

NEW CANONICAL TRIPLE COVERS OF SURFACES

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ABSTRACT. We construct a surface of general type with canonical map of degree 12 which factors as a triple cover and a bidouble cover of \mathbb{P}^2 . We also show the existence of a smooth surface with $q = 0$, $\chi = 13$ and $K^2 = 9\chi$ such that its canonical map is either of degree 3 onto a surface of general type or of degree 9 onto a rational surface.

1. INTRODUCTION

Let S be a smooth minimal surface of general type. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following Beauville's result is well known.

Theorem 1.1 ([3]). *If the canonical image $\Sigma := \phi(S)$ is a surface, then either:*

- (i) $p_g(\Sigma) = 0$, or
- (ii) Σ is a canonical surface (in particular $p_g(\Sigma) = p_g(S)$).

Moreover, in case (i) $d \leq 36$ and in case (ii) $d \leq 9$.

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$ and $p_g(\Sigma) = 0$. Although this is a classical problem, for $d > 8$ the number of known examples drops drastically: only Tan's example [18, §5] with $d = K^2 = 9$, $\chi = 4$ and Persson's example [14] with $d = K^2 = 16$, $\chi = 4$ are known. More recently, Du and Gao [7] claimed that if the canonical map is an abelian cover of \mathbb{P}^2 , then these are the only possibilities for $d > 8$.

In this note we construct an example with $d = K^2 = 12$ which factors as a triple cover and a bidouble cover of \mathbb{P}^2 .

Known examples for case (ii) with $d = 3$ date to 1991/2: Pardini's example [13] with $K^2 = 27$, $\chi = 6$ and Tan's examples [18] with $K^2 \leq 6\chi$, $5 \leq \chi \leq 9$. Later Barth [1, unpublished] have shown that the quintic surface constructed by van der Geer and Zagier in 1977 [20] is also the image of a triple canonical cover with $\chi = 5$ (see also [16]).

Nowadays the case $d = 3$ is still mysterious. On the one hand no one has given a bound for χ , on the other hand there are no examples for other values of the invariants.

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More generally, for the case where the canonical map factors through a triple cover of a surface of general type and $d = 6$, we have only the family given in [6, Example 3.4] with invariants on the Noether’s line $K^2 = 2p_g - 4$.

Here we show the existence of a smooth regular surface S with $\chi = 13$ and $K^2 = 9\chi$ such that its canonical map ϕ factors through a triple cover of a surface of general type. This is the first example on the border line $K^2 = 9\chi$ (recall that, for a surface of general type, one always has $2p_g - 4 \leq K^2 \leq 9\chi$).

This surface is an unramified cover of a fake projective plane. We show that if $\deg(\phi) \neq 3$, then ϕ is of degree 9 onto a rational surface. If this is the case, then one might expect to be able to recover the construction of the rigid surface S as a covering of \mathbb{P}^2 , which would be interesting. Since it seems very difficult to provide a geometric construction of a fake projective plane, we conjecture that $d = 3$.

Notation. We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$ -curve on a surface is a curve isomorphic to \mathbb{P}^1 with self-intersection $-n$. Linear equivalence of divisors is denoted by \equiv .

If X is a normal surface and Y is a smooth minimal model of X , then $\chi(\mathcal{O}_X)$, $q(X)$ and $p_g(X)$ mean $\chi(\mathcal{O}_Y)$, $q(Y)$ and $p_g(Y)$ (holomorphic Euler characteristic, irregularity and geometric genus, respectively).

The rest of the notation is standard in Algebraic Geometry.

2. BASICS ON GALOIS TRIPLE COVERS

Our references for triple covers are [17], [18] or [11].

Let X be a smooth surface. A Galois triple cover $\pi : Y \rightarrow X$ is determined by divisors L, M, B, C on X such that $B \in |2L - M|$ and $C \in |2M - L|$. The branch locus of π is $B + C$ and $3L \equiv 2B + C$, $3M \equiv B + 2C$. The surface Y is normal iff $B + C$ is reduced. The singularities of Y lie over the singularities of $B + C$.

