathcal{B}(X)$ and $Q_{\phi,\alpha} \in \mathcal{B}(X_\phi)$ as

$$Q_\alpha := \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T(s) \widehat{a}(T) \, \mathrm{d}\mu(s) \tag{3.3}$$

and

$$Q_{\phi,\alpha} := \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_\phi(s) \widehat{a}(T_\phi) \, \mathrm{d}\mu(s).$$

Then $\pi_\phi Q_\alpha = Q_{\phi,\alpha}\pi_\phi$, from which it follows that $\|Q_{\phi,\alpha}\| \le \|Q_\alpha\|$. In particular, $\|Q_{\phi,\alpha}\| \to 0$ as $\alpha \to \infty$. Identifying $L^1(S)$ in the natural way with a subset of $L^1(G)$, it follows from Proposition 3.5 applied to T_ϕ, X_ϕ and a that $\widehat{a}(T_\phi) = 0$. In particular, $\pi_\phi(\widehat{a}(T)x) = \widehat{a}(T_\phi)\pi_\phi(x) = 0$ for any $x \in X$, so the result follows from the description of $\operatorname{Ker}\pi_\phi$. □

Proposition 3.8. *Let T be a bounded representation of a semigroup S on a Hilbert space X, and let $a \in L^1(S)$. Suppose that, for some Følner net (Ω_α) for S, equation* (3.1) *is satisfied for all $\chi \in \operatorname{Sp_u}(T)$. Then $\|T(s)\widehat{a}(T)\| \to 0$ as $s \to \infty$.*

Proof. Suppose, for the sake of contradiction, that (3.1) holds for all $\chi \in \operatorname{Sp_u}(T)$ and some Følner net (Ω_α) but that there exist $\varepsilon > 0$ and a net (s_β) in S, with indexing set A say, such that $s_\beta \to \infty$ as $\beta \to \infty$ and, for some suitable sequence (y_β) of unit vectors in X, $\|T(s_\beta)\widehat{a}(T)y_\beta\| \ge \varepsilon$ for all $\beta \in A$. Letting $M := \sup\{\|T(s)\| : s \in S\}$, it follows that $\|T(s)\widehat{a}(T)y_\beta\| \ge \varepsilon M^{-1}$ whenever $s_\beta - s \in S$. Fix $t \in S^\circ$ and, essentially as in [27, Section 1.1.8], let $b \in L^1(S)$ satisfy $\|b\|_1 = 1$ and $\|a*b - a_t\|_1 < \varepsilon/2M^3$, where $a_t \in L^1(S)$ is given by $a_t(s) = a(s-t)$ if $s - t \in S$ and $a_t(s) = 0$ otherwise. Now, by a construction similar to those contained in [16], [24] and [26], there exist

(a) a Banach space X_A^∞, which is contained in the set $\ell^\infty(A;X)$ of X-valued nets indexed by A and contains all nets of the form $(\widehat{c}(T)x_\alpha)$ with $c \in L^1(S)$ and $(x_\alpha) \in \ell^\infty(A;X)$, and a bounded representation T_A^∞ of S on X_A^∞ with $\operatorname{Sp}(T_A^\infty) = \operatorname{Sp}(T)$;

(b) a Hilbert space X_A, a bounded representation T_A of S on X_A with $\operatorname{Sp}(T_A) \subset \operatorname{Sp}(T_A^\infty)$ and a surjective intertwining operator $\pi_A : X_A^\infty \to X_A$ which is contractive and such that $\|\pi_A(x_\alpha)\|$ is given, for each $(x_\alpha) \in X_A^\infty$, by the limit of the net $(\|x_\alpha\|)$ along some ultrafilter on A which contains the filter generated by the sets $\{\alpha \in A : \alpha \succeq \beta\}$ with $\beta \in A$.

