PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 12, December 2015, Pages 5437–5453
http://dx.doi.org/10.1090/proc/12666
Article electronically published on April 14, 2015

RATIONAL MODEL OF THE CONFIGURATION SPACE OF TWO POINTS IN A SIMPLY CONNECTED CLOSED MANIFOLD

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(Communicated by Michael A. Mandell)

Abstract. Let M be a simply connected closed manifold of dimension n. We study the rational homotopy type of the configuration space of two points in M, $F(M,2)$. When M is even dimensional, we prove that the rational homotopy type of $F(M,2)$ depends only on the rational homotopy type of M. When the dimension of M is odd, for every $x \in H^{n-2}(M,\mathbb{Q})$, we construct a commutative differential graded algebra $C(x)$. We prove that for some $x \in H^{n-2}(M;\mathbb{Q})$, $C(x)$ encodes completely the rational homotopy type of $F(M,2)$. For some class of manifolds, we show that we can take $x = 0$.

1. Introduction

Let M be a simply connected closed manifold of dimension n. The configuration space of two points in M is the space

$$F(M,2) = \{(x,y) \in M \times M : x \neq y\} = M \times M \setminus \Delta(M)$$

where $\Delta\colon M \hookrightarrow M \times M$ is the diagonal embedding. We have an obvious inclusion $F(M,2) \hookrightarrow M \times M$.

Our goal in this paper is to study the rational homotopy type of $F(M,2)$. Recall that by the theory of Sullivan, the rational homotopy type of a simply connected space is encoded by a commutative differential graded algebra (CDGA for short), which is called a *rational model* of the space. By Poincaré duality of the manifold, M admits a Poincaré duality CDGA model (A,d) (see Section 2). There exists an element called the *diagonal class* $\Delta \in (A \otimes A)^n$ generalizing the classical diagonal class in $H^*(M;\mathbb{Q}) \otimes H^*(M;\mathbb{Q})$.

In [4] it is shown that $A \otimes A/(\Delta)$, where (Δ) is the ideal generated by Δ in $A \otimes A$, is a CDGA model of $F(M,2)$ when M is at least 2-*connected.* This result implies that the rational homotopy type of $F(M,2)$ depends only on the rational homotopy type of M for a 2-connected closed manifold. On the other side, Longoni and Salvatore [7] have constructed an example of two connected (but not simply connected) closed manifolds that are homotopy equivalent but such that their configuration spaces of two points are not, even rationally.

The goal of this paper is to discuss the rational homotopy type of $F(M,2)$ where M is 1-connected. When the dimension of M is *even* we show that (see Theorem 4.1), as in the 2-connected case, the CDGA $A \otimes A/(\Delta)$ is a rational model of

Received by the editors February 20, 2014 and, in revised form, October 27, 2014 and October 28, 2014.
2010 *Mathematics Subject Classification.* Primary 55P62.

$F(M,2)$. Actually we show that $A\otimes A \to A\otimes A/(\Delta)$ is a CDGA model of the inclusion $F(M,2)\hookrightarrow M\times M$.

The *odd* dimension case is more complicated. For any element $x\in H^{n-2}(M;\mathbb{Q})$ we will construct a CDGA $C(x)$ and a map $A\otimes A\to C(x)$. One of the main results of this paper (see Theorem 5.3 and Corollary 7.6) is that for *some* $x\in H^{n-2}(M;\mathbb{Q})$, this map is a rational model of $F(M,2)\hookrightarrow M\times M$. Note that, for $x=0$, $C(0)$ is equivalent to the CDGA $A\otimes A/(\Delta)$; when $x\neq 0$, $C(x)$ is a multiplicatively twisted version of such a quotient (see Definition 5.2).

In Section 6 we introduce the notion of an *untwisted manifold.* We will say that a manifold M is *untwisted* if $C(0)$ is a CDGA model of $F(M,2)$ (see Definition 6.1). In [4] it is shown that all 2-connected closed manifolds are untwisted and in Section 4 we will show that every simply connected closed manifold of even dimension is untwisted. We prove in Section 6 that the product of two untwisted manifolds is untwisted. It is still an open question to know if all manifolds are untwisted.

2. Basic notions

In this section we review the definition of a mapping cone of a module map and we describe a CGA structure on it. We also recall the definition and properties of a Poincaré duality CDGA.

In this paper we will use the usual tools of rational homotopy theory as developped for example in [2].

2.1. **Mapping cones.** Let R be a CDGA and let A be an R-dgmodule. We will denote by s^kA the k-th suspension of A defined by $(s^kA)^n\cong A^{n+k}$ as a vector space and with an R-dgmodule structure defined by $r\cdot(s^ka)=(-1)^{k|r|}s^k(r\cdot a)$ and $d(s^ka)=(-1)^ks^k(da)$ for $a\in A$ and $r\in R$. If $f\colon B\to A$ is an R-dgmodule morphism, *the mapping cone* of f is the R-dgmodule

$$C(f):=(A\oplus_f sB,\delta)$$

defined by $A\oplus sB$ as an R-dgmodule and $\delta(a,sb)=(d_A(a)+f(b),-sd_B(b))$. This R-dgmodule can be equipped with a commutative graded algebra (CGA) structure, that respects the R-dgmodule structure, characterised by the fact that $(sb)\cdot(sb')=0$, for $b,b'\in B$. Precisely, we have the multiplication

$$\mu: C(f)\otimes C(f)\to C(f)\ ,\ c_1\otimes c_2\mapsto c_1\cdot c_2,$$

such that, for homogeneous elements $a,a'\in A$ and $b,b'\in B$.