If $B + C$ is smooth, we have

$$(2.1) \quad \chi(\mathcal{O}_Y) = 3\chi(\mathcal{O}_X) + \frac{1}{2}(L^2 + K_X L) + \frac{1}{2}(M^2 + K_X M),$$

$$(2.2) \quad K_Y^2 = 3K_X^2 + 4(L^2 + K_X L) + 4(M^2 + K_X M) - 4LM,$$

$$(2.3) \quad q(Y) = q(X) + h^1(X, \mathcal{O}_X(K_X + L)) + h^1(X, \mathcal{O}_X(K_X + M)),$$

$$(2.4) \quad p_g(Y) = p_g(X) + h^0(X, \mathcal{O}_X(K_X + L)) + h^0(X, \mathcal{O}_X(K_X + M)).$$

Now suppose that $\sigma : X \rightarrow X'$ is the minimal resolution of a normal surface X' with a set $s = \{s_1, \dots, s_n\}$ of ordinary cusps (singularities of type A_2). If the (-2) -curves A_i, A'_i satisfying $\sigma^{-1}(s_i) = A_i + A'_i$ can be labelled such that

$$\sum_1^n (2A_i + A'_i) \equiv 3J,$$

for some divisor J , then we say that s is a 3-divisible set of cusps.

Proposition 2.1. *Let X' be a minimal surface of general type containing a 3-divisible set as above as only singularities. Let $\phi : Y \rightarrow X$ be a Galois triple cover with branch locus $\sum_1^n (A_i + A'_i)$.*

If $n = 3\chi(\mathcal{O}_X)$, then $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_X)$ and $K_{Y'}^2 = 3K_X^2$, where Y' is the smooth minimal model of Y .

Proof. Let $\tilde{X} \rightarrow X$ be the blow-up at the singular points of $\sum_1^n (A_i + A'_i)$. Denote by $\widehat{A}_i, \widehat{A}'_i$ the (-3) -curves which are the strict transforms of $A_i, A'_i, i = 1, \dots, n$. The surface Y' is the minimal model of the Galois triple cover of \tilde{X} with branch locus $\sum_1^n (\widehat{A}_i + \widehat{A}'_i)$. The result follows from (2.1) and (2.2) (notice that $K_{\tilde{X}}^2 = K_X^2 - n$ and $K_{Y'}^2 = 3K_{\tilde{X}}^2 + 3n$). \square

Remark 2.2. Note that the cusps induce smooth points on the covering surface, i.e., the pullback of the divisor $\sum_1^n (A_i + A'_i)$ is contracted to smooth points of Y' .

3. A SURFACE WITH CANONICAL MAP OF DEGREE 12

The following result has been shown by Tan [19, Thm 6.2.1], using codes. Here we give an alternative proof.

Lemma 3.1. *Let X' be a double cover of \mathbb{P}^2 ramified over a quartic curve with 3 cusps. Then the 3 cusps of X' are 3-divisible.*

Proof. Let $B \subset \mathbb{P}^2$ be a quartic curve with 3 cusps at points p_1, p_2, p_3 (it is well known that such a curve exists; it is unique up to projective equivalence). Consider the canonical resolution $X \rightarrow X'$. The strict transform of the lines through p_1p_2, p_2p_3 and p_1p_3 is a union of disjoint (-1) -curves $E_i, E'_i \subset X, i = 1, 2, 3$. These curves and the (-2) -curves $A_i, A'_i, i = 1, 2, 3$, which contract to the cusps of X' can be labelled such that the intersection matrix of the curves $A_1, A'_1, A_2, A'_2, A_3, A'_3, E_1$ and E_2 is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

This matrix has determinant zero. Since

$$b_2(X) = 12\chi(\mathcal{O}_X) - K_X^2 + 4q(X) - 2 = 8,$$

these 8 curves are dependent in $\text{Num}(X)$, and this relation has to be expressed in the nullspace of the matrix. Using computer algebra we get that this nullspace has basis

$$(2 \ 1 \ 1 \ -1 \ -1 \ -2 \ 3 \ -3).$$

Thus $2A_1 + A'_1 + A_2 - A'_2 - A_3 - 2A'_3 + 3E_1 - 3E_2 = 0$ in $\text{NS}(X)$ (notice that X has no non-trivial torsion). One has $\text{NS}(X) = \text{Pic}(X)$ for regular surfaces X (Castelnuovo), hence there exists a divisor L such that