Note, in particular, that $\operatorname{Sp}(T_A) \subset \operatorname{Sp}(T)$. Consider the element (x_β) of X_A^∞, where $x_\beta := \widehat{b}(T)y_\beta$. Then, writing $c := a * b - a_t$,

$$\begin{aligned}
\|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| &= \|\pi_A T_A^\infty(s)\widehat{a}(T_A^\infty)(x_\beta)\| \\
&= \big\|\pi_A\big(T(s)\widehat{a*b}(T)y_\beta\big)\big\| \\
&\ge \big\|\pi_A\big(T(s+t)\widehat{a}(T)y_\beta\big)\big\| - \big\|\pi_A\big(T(s)\widehat{c}(T)y_\beta\big)\big\| \\
&\ge \liminf_{\beta\to\infty} \|T(s+t)\widehat{a}(T)y_\beta\| - M^2\|c\|_1
\end{aligned}$$

for all $s \in S$, where the last line follows from the definition of the norm on X_A. Thus $\|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| \ge \varepsilon/2M$ for all $s \in S$.

Fix $\chi \in \operatorname{Sp_u}(T_A)$ and define the operators $Q_{A,\alpha}^\infty \in \mathcal{B}(X_A^\infty)$ and $Q_{A,\alpha} \in \mathcal{B}(X_A)$ as

$$Q_{A,\alpha}^\infty := \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_A^\infty(s)\widehat{a}(T_A^\infty)\,\mathrm{d}\mu(s)$$

and

$$Q_{A,\alpha} := \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_A(s)\widehat{a}(T_A)\,\mathrm{d}\mu(s).$$

Then $\pi_A Q_{A,\alpha}^\infty = Q_{A,\alpha}\pi_A$, so the properties of π_A and the fact that $Q_{A,\alpha}^\infty$ acts on X_A^∞ by entrywise application of the operator Q_α, as defined in equation (3.3), imply that $\|Q_{A,\alpha}\| \le \|Q_\alpha\|$. Hence $\|Q_{A,\alpha}\| \to 0$ as $\alpha \to \infty$, and Proposition 3.7 applied to T_A and X_A leads to the required contradiction. □

Corollary 3.4 and Proposition 3.8 together prove the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) of Theorem 3.1. The following simple lemma, which follows immediately from the definition of the spectrum of a semigroup representation T along with the observation that $\widehat{a_s}(T) = T(s)\widehat{a}(T)$ for all $a \in L^1(S)$ and $s \in S$, shows that (iii) $\Longrightarrow$ (i), thus completing the proof of the main result.

Lemma 3.9. *Let T be a bounded representation of a semigroup S on a Banach space X, and let $a \in L^1(S)$. Then $|\widehat{a}(\chi)| \leq \|T(s)\widehat{a}(T)\|$ for all $\chi \in \mathrm{Sp_u}(T)$ and all $s \in S$.*

Remark 3.10. There is a direct proof of the implication (iii) $\Longrightarrow$ (ii) in Theorem 3.1. Indeed, if T is a bounded representation of a semigroup S on any Banach space X, if $a \in L^1(S)$ and if (Ω_α) is any Følner net for S, then

$$\|Q_\alpha\| \leq \sup\{\|T(s)\widehat{a}(T)\| : s \succeq t\} + M^2\|a\|_1 \frac{\mu\big(\Omega_\alpha \triangle (\Omega_\alpha + t)\big)}{\mu(\Omega_\alpha)}$$

for any $t \in S$, where Q_α is as in (3.3) and $M = \sup\{\|T(s)\| : s \in S\}$. Hence (iii) $\Longrightarrow$ (ii) by definition of a Følner net. Moreover, it is possible, at least in special cases, to show directly that (ii) $\Longrightarrow$ (i). When $S = \mathbb{Z}_+$, this follows from Corollary 3.4 and the uniform ergodic theorem (see [21, Corollary 2.3]), and a similar argument works when $S = \mathbb{R}_+$.

If one is interested in establishing only the equivalence of statements (i) and (iii) of Theorem 3.1, there is a shorter argument which may be of independent interest. Recall the classical fact that, given a representation T of a group G by isometries on a Banach space X, one has $\widehat{a}(T) = 0$ for all $a \in L^1(G)$ which are of spectral synthesis with respect to $\mathrm{Sp}(T)$. This is a simple consequence of the definition of the spectrum (see also [12, Chapter 8], [22, Chapter 5] and [28, Lemma 2.4.3]) and is used (together with constructions analogous to those used in the proof of Proposition 3.7) in [6, Theorem 4.3] to derive a general form of the Katznelson-Tzafriri theorem on Banach space. Corollary 3.13 below, which is an improved version of this result when X is a Hilbert space, makes it possible to obtain the implication (i) $\Longrightarrow$ (iii) of Theorem 3.1 by an analogous argument which bypasses Proposition 3.5. It is a special case of the following more general statement.