(*i*) $\mu(a\otimes a')=a\cdot a'$,

(*ii*) $\mu(a\otimes sb')=(-1)^{|a|}s(a\cdot b')$,

(*iii*) $\mu(sb\otimes a')=(-1)^{|b||a'|}s(a'\cdot b)$,

(*iv*) $\mu(sb\otimes sb')=0.$

We will call this structure the *semi-trivial structure* on the mapping cone.

2.2. **Poincaré duality CDGA.** A *Poincaré duality* CDGA of formal dimension n is a triple (A,d,ϵ) such that

- (A,d) is a CDGA;
- $\epsilon\colon A^n\to\mathbb{Q}$ is a linear map such that $\epsilon(dA^{n-1})=0$ (one can think of ϵ as an orientation of the Poincaré duality CDGA A);

- for each $k \in \mathbb{Z}$
$$\begin{array}{ccc} A^k \otimes A^{n-k} & \to & \mathbb{Q} \\ a \otimes b & \mapsto & \epsilon(a \cdot b) \end{array}$$
is non-degenerate, i.e. if $a \in A^k$ and $a \neq 0$, then there exists $b \in A^{n-k}$ such that $\epsilon(a \cdot b) \neq 0$.

Let $\{a_i\}_{i=1}^N$ be a homogeneous basis of A. There exists a *Poincaré dual* basis $\{a_j^*\}_{j=1}^N$ characterised by the fact that $\epsilon(a_i \cdot a_j^*) = \delta_{ij}$.

One of the main results concerning this algebra is the following one.

Theorem 2.1 ([5, Theorem 1.1])**.** *Let (A, d) be a CDGA such that $H^*(A, d)$ is a simply connected Poincaré duality algebra of formal dimension n. Then there exists (A', d') a Poincaré duality CDGA of formal dimension n weakly equivalent to (A, d).*

As a direct consequence of this result we have that every simply connected closed manifold admits a Poincare duality CDGA model.

3. A dgmodule model of $F(M, 2)$

The following result is an evident reformulation of the results of [6, Theorem 10.1] and [4].

Let (A, d, ϵ) be a Poincaré duality CDGA of formal dimension n. Let $\{a_i\}_{i=1}^N$ be a homogeneous basis of A and denote by $\{a_i^*\}_{i=1}^N$ its Poincaré dual basis. Denote by
$$\Delta := \sum_{i=1}^N (-1)^{|a_i|} a_i \otimes a_i^* \in (A \otimes A)^n$$
the diagonal class in $A \otimes A$. There is an obvious $A \otimes A$-module structure on $s^{-n}A$ defined, for homogeneous elements $a, x, y \in A$, by
$$(x \otimes y) \cdot s^{-n}a = (-1)^{(n|x|+n|y|+|a||y|)} s^{-n} x \cdot a \cdot y.$$

The map
$$\Delta^! \colon s^{-n}A \to A \otimes A$$
defined by $\Delta^!(s^{-n}a) = \Delta \cdot (1 \otimes a)$ is an $A \otimes A$-dgmodule map (see [4, Proposition 5.1]). Therefore the mapping cone $C(\Delta^!) = A \otimes A \oplus_{\Delta^!} ss^{-n}A$ is an $A \otimes A$-dgmodule.

As a direct consequence of [6, Theorem 10.1] we obtain the following result.

Proposition 3.1. *Let (A, d, ϵ) be a Poincaré duality model of M of formal dimension n. Let $\hat{A}$ be a CDGA such that we have*
$$(A, d) \xleftarrow{\simeq} \hat{A} \xrightarrow{\simeq} A_{PL}(M)$$
a zig-zag of CDGA quasi-isomorphisms. The CGDAs $A \otimes A$, $A_{PL}(M \times M)$ and $A_{PL}(F(M, 2))$ inherit an $\hat{A} \otimes \hat{A}$-dgmodule structure. Denote by Δ the diagonal class in $A \otimes A$ and let $\Delta^! \colon s^{-n}A \to A \otimes A$ be the morphism defined by $\Delta^!(s^{-n}a) = \Delta(1 \otimes a)$ for all $a \in A$. Then the mapping cone
$$C(\Delta^!) = A \otimes A \oplus_{\Delta^!} ss^{-n}A$$
is an $\hat{A} \otimes \hat{A}$-dgmodule weakly equivalent to $A_{PL}(F(M, 2))$.

We will often use the following result.

Lemma 3.2 ([4, Lemma 5.2]). *Let (A, d, ϵ) be a Poincaré duality CDGA. The mapping cone $C(\Delta^!) = A \otimes A \oplus_{\Delta^!} ss^{-n}A$ equipped with the semi-trivial structure is a CDGA .*

This suggests that $C(\Delta^!)$ equipped with the semi-trivial structure (see Section 2.1) is a natural candidate to be a CDGA model of $F(M, 2)$. The main result of [4] proves that it is when the manifold is 2-connected. In the following section we show that it also is when the manifold is 1-connected and of even dimension.