$$2A_1 + A'_1 + A_2 + 2A'_2 + 2A_3 + A'_3 \equiv 3L.$$

\square

Let $Q_1, Q_2 \subset \mathbb{P}^2$ be quartic curves with 3 cusps each such that $Q_1 + Q_2$ has 6 cusps and 16 nodes. Let V be the bidouble cover of \mathbb{P}^2 defined by the divisors $Q_1, Q_2, Q_3 := 0$ (for information on bidouble covers see e.g. [5] or [12]). Consider the divisors J_1, J_2, J_3 such that $2J_1 \equiv Q_2 + Q_3, 2J_2 \equiv Q_1 + Q_3, 2J_3 \equiv Q_1 + Q_2$. Notice that since all singularities of V are rational double points, its invariants do not change under the canonical resolution (see e.g. [2, V.22]). So we have

$$p_g(V) = p_g(\mathbb{P}^2) + \sum_1^3 h^0(\mathbb{P}^2, K_{\mathbb{P}^2} + J_i) = 3,$$

$$\chi(\mathcal{O}_V) = 4\chi(\mathcal{O}_{\mathbb{P}^2}) + \frac{1}{2} \sum_1^3 J_i(K_{\mathbb{P}^2} + J_i) = 4$$

and V has 12 ordinary cusps and no other singularities.

Denote by W_1, W_2, W_3 the double covers of \mathbb{P}^2 with branch curves $Q_2, Q_1, Q_1 + Q_2$, respectively (the intermediate surfaces of the bidouble cover). The canonical map of V factors through maps $V \rightarrow W_i, i = 1, 2, 3$, hence it is of degree 4. We get from Lemma 3.1 that the 3 cusps of W_1 and the 3 cusps of W_2 are 3-divisible, therefore the 12 cusps of V are also 3-divisible.

Now let $S \rightarrow V$ be the Galois triple cover ramified over the 12 cusps. We **claim** that $p_g(S) = p_g(V) = 3$. This implies that the canonical map of S factors through the triple cover, thus it is of degree 12. From Proposition 2.1, $q(S) = 0$ and $K_S^2 = 12$.

So it remains to prove the claim. Let \tilde{V} be the smooth minimal resolution of V . The cusps of V correspond to configurations of (-2) -curves $A_i + A'_i \subset \tilde{V}, i = 1, \dots, 12$. These can be labelled such that there exist divisors L, M satisfying $2B + C \equiv 3L, B + 2C \equiv 3M$, where $B := \sum A_i$ and $C := \sum A'_i$.

Below we use the notation $D \geq 0$ for $h^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(D)) > 0$.

From (2.4) we need to show that $K_{\tilde{V}} + L \not\geq 0$ and $K_{\tilde{V}} + M \not\geq 0$. Suppose first that $K_{\tilde{V}} + L \geq 0$. Then

$$(K_{\tilde{V}} + L)A_i = -1, \forall i \implies K_{\tilde{V}} + L - B \geq 0$$

and

$$(K_{\tilde{V}} + L - B)A'_i = -1, \forall i \implies K_{\tilde{V}} + L - B - C \geq 0.$$

From $3K_{\tilde{V}} + 3L - 3B - 3C \geq 0$ and $3(L + M) \equiv 3(B + C)$ one gets $3K_{\tilde{V}} - 3M \geq 0$, i.e., $3K_{\tilde{V}} - B - 2C \geq 0$. This implies the existence of an element in the linear system $|3K_V|$ having multiplicity > 1 at each of the 12 cusps of V .

Let q_1, q_2 be the defining equations of Q_1, Q_2 . The surface V is given by equations $w^2 = q_1, t^2 = q_2$ in the weighted projective space $\mathbb{P}(x^1, y^1, z^1, w^2, t^2)$. It is easy to see that the linear system of polynomials of degree 3 has no element with multiplicity > 1 at the cusps of V (for instance using computer algebra).

The case $K_{\tilde{V}} + M \geq 0$ is analogous.