Proposition 3.11. *Let T be a representation of a group G by unitary operators on a Hilbert space X, and let $a \in L^1(G)$. Then $\|\widehat{a}(T)\| = \sup\{|\widehat{a}(\chi)| : \chi \in \mathrm{Sp}(T)\}$.*

Proof. Let $\mathcal{A}_T$ denote the norm closure in $\mathcal{B}(X)$ of $\{\widehat{b}(T) : g \in L^1(G)\}$. Then $\mathcal{A}_T$ is a commutative C^*-algebra and hence, writing $\Delta(\mathcal{A}_T)$ for the character space of $\mathcal{A}_T$, the Gelfand transform $\Phi : \mathcal{A}_T \to C_0(\Delta(\mathcal{A}_T))$ is an isometric $*$-isomorphism. By [6, Proposition 2.4], the map sending a character $\chi \in \mathrm{Sp}(T)$ to the character ξ_χ on $\mathcal{A}_T$ defined, on the dense subspace $\{\widehat{b}(T) : g \in L^1(G)\}$, by $\xi_\chi(\widehat{b}(T)) := \widehat{b}(\chi)$ is a bijection, and hence $\|\widehat{a}(T)\| = \|\Phi(\widehat{a}(T))\|_\infty = \sup\{|\widehat{a}(\chi)| : \chi \in \mathrm{Sp}(T)\}$. $\square$

Remark 3.12. This result can also be proved using (3.2) with $\Lambda = \mathrm{Sp}(T)$.

Corollary 3.13. *Let T be a representation of a group G by unitary operators on a Hilbert space X with spectrum $\Lambda := \mathrm{Sp}(T)$, and let $a \in L^1(G)$. Then $\widehat{a}(T) = 0$ if and only if $a \in K_\Lambda$.*

Remark 3.14. This follows also from Corollary 3.4 and Proposition 3.5 together with Lemma 3.9.

4. Quantified results for contractive representations

The purpose of this section is to study the limit of $\|T(s)\widehat{a}(T)\|$ as $s \to \infty$ in the case where T is a contractive representation of a semigroup S on a Hilbert space X and a is an element of $L^1(S)$ whose Fourier transform $\widehat{a}$ does not necessarily vanish on the unitary spectrum $\mathrm{Sp_u}(T)$ of T. Theorem 4.2 below constitutes an important step towards this aim and can be viewed as a sharper version of [3, Proposition 5.5] which holds on general Banach space. It follows from the following result for individual orbits.

Proposition 4.1. *Let T be a representation of a semigroup S by contractions on a Hilbert space X with unitary spectrum $\Lambda := \mathrm{Sp_u}(T)$, and let $a \in L^1(S)$. Then*

$$\lim_{s\to\infty} \|T(s)\widehat{a}(T)x\| \le \|a + K_\Lambda\|\|x\|$$

for all $x \in X$.

Proof. Fix any Banach limit ϕ on $L^\infty(S)$ and let X_ϕ, T_ϕ and π_ϕ be as in the proof of Proposition 3.7. Then, since T is contractive, $\|\pi_\phi(x)\| = \lim_{s\to\infty} \|T(s)x\|$ for all $x \in X$ and it is possible, as before, to assume that T_ϕ is in fact a representation of the group $G = S - S$ on X_ϕ by unitary operators. It follows from Corollary 3.13 that $\widehat{b}(T_\phi) = 0$ for all $b \in K_\Lambda$. Hence $\|\widehat{a}(T_\phi)\| \le \|a - b\|_1$ for any such b, which implies that $\|\widehat{a}(T_\phi)\| \le \|a + K_\Lambda(G)\|$. Thus, given any $x \in X$,

$$\begin{aligned}\lim_{s\to\infty} \|T(s)\widehat{a}(T)x\| &= \|\pi_\phi(\widehat{a}(T)x)\| \\ &= \|\widehat{a}(T_\phi)\pi_\phi(x)\| \\ &\le \|a + K_\Lambda(G)\|\|\pi_\phi(x)\|,\end{aligned}$$