4. Rational model of $F(M, 2)$ for a simply connected manifold of even dimension

Let M be a simply connected closed manifold of even dimension n. We will prove the following result:

Theorem 4.1. *Let M be a simply connected closed manifold of even dimension n. Let A be a 1-connected Poincaré duality model of M of formal dimension n and denote by Δ the diagonal class in $A \otimes A$. Let $\Delta^! \colon s^{-n}A \to A \otimes A$ be the morphism such that $\Delta^!(s^{-n}a) = \Delta(1 \otimes a)$ for $a \in A$. The mapping cone of $\Delta^!$*

$$C(\Delta^!) = A \otimes A \oplus_{\Delta^!} ss^{-n}A$$

equipped with the semi-trivial structure is a CDGA and the map

$$\mathrm{id} \oplus 0 \colon A \otimes A \to C(\Delta^!)$$

defined by $\mathrm{id} \oplus 0(a \otimes b) = (a \otimes b, 0)$ is a CDGA model of the inclusion $F(M, 2) \hookrightarrow M \times M$.

A direct corollary of the theorem is the rational homotopy invariance of $F(M, 2)$. Specifically:

Corollary 4.2. *Let M be a simply connected closed manifold of even dimension. The rational homotopy type of $F(M, 2)$ is completely determined by the rational homotopy type of M.*

Proof. Let $n = \dim M$. If $n = 2$, then $M \cong S^2$ and the result is obvious. If $n \geq 4$, a transversality argument shows that, if M is simply connected, the configuration space $F(M, 2)$ is simply connected. Hence the result is a direct consequence of Theorem 4.1. $\square$

The rest of this section is devoted to the proof of Theorem 4.1. For this we use the following obvious result.

Lemma 4.3. *Let $\rho \colon R \to Q$ be a CDGA morphism and $R \otimes \Lambda Z$ be a relative Sullivan algebra. Let $\varphi : R \otimes \Lambda Z \to Q$ be an R-dgmodules morphism such that $\varphi|_R = \rho$. The morphism φ is a CDGA morphism if for all $z, z' \in \Lambda^+ Z$, $\varphi(z \cdot z') = \varphi(z) \cdot \varphi(z')$.*

Let us prove Theorem 4.1.

Proof of Theorem 4.1. Let A be a 1-connected (i.e. $A^0 = \mathbb{Q}$, $A^1 = 0$) Poincaré duality model of M of formal dimension n. Take $\hat{A}$ a CDGA such that we have a zig-zag

$$(A, d) \xleftarrow{\simeq} \hat{A} \xrightarrow{\simeq} A_{PL}(M)$$

of CDGA quasi-isomorphisms.

The CDGAs A, $A\otimes A$, $A_{PL}(M\times M)$ and $A_{PL}(F(M,2))$ inherit an $\hat{A}\otimes\hat{A}$-dgmodule structure. Taking a relative Sullivan model of

$$f\colon \hat{A}\otimes\hat{A}\to A_{PL}(M\times M)\to A_{PL}(F(M,2)),$$

we get the following commutative diagram:

$$\begin{tikzcd}
\hat{A}\otimes\hat{A} \arrow[r] \arrow[dr] & A_{PL}(M\times M) \arrow[r] & A_{PL}(F(M,2)) \\
 & (\hat{A}\otimes\hat{A}\otimes\Lambda Z, D). \arrow[ur, "\simeq", "m"'] &
\end{tikzcd}$$

Since $H^{<n}(M\times M,F(M,2))=0$ and $H^n(M\times M,F(M,2);\mathbb{Q})\cong\mathbb{Q}$ the vector space Z can be taken such that $Z^{<n-1}=0$ and $Z^{n-1}\cong\mathbb{Q}$. Let u denote a generator of Z^{n-1}. By Proposition 3.1, $A_{PL}(F(M,2))$ and $C(\Delta^!)$ are weakly equivalent as $\hat{A}\otimes\hat{A}$-dgmodules. Since $(\hat{A}\otimes\hat{A}\otimes\Lambda Z,D)$ is a cofibrant $\hat{A}\otimes\hat{A}$-dgmodule we have a direct quasi-isomorphism of $\hat{A}\otimes\hat{A}$-dgmodules

$$\theta\colon (\hat{A}\otimes\hat{A}\otimes\Lambda Z,D)\xrightarrow{\simeq} C(\Delta^!).$$

Denote by $\omega\in A^n$ a generator of the vector space $A^n\cong\mathbb{Q}$. The differential ideal $I=\langle\omega\otimes\omega, ss^{-n}\omega\rangle\subset C(\Delta^!)$ is acyclic and the projection $\pi\colon C(\Delta^!)\to C(\Delta^!)/I$ is a CDGA quasi-isomorphism. Therefore the composition

$$\varphi:=\pi\circ\theta\colon (\hat{A}\otimes\hat{A}\otimes\Lambda Z,D)\to C(\Delta^!)/I$$

is a quasi-isomorphism of $\hat{A}\otimes\hat{A}$-dgmodules. To show that it is a quasi-isomorphism of CDGA it suffices to show, by Lemma 4.3, that for all $z,z'\in\Lambda^+Z$:

$$\varphi(z\cdot z')=\varphi(z)\varphi(z'). \tag{4.1}$$

Recall that $Z^{<n-1}=0$,

- If $|z|>n-1$ or $|z'|>n-1$, then the degree of expression (4.1) is $\geq 2n-1$, hence both terms are zero since $(C(\Delta^!)/I)^{\geq 2n-1}=0$.
- If $|z|=n-1$ and $|z'|=n-1$, by linearity and since $Z^{n-1}=\mathbb{Q}\cdot u$ it suffices to look at the case $z=z'=u$. Since $n-1$ is odd, by graded commutativity, $u^2=0$ and $\varphi(u^2)=\varphi(0)=\varphi(u)\cdot\varphi(u)=0$.