4. A SURFACE WITH $K^2 = 9\chi$

Based on the work of Prasad and Yeung [15], Cartwright and Steger [4] constructed a fake projective plane F with an automorphism j of order 3 such that F/j is a surface of general type with $\chi = 1, p_g = 0, K^2 = 3$ and fundamental group \mathbb{Z}_{13} . This surface has a set of three 3-divisible cusps (cf. [9]). Denote by B the unit ball in \mathbb{C}^2 and let P, H and G be the groups such that $F = B/P, F/j = B/H$ and

the universal cover of F/j is B/G . We have the following commutative diagram, where the vertical arrows denote unramified \mathbb{Z}_{13} covers and the horizontal arrows denote \mathbb{Z}_3 covers ramified over cusps:

$$\begin{CD} B/(G \cap P) @>p>> B/G \\ @V13:1VV @VV13:1V \\ B/P @>3:1>> B/H. \end{CD}$$

Let $S := B/(G \cap P)$. Proposition 2.1 implies $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{B/G})$. We show that the surface S is regular, hence B/G is also regular and then $p_g(S) = p_g(B/G)$. This implies that the canonical map ϕ of S factors through the triple cover p . Since $G \cap P$ is the fundamental group of S , the commutator quotient $(G \cap P)/[G \cap P, G \cap P]$ is isomorphic to $H_1(S, \mathbb{Z})$. The first Betti number $b_1(S) = 2q(S)$ is the minimal number of generators of $H_1(S, \mathbb{Z})$ modulo elements of finite order. Thus $q(S) = 0$ if $(G \cap P)/[G \cap P, G \cap P]$ is finite. This is shown in the Appendix, where we use computational GAP [8] data from Cartwright and Steger to compute $G \cap P$.

Now notice that ϕ is not composed with a pencil. In fact otherwise the canonical map of B/G is composed with a pencil and then

$$39 = K_{B/G}^2 \geq 4\chi(\mathcal{O}_{B/G}) - 10 = 42,$$

from [21, Theorem A] (see also [10, Corollary 3.4]).

Finally we prove that if $d := \deg(\phi) \neq 3$, then $d = 9$ and the canonical image $\phi(S)$ is a rational surface. As in the proof of Proposition 4.1 of [3], we have

$$9\chi(\mathcal{O}_S) \geq K_S^2 \geq d \deg(\phi(S)) \geq nd(p_g(S) - 2)$$

where $n = 2$ if $\phi(S)$ is not ruled and $n = 1$ otherwise. This gives $d < 6$ if $\phi(S)$ is not ruled and $d < 12$ otherwise. Since $d \equiv 0 \pmod{3}$, then $d = 6$ or 9 and $\phi(S)$ is a rational surface. If $d = 6$, the canonical map of B/G is of degree 2 and then B/G has an involution. But, as seen in the Appendix, the structure of the automorphism group of B/G is the semidirect product $\mathbb{Z}_{13} : \mathbb{Z}_3$, so there is no involution on B/G .

APPENDIX: GAP CODE

```
#We are using data from
#http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/
#C18p3/C18p3-0-FP.gap
#namely the groups 'index9aFP', 'index3aFP' and
#the functions 'FundGp', 'AutGp'.

P:=index9aFP;
H:=index3aFP;
#B/P is a Fake projective plane.
#B/H is a quotient of the fake p.p. B/P by an order 3 automorphism.
#B/H is a surface with p_g=0 and K^2=3 having 3 cusps.

#Using the Cartwright-Steger function 'FundGp', we see that the
#fundamental group of B/H is "C13". We want to find the group G such
#that B/G is the universal cover of B/H. This is also computed by the
#function 'FundGp'.
```

#The following function is equal to 'FundGp' except that outputs G.

```
Grp:=function(G,FOList)
  local e1,e2,e3,GF0,G0,GO,FCA;
  e1:=GeneratorsOfGroup(GammaBarFP);
  e2:=Concatenation(e1,List(e1,Inverse));
  Add(e2,One(G));
  e3:=ListX(e2,e2,\*);
  GF0:=Filtered(FOList,fo->fo in G);
  G0:=Group(ListX(e3,GF0,function(elt,fo) return fo^elt; end));
  Print(ForAll(GF0,fo->fo in G0),"\n");
  Print(IsNormal(G,G0),"\n");
  return G0;
end;
```

```
G:=Grp(H,FOList);
GP:=Intersection(G,P);
```

```
#The "commutative diagram":
StructureDescription(FactorGroup(P,GP)) = "C13";
StructureDescription(FactorGroup(H,G)) = "C13";
StructureDescription(FactorGroup(G,GP)) = "C3";
StructureDescription(FactorGroup(H,P)) = "C3";
```

```
#The first Betti number  $b_1(B/GP) = 0$ :
Index(GP,CommutatorSubgroup(GP,GP)) = 2916; #It is finite.
```

```
#The automorphism group of B/G:
StructureDescription(AutGp(G)) = "C13 : C3";
```

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