and the result follows since π_ϕ is a contraction. □

Theorem 4.2. *Let T be a representation of S by contractions on a Hilbert space X with unitary spectrum $\Lambda := \mathrm{Sp_u}(T)$, and let $a \in L^1(S)$. Then*

$$\lim_{s\to\infty} \|T(s)\widehat{a}(T)\| \le \|a + K_\Lambda\|.$$

Proof. Suppose not. Then there exist $\varepsilon > 0$ and a net (s_β) in S, with indexing set A, say, and satisfying $s_\beta \to \infty$ as $\beta \to \infty$, as well as a net of unit vectors (y_β) in X such that $\|T(s_\beta)\widehat{a}(T)y_\beta\| \ge \|a + K_\Lambda\| + \varepsilon$ for all $\beta \in A$. In particular, $\|T(s)\widehat{a}(T)y_\beta\| \ge \|a + K_\Lambda\| + \varepsilon$ whenever $s_\beta - s \in S$.

Let X_A, X_A^∞ , T_A and π_A be as in the proof of Proposition 3.8, choose, for a fixed $t \in S^\circ$, $b \in L^1(S)$ such that $\|b\|_1 = 1$ and $\|a * b - a_t\|_1 < \varepsilon/2$, and again define $(x_\beta) \in X_A^\infty$ by setting $x_\beta := \widehat{b}(T)y_\beta$. It then follows from a calculation analogous to the one in the proof of Proposition 3.8 that $\|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| > \|a + K_\Lambda\| + \varepsilon/2$ for all $s \in S$. However, applying Proposition 4.1 to the contractive representation T_A of S on X_A, and using the fact that $\|\pi_A(x_\beta)\| \le \|\widehat{b}(T)\| \le 1$, leads to

$$\lim_{s\to\infty} \|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| \le \|a + K_{\Lambda_A}\|\|\pi_A(x_\beta)\| \le \|a + K_\Lambda\|,$$

where $\Lambda_A := \mathrm{Sp_u}(T_A) \subset \Lambda$. This gives the required contradiction. □

The final result is a special instance of Theorem 4.2 which applies to representations whose unitary spectrum is a Helson set. It is a simple consequence of the definition of a Helson set and should be compared with the results in [31, Section 5],

which hold on Banach space, but, in addition, assume the unitary spectrum to be of spectral synthesis.

Corollary 4.3. *Let T be a representation of a semigroup S by contractions on a Hilbert space X, and let $a \in L^1(S)$. Suppose that the unitary spectrum $\Lambda := \mathrm{Sp_u}(T)$ of T is a Helson set. Then*

$$\sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\} \leq \lim_{s\to\infty} \|T(s)\widehat{a}(T)\| \leq \alpha(\Lambda) \sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\}.$$

Remark 4.4. The first inequality holds irrespective of whether the unitary spectrum is a Helson set, and indeed of whether X is a Hilbert space. It is an immediate consequence of Lemma 3.9.

5. Local results

This final section gives a brief account of how some of the aforementioned improvements of the Katznelson-Tzafriri theorem on Hilbert space carry over to orbitwise, or 'local', versions of the result. More specifically, the aim is to obtain spectral conditions which ensure, given a bounded representation T of a semigroup on a Hilbert space X and an element a of $L^1(S)$, that $\|T(s)\widehat{a}(T)x\| \to 0$ as $s \to \infty$ for a *particular* point $x \in X$. Such results are of particular interest in the context of C_0-semigroups, where orbits correspond to solutions of the associated Cauchy problem, and they have been studied for instance in [5], [7, Section 4] and [20].