So, we have the following zig-zag of CDGA quasi-isomorphisms:

$$A_{PL}(F(M,2))\xleftarrow[\simeq]{m}(\hat{A}\otimes\hat{A}\otimes\Lambda Z,D)\xrightarrow[\simeq]{\pi\circ\theta}C(\Delta^!)/I\xleftarrow[\simeq]{\pi}C(\Delta^!),$$

which proves that $C(\Delta^!)$ is a rational model of $F(M,2)$.

Multiplying by a non-zero rational number we can suppose that $\theta(1)=1$ and since θ is an $\hat{A}\otimes\hat{A}$-dgmodules map we have that for $a,b\in\hat{A}$

$$\theta(a\otimes b)=(a\otimes b)\cdot\theta(1)=\rho(a\otimes b).1=((\mathrm{id}\oplus 0)\circ\rho)\,(a\otimes b).$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
\hat{A}\otimes\hat{A} \arrow[r, "\rho"] \arrow[d] & A\otimes A \arrow[d, "\mathrm{id}\oplus 0"] \\
(\hat{A}\otimes\hat{A}\otimes\Lambda Z,D) \arrow[r, "\theta"] & C(\Delta^!),
\end{tikzcd}$$

and proves that the morphism $\mathrm{id} \oplus 0 \colon A \otimes A \to C(\Delta^!)$ is a CDGA model morphism of $A_{PL}(M \times M) \to A_{PL}(F(M,2))$. □

In [4, Lemma 5.4 and Lemma 5.5] it is shown that, if (Δ) denotes the ideal generated by $\Delta \in A \otimes A$, $\frac{A \otimes A}{(\Delta)}$ is a CDGA quasi-isomorphic to the CDGA $C(\Delta^!)$ endowed with the semi-trivial structure.

Corollary 4.4. *Let M be a 1-connected closed manifold of even dimension. Let A be a 1-connected Poincaré duality CDGA model of M. Then*

$$A \otimes A \to \frac{A \otimes A}{(\Delta)}$$

is a CDGA model of $F(M,2) \hookrightarrow M \times M$.

5. Rational model of $F(M,2)$ for a simply connected manifold of odd dimension

Let M be a closed, simply connected manifold of odd dimension n. Let A be a Poincaré duality model of M. The construction of a rational model for $F(M,2)$ in this case is more complicated. To construct such a model we will introduce dgmodules weakly equivalent to $C(\Delta^!)$ that admit multiplicative structures other than the semi-trivial structure used above. Explicitly, for every $\xi \in (A \otimes A)^{2n-2}$ we will construct a CDGA $C(\xi)$ where the multiplicative structure depends on ξ (see Definition 5.2). We will show that one of these CDGAs $C(\xi)$ is the rational model of the configuration space $F(M,2)$. Later on we will discuss the fact that the choice of such a multiplicative structure is determined by the choice of an element in $H^{n-2}(M;\mathbb{Q})$.

5.1. **The algebra $C(\xi)$.** Let A be a 1-connected Poincaré duality CDGA of formal dimension n odd. Denote by Δ the diagonal class in $A \otimes A$. Recall that the morphism

$$\Delta^! \colon s^{-n}A \to A \otimes A \; ; \; s^{-n}a \mapsto \Delta(1 \otimes a)$$

is an $A \otimes A$-dgmodule morphism. Hence, the mapping cone

$$C(\Delta^!) = A \otimes A \oplus_{\Delta^!} ss^{-n}A$$

is an $A \otimes A$-dgmodule.

We will show subsequently that a quotient of the mapping cone $C(\Delta^!)$ by an acyclic $A \otimes A$-subdgmodule admits other CDGA structures than the semi-trivial one. It is easy to see that the $A \otimes A$-subdgmodule $(C(\Delta^!))^{\geq 2n-1} = \langle \omega \otimes \omega, ss^{-n}\omega \rangle \subset C(\Delta^!)$ is acyclic. Hence, the quotient

$$C := C(\Delta^!)/C(\Delta^!)^{\geq 2n-1}$$

is an $A \otimes A$-dgmodule and the canonical projection $\pi : C(\Delta^!) \to C$ is a quasi-isomorphism of $A \otimes A$-dgmodules.

Let us define a multiplication over C such that it becomes a CDGA. We want this multiplication to be compatible with the $A \otimes A$-dgmodule structure, namely that the composition

$$A \otimes A \hookrightarrow C(\Delta^!) \xrightarrow{\pi} C$$

is a CDGA morphism.

Since C is the quotient of $A\otimes A\oplus_{\Delta^!} ss^{-n}A$ by the vector space $\langle \omega\otimes\omega, ss^{-n}\omega\rangle$, we have an obvious isomorphism

$$C\cong (A\otimes A)^{<2n}\oplus_{\Delta^!} ss^{-n}(A^{<n}).$$

To define a multiplication over C :

$$\mu : C\otimes C\to C\ ,\ c_1\otimes c_2\mapsto c_1\cdot c_2,$$

it suffices to define the products

(i) $(a\otimes b)\cdot(a'\otimes b')$,

(ii) $(a\otimes b)\cdot ss^{-n}x'$ and $ss^{-n}x\cdot(a'\otimes b')$,

(iii) $(ss^{-n}x)\cdot(ss^{-n}x')$,

for $a\otimes b$, $a'\otimes b'\in (A\otimes A)^{<2n}$ and $x,x'\in A^{<n}$ homogeneous elements.

The products (i) and (ii) are completely determined by the compatibility with the $A\otimes A$-module structure and the graded commutativity. Specifically:

(i) $(a\otimes b)\cdot(a'\otimes b')=(-1)^{|a'||b|}a\cdot a'\otimes b\cdot b'$,

(ii) $(a\otimes b)\cdot ss^{-n}x'=(-1)^{(n-1)(|a|+|b|)+|b||x|}ss^{-n}a\cdot x'\cdot b$
and $(ss^{-n}x)(a'\otimes b')=(-1)^{|a'||x|}ss^{-n}a'\cdot x\cdot b'$.

For (iii), note that, if $|x|+|x'|\geq 1$, then for degree reasons

(iii) $(ss^{-n}x)\cdot(ss^{-n}x')=0$.

Hence, the only still undefined product is

(iv) $(ss^{-n}1)\cdot(ss^{-n}1)\in C^{2n-2}=(A\otimes A)^{2n-2}$.

The following result shows that if n is odd, the product $ss^{-n}1\cdot ss^{-n}1$ can be arbitrarily chosen in $(A\otimes A)^{2n-2}$.

Proposition 5.1. *Let $\xi\in(A\otimes A)^{2n-2}$. If n is odd there exists a CDGA structure on C compatible with the $A\otimes A$-dgmodule structure such that*

$$ss^{-n}1\cdot ss^{-n}1=\xi.$$

Proof. Endow C with the multiplicative structure described above setting that $(ss^{-n}1)^2=\xi$. Let us show that this multiplication endows C with a CDGA structure.

- Associativity: For $x,y,z\in A$, for degree reasons

$$(ss^{-n}x\cdot ss^{-n}y)\cdot ss^{-n}z=0=ss^{-n}x\cdot(ss^{-n}y\cdot ss^{-n}z).$$

  For $x,y\in A$ and $a\otimes b\in A\otimes A$ such that $|a\otimes b|>0$, then $|(a\otimes b)|+|ss^{-n}x|+|ss^{-n}y|\geq 2n-1$. Therefore, for degree reasons

$$(a\otimes b)\cdot(ss^{-n}x\cdot ss^{-n}y)=(a\otimes b\cdot ss^{-n}x)\cdot ss^{-n}y=0.$$

  In the other cases the associativity is granted by the $A\otimes A$-dgmodule structure of C.
- Commutativity: The multiplicative structure is defined in such a way that the multiplication is commutative.

- The Leibniz rule for the differential: Denote by $\bar{\delta}$ the differential in C. Let us show that for c, c' homogeneous elements in C,

$$\bar{\delta}(c \cdot c') - (\bar{\delta}(c) \cdot c') - (-1)^{|c|}(c \cdot \bar{\delta}(c')) = 0. \tag{5.1}$$

  - If $c, c' \in A \otimes A$, then (5.1) is true since $A \otimes A$ is a CDGA and $\bar{\delta}|_{A\otimes A} = d|_{A\otimes A}$.
  - If $c \in A \otimes A$ and $c' \in ss^{-n}A$, the left term of the equation (5.1) is zero since $\bar{\delta}$ is a differential of $A \otimes A$-dgmodules.
  - If $c \in ss^{-n}A$ and $c' \in A \otimes A$, (5.1) is established by the commutativity of the multiplication.
  - If $c, c' \in ss^{-n}A$, then the degree of the expression is at least $2n-2+1 = 2n-1$. Therefore, for degree reasons, the left term of the equation is zero. □

Definition 5.2. Let $\xi \in (A \otimes A)^{2n-2}$. The CDGA $C(\xi)$ is the $A \otimes A$-dgmodule $C(\Delta^!)/C(\Delta^!)^{\geq 2n-1}$ equipped with the only structure of an algebra compatible with the $A \otimes A$-dgmodule structure and such that $ss^{-n}1 \cdot ss^{-n}1 = \xi$. We will denote by

$$\overline{\mathrm{id}} \oplus 0\colon A \otimes A \to C(\xi)$$

the composition of the inclusion $\mathrm{id} \oplus 0 : A \otimes A \to A \otimes A \oplus_{\Delta^!} ss^{-n}A$ and the canonical projection $C(\Delta^!) \to C(\xi)$.

5.2. **CDGA model of $F(M,2)$ for a simply connected manifold of odd dimension.** Let M be a simply connected closed manifold of odd dimension n. Let A be a 1-connected Poincaré duality model of M. In the previous sections we have showed that for all $\xi \in (A \otimes A)^{2n-2}$ we can construct a CDGA $C(\xi)$ characterised by $(ss^{-n}1)^2 = \xi$. The objective of this section is to show that there exists $\xi \in (A \otimes A)^{2n-2}$ such that $C(\xi)$ is a rational model of $F(M,2)$. Specifically we will prove the following:

Theorem 5.3. *Let M be a simply connected closed manifold of odd dimension n and let A be a 1-connected Poincaré duality model of M of formal dimension n. There exists $\xi \in (A \otimes A)^{2n-2}$, such that the map*

$$\overline{\mathrm{id}} \oplus 0\colon A \otimes A \to C(\xi)$$

is a CDGA model of the inclusion $F(M,2) \hookrightarrow M \times M$.

Remark 5.4. The model $A \otimes A$ of $M \times M$ that is mentioned in the theorem is the one naturally induced by the CDGA model A of M, i.e., the zig-zag of CDGA quasi-isomorphism linking $A \otimes A$ and $A_{PL}(M \times M)$ is naturally induced by the zig-zag of quasi-isomorphisms between A and $A_{PL}(M)$. We will use this fact in Section 7 to introduce Definition 7.1.