The notion of *spectrum* that is most appropriate to this context goes back to [1]. Consider a bounded representation T of a semigroup S on some Banach space X, and let $x \in X$ be given. A character $\chi \in \Gamma$ will be said to be *locally regular at x* if there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in L^1(S)$, a neighbourhood Ω of the point $\lambda_0 := (\widehat{a_1}(\chi), \ldots, \widehat{a_n}(\chi))$ in $\mathbb{C}^n$ and holomorphic functions $g_1, \ldots, g_n : \Omega \to X$ such that $\sum_{k=0}^n (\lambda_k - \widehat{a_k}(T))g_k(\lambda) = x$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Omega$. The *unitary local (Albrecht) spectrum* $\mathrm{Sp_u}(T; x)$ of T at x is then defined as the set of all characters $\chi \in \Gamma$ which fail to be locally regular at x. It is easy to see that $\mathrm{Sp_u}(T; x) \subset \mathrm{Sp_u}(T)$ for each $x \in X$. For further details on the unitary local spectrum and its relation to other natural notions of local spectrum, see [7, Section 4].

The main result of this section is the following theorem, which improves [5, Theorem 5.1] in the Hilbert space setting.

Theorem 5.1. *Let T be a bounded representation of a semigroup S on a Hilbert space X. Furthermore, let $x \in X$ and $a \in L^1(S)$. Then $\|T(s)\widehat{a}(T)x\| \to 0$ as $s \to \infty$ provided $\widehat{a}(\chi) = 0$ for all $\chi \in \mathrm{Sp_u}(T; x)$.*

Proof. Fix a Banach limit ϕ on $L^\infty(S)$ and let X_ϕ, T_ϕ and π_ϕ be as in the proof of Proposition 3.7, so that T_ϕ may again be assumed to be a representation of the group $G = S - S$ on X_ϕ by unitary operators. By [7, Proposition 5.1], $\mathrm{Sp_u}(T_\phi; \pi_\phi(x)) \subset \mathrm{Sp_u}(T; x)$. Moreover, writing X_x for the closed linear span of the set $\{T_\phi(s)\pi_\phi(x) : s \in G\}$ in X_ϕ and T_x for the representation of G on X_x obtained by restricting T_ϕ, [7, Theorem 4.5] gives $\mathrm{Sp}(T_x) = \mathrm{Sp_u}(T_\phi; \pi_\phi(x))$ and hence $\mathrm{Sp}(T_x) \subset \mathrm{Sp_u}(T; x)$. Thus the assumption on a implies that $\widehat{a}(\chi) = 0$ for all $\chi \in \mathrm{Sp}(T_x)$, from which it follows by Corollary 3.13 that $\widehat{a}(T_x) = 0$. Hence $\pi_\phi(\widehat{a}(T)x) = \widehat{a}(T_\phi)\pi_\phi(x) = 0$, which is to say $\widehat{a}(T)x \in \operatorname{Ker} \pi_\phi$, as required. □

Theorem 5.1 in fact holds even for certain unbounded representations provided the growth of the norm is sufficiently slow and regular. Given a semigroup S,

a measurable function $w : S \to [1, \infty)$ is said to be a *weight* if it is bounded on compact subsets of S and satisfies $w(s+t) \leq w(s)w(t)$ for all $s, t \in S$. Given a weight w and a representation T of S on a Banach space X which satisfies $\|T(s)\| \leq w(s)$ for all $s \in S$, it is possible, essentially by replacing any occurrence of $L^1(S)$ with the Beurling algebra $L^1_w(S)$, to define the modified unitary local (Albrecht) spectrum $\mathrm{Sp}^w_{\mathrm{u}}(T; x)$ for any point $x \in X$; see [8] and [22] for details.

An argument entirely analogous to the proof of Theorem 5.1, but this time using the full strength of the results in [7], then leads to the following result. Here the additional regularity assumption on the weight w ensures that the representation corresponding to T_ϕ in the above proof is again isometric; see [7, Proposition 3.1]. Similar non-local results may be found in [7, Theorem 3.4] and [30, Theorem 8].

Theorem 5.2. *Let T be a representation of a semigroup S on a Hilbert space X and suppose that T is dominated by a weight w such that, for every $t \in S$, $w(s)^{-1}w(s+t) \to 1$ as $s \to \infty$. Furthermore, let $x \in X$ and suppose that $a \in L^1_w(S)$ is such that $\widehat{a}(\chi) = 0$ for all $\chi \in \mathrm{Sp}^w_{\mathrm{u}}(T; x)$. Then $\|T(s)\widehat{a}(T)x\| = o(w(s))$ as $s \to \infty$.*

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