Remark 5.5. We will see in Corollay 7.6 that actually the model $C(\xi)$ is determined by a cohomology class $x \in H^{n-2}(M;\mathbb{Q})$.

Proof of Theorem 5.3. Take $\hat{A}$ a CDGA such that we have

$$A \xleftarrow{\simeq} \hat{A} \xrightarrow{\simeq} A_{PL}(M)$$

a zig-zag of surjective CDGA quasi-isomorphisms. This can be done easily using standard techniques of model categories. The CDGAs A, $A \otimes A$, $A_{PL}(M \times M)$ and $A_{PL}(F(M,2))$ inherit an $\hat{A} \otimes \hat{A}$-dgmodule structure.

Taking a relative Sullivan model of the map $f\colon \hat{A}\otimes\hat{A} \xrightarrow{\simeq} A_{PL}(M\times M) \to A_{PL}(F(M,2))$ we obtain the following commutative diagram:

$$\begin{tikzcd} \hat{A}\otimes\hat{A} \arrow[r, "\simeq"] \arrow[dr] & A_{PL}(M\times M) \arrow[r] & A_{PL}(F(M,2)) \\ & (\hat{A}\otimes\hat{A}\otimes\Lambda Z, D). \arrow[ur, "\simeq"', "m"] & \end{tikzcd}$$

We can take Z such that $Z^{<n-1}=0$ and $Z^{n-1}=u\cdot\mathbb{Q}$. By construction, $Du=\tilde{\Delta}$ where $\tilde{\Delta}\in\hat{A}\otimes\hat{A}$ is such that $[\tilde{\Delta}]$ is a generator of $\ker H^*(f)$. The surjectivity of $\rho\colon \hat{A}\otimes\hat{A} \xrightarrow{\simeq} A\otimes A$ allows us to take $\tilde{\Delta}$ such that $\rho(\tilde{\Delta})=\Delta$. Indeed, let $\hat{\Delta}\in A\otimes A\cap\ker d$ be a generator of $\ker H^*(f)$. Since $\hat{\Delta}$ represents the diagonal class in $H^*(\hat{A}\otimes\hat{A})\cong H^*(M\times M)$,

$$H^*(\rho)([\hat{\Delta}])=[\Delta]\in H^*(A\otimes A).$$

Hence, $\rho(\hat{\Delta})=\Delta+d\epsilon$, where $\epsilon\in(A\otimes A)^{n-1}$. Since ρ is surjective, there exists $\gamma\in\hat{A}\otimes\hat{A}$ such that $\rho(\gamma)=\epsilon$. Set $\tilde{\Delta}=(\hat{\Delta}-d\gamma)$. We have that $d\tilde{\Delta}=0$ and $\rho(\tilde{\Delta})=\Delta+d\epsilon-d\epsilon=\Delta$.

By Proposition 3.1, $C(\Delta^!)=A\otimes A\oplus_{\Delta^!} ss^{-n}A$ and $A_{PL}(F(M,2))$ are weakly equivalent as $\hat{A}\otimes\hat{A}$-dgmodules. As $(\hat{A}\otimes\hat{A}\otimes\Lambda Z,D)$ is a cofibrant $\hat{A}\otimes\hat{A}$-dgmodule we have an $\hat{A}\otimes\hat{A}$-dgmodules quasi-isomorphism

$$\theta\colon (\hat{A}\otimes\hat{A}\otimes\Lambda Z,D) \xrightarrow{\simeq} C(\Delta^!).$$

Multiplying θ by a non-zero rational number we can assume that $\theta(1)=1$ and as in the end of the proof of Theorem 4.1 we show that the following diagram commutes:

(5.2)
$$\begin{tikzcd} \hat{A}\otimes\hat{A} \arrow[r, "\rho"] \arrow[d] & A\otimes A \arrow[d, "\mathrm{id}\oplus 0"] \\ (\hat{A}\otimes\hat{A}\otimes\Lambda Z,D) \arrow[r, "\theta"] & C(\Delta^!). \end{tikzcd}$$

Let us show that $\theta(u)=ss^{-n}1+\tau$ with $\tau\in(A\otimes A)^{n-1}\cap\ker d$. Indeed, we remark that for degree reasons

$$\theta(u)=qss^{-n}1+\tau$$

with $q\in\mathbb{Q}$ and $\tau\in(A\otimes A)^{n-1}$. Let us denote by δ the differential in $C(\Delta^!)$. Since θ commutes with the differentials we have

$$\begin{aligned} \delta(\theta(u)) &= \theta(Du), \\ \delta(qss^{-n}1+\tau) &= \theta(\tilde{\Delta}), \\ q\Delta+d\tau &= \rho(\tilde{\Delta})=\Delta. \end{aligned}$$

Hence,

$$d\tau=(1-q)\Delta. \tag{5.3}$$

Since $[\Delta]\neq 0$ in $H^*(A\otimes A;\mathbb{Q})$, the equation (5.3) is satisfied if and only if $q=1$ and thus $\theta(u)=ss^{-n}1+\tau$ with $d\tau=0$.

Set $\xi = \theta(u^2) - \tau^2$ so $|\xi| = 2n-2$. Since A is 1-connected $(ss^{-n}A)^{2n-2} \cong A^{n-1} = 0$, thus $\xi \in C(\Delta^!)$ can be identified with an element of $(A \otimes A)^{2n-2}$. Take $C(\xi)$ the CDGA of Definition 5.2. The composition

$$\varphi\colon (\hat{A} \otimes \hat{A} \otimes \Lambda Z, D) \xrightarrow{\theta} C(\Delta^!) \xrightarrow{\pi} C(\xi)$$

is an $\hat{A} \otimes \hat{A}$-dgmodules quasi-isomorphism. Let us show that φ is an algebra morphism. By Lemma 4.3, it suffices to show that for z, $z' \in \Lambda^+ Z$

$$\varphi(z \cdot z') = \varphi(z)\varphi(z'). \tag{5.4}$$

We study the following cases:

- If $|z| > n-1$ or $|z'| > n-1$, then the degree of the expression is $\geq 2n-1$, and so both terms of the equation are zero.
- If $|z| = n-1$ and $|z'| = n-1$, by linearity we can suppose that $z = z' = u$. Note that, since A is a 1-connected Poincaré duality CDGA, $\tau \cdot ss^{-n}1 \in ss^{-n}(A^{n-1}) = 0$. We compute that

  $$(\varphi(u))^2 = (ss^{-n}1 + \tau)^2 = (ss^{-n}1)^2 + \tau^2 = \xi + \tau^2 = \pi(\theta(u^2))$$

  and

  $$\varphi(u^2) = \pi(\theta(u^2)).$$

This proves that φ is an algebra morphism.

Since φ is an $\hat{A} \otimes \hat{A}$-dgmodules quasi-isomorphism and is a CDGA map, it is a CDGA quasi-isomorphism. The following zig-zag of CDGA quasi-isomorphisms:

$$A_{PL}(F(M,2)) \xleftarrow[m]{\simeq} (\hat{A} \otimes \hat{A} \otimes \Lambda Z, D) \xrightarrow[\varphi]{\simeq} C(\xi),$$

proves that $C(\xi)$ is a CDGA model of $F(M,2)$. Since the diagram (5.2) commutes, $A \otimes A \to C(\xi)$ is a CDGA model morphism of $A_{PL}(M \times M) \to A_{PL}(F(M,2))$. □

6. Untwisted manifolds

It is well known that for M a simply connected closed manifold

$$H^*(F(M,2)) \cong \frac{H^*(M) \otimes H^*(M)}{([\Delta])},$$

where $[\Delta]$ is the diagonal class in $H^*(M) \otimes H^*(M)$. It is easy to see that for $\xi = 0$ the CDGA $C(\xi)$ (see Definition 5.2) is quasi-isomorphic to $\frac{A \otimes A}{(\Delta)}$ as a CDGA. This suggests "naively" that $C(0)$ is the good candidate to be the rational model of $F(M,2)$ and therefore it suggests the following definition:

Definition 6.1. A closed manifold M is *untwisted* if $A \otimes A \to A \otimes A/(\Delta)$ is a CDGA model of $F(M,2) \hookrightarrow M \times M$, where A is a 1-connected Poincaré duality CDGA model of M.

In [4] it is shown that all 2-connected closed manifolds are untwisted and in Section 4 we showed that every simply connected closed manifold of even dimension is untwisted. We also have the following result:

Proposition 6.2. *The product of two untwisted manifolds is untwisted.*

Proof. Notice that if X and Y are topological spaces, $F(X \times Y, 2)$ can be viewed as the pushout of the diagram

$$X^2 \times F(Y,2) \hookleftarrow F(X,2) \times F(Y,2) \hookrightarrow F(X,2) \times Y^2 . \tag{6.1}$$

Replacing $\Delta(X) \subset X \times X$ by an open tubular neighborhood in $F(X,2)$ and similarly replacing $\Delta(Y) \subset Y \times Y$ by an open tubular neighborhood in $F(Y,2)$ we can look at pushout (6.1) as a homotopy pushout.

Suppose that X and Y are untwisted manifolds, let C be a Poincaré duality CDGA model of X and B a Poincaré duality CDGA model of Y. By definition of untwisted manifolds $\frac{C \otimes C}{(\Delta_C)}$ is a CDGA model of $F(X,2)$ and $\frac{B \otimes B}{(\Delta_B)}$ is a CDGA model of $F(Y,2)$. Therefore a CDGA model of diagram (6.1) is given by

$$B \otimes B \otimes \tfrac{C \otimes C}{(\Delta_C)} \twoheadrightarrow \tfrac{B \otimes B}{(\Delta_B)} \otimes \tfrac{C \otimes C}{(\Delta_C)} \twoheadleftarrow \tfrac{B \otimes B}{(\Delta_B)} \otimes C \otimes C. \tag{6.2}$$

As both maps are surjective the homotopy pullback of diagram (6.2) is the actual pullback of the diagram and hence this pullback is a CDGA model of $F(X \times Y, 2)$.

Let us denote by PB this pullback. The universal property guarantees the existence of a unique map

$$\alpha \colon C \otimes C \otimes B \otimes B \to PB.$$

It is easy to show that α is surjective and that $\ker \alpha = (\Delta_C \otimes \Delta_B)$. Thus,

$$PB \cong C \otimes C \otimes B \otimes B/(\Delta_C \otimes \Delta_B).$$

The CDGA $A = C \otimes B$ is a CDGA model of $X \times Y$ and denote by Δ_A the diagonal class in A. There is a natural isomorphim $C \otimes C \otimes B \otimes B \cong A \otimes A$ which, by an adequate choice of basis, sends $\Delta_C \otimes \Delta_B$ to Δ_A. So $A \otimes A/(\Delta_A)$ is a CDGA model of $F(X \times Y, 2)$. ☐

As a consequence we have that, if N is a 2-connected odd dimensional closed manifold, then $N \times \underbrace{S^2 \times \cdots \times S^2}_{k \text{ times}}$ is untwisted. We conjecture the following:

Conjecture 6.3. *Every simply connected closed manifold is untwisted.*

7. Models $A \otimes A \to C(\xi)$ depend only on a class $x \in H^{n-2}(M; \mathbb{Q})$

A natural question is whether two different $\xi, \xi' \in (A \otimes A)^{2n-2}$ induce isomorphic CDGAs $C(\xi) \cong C(\xi')$ (at the end of this section we show that this is not always the case). More precisely, in view of Theorem 9 where $C(\xi)$ is seen as a CDGA under $A \otimes A$, the right question is whether $C(\xi)$ and $C(\xi')$ are equivalent in the following sense:

Definition 7.1. Let A be a Poincaré duality CDGA of formal dimension n odd and $\xi, \xi' \in (A \otimes A)^{2n-2}$. We say that the CDGAs $C(\xi)$ and $C(\xi')$ are *weakly equivalent* *under* $A \otimes A$ and we write $C(\xi) \simeq_{A\otimes A} C(\xi')$, if there exists a commutative CDGA diagram

$$\begin{tikzcd} & C(\xi) \\ A\otimes A \arrow[ur, "\bar{\mathrm{id}}\oplus 0"] \arrow[r, tail] \arrow[dr, "\bar{\mathrm{id}}\oplus 0"'] & (A\otimes A\otimes \Lambda Z, D) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ & C(\xi'), \end{tikzcd}$$

where $(A\otimes A\otimes \Lambda Z, D)$ is a relative Sullivan algebra.

We have the following result:

Proposition 7.2. *Let A be a 1-connected Poincaré duality CDGA of formal dimension n odd, and Δ be the diagonal class in $A\otimes A$. If $\xi, \xi' \in (A\otimes A)^{2n-2}$ are such that $[\xi] = [\xi']$ in $\frac{H^{2n-2}(A\otimes A)}{([\Delta])}$, then $C(\xi) \simeq_{A\otimes A} C(\xi')$.*

Proof. We have a canonical $A\otimes A$-dgmodule isomorphism $C(\xi) \cong C(\xi')$. Abusing notation, we can look at this as an identification since these two CDGA differ simply by the value of the square $(ss^{-n}1)^2$.

As $A\otimes A$-dgmodules we have

$$C(\xi) = ((A\otimes A)^{<2n-1} \oplus ss^{-n}(A^{<n-1}), \delta)$$

with $\delta(ss^{-n}a) = \delta(a\cdot ss^{-n}1) = da\cdot ss^{-n}1 + (-1)^{|a|}(a\otimes 1)\cdot \Delta$. Set

$$(\Delta)^{>n} = \{(a\otimes b)\cdot \Delta \mid a\otimes b \in (A\otimes A)^+\}.$$

Since the map $a \mapsto (a\otimes 1)\cdot\Delta$ is injective (see [4, Lemma 5.4]), it is clear that

$$(\Delta)^{>n} \oplus ss^{-n}1\cdot A^+$$

is an acyclic differential ideal in $C(\Delta^!)$ and then, the natural projection of this ideal is also a differential ideal in $C(\xi)$ and $C(\xi')$. Consider S a supplementary of cocycles in $(A\otimes A)^{2n-3}$. The ideal,

$$I = S + d(S) + (\Delta)^{>n} + ss^{-n}\cdot A^+,$$

is an acyclic differential ideal in $C(\Delta^!)$ and so the natural projection of this ideal is also an acyclic ideal in $C(\xi)$ and $C(\xi')$. We have two CDGA quasi-isomorphisms $C(\xi) \xrightarrow{\simeq} C(\xi)/I$ and $C(\xi') \xrightarrow{\simeq} C(\xi')/I$. Since $\xi - \xi' \in (\Delta) + d\left((A\otimes A)^{2n-3}\right)$, $\xi = \xi'$ mod I. Therefore, the identification of $A\otimes A$-dgmodules $C(\xi) = C(\xi')$ induces a CDGA isomorphism $C(\xi)/I \cong C(\xi')/I$. We have then the following CDGA diagram:

$$\begin{tikzcd} & C(\xi) \arrow[r, "\simeq"] & C(\xi)/I \arrow[dd, "\cong"'] \\ A\otimes A \arrow[ur] \arrow[dr] & & \\ & C(\xi') \arrow[r, "\simeq"] & C(\xi')/I. \end{tikzcd}$$

Thereafter, taking a relative Sullivan model of $A \otimes A \to C(\xi)$ and using the lifting lemma we obtain the following CDGA diagram:

$$\begin{tikzcd} A \otimes A \arrow[rr, rightarrowtail] \arrow[d, rightarrowtail] & & C(\xi') \arrow[d, two heads, "\simeq"] \\ (A \otimes A \otimes \Lambda Z, D) \arrow[urr, dashed, "\lambda"] \arrow[r, "\simeq"'] & C(\xi) \arrow[r, "\simeq"'] & C(\xi)/I \cong C(\xi')/I. \end{tikzcd}$$

Hence,

$$C(\xi) \simeq_{A\otimes A} C(\xi').$$

□

As a corollary we have:

Corollary 7.3. *For A a 1-connected Poincaré duality CDGA of formal dimension n odd, we have a natural surjection*

$$\psi\colon H^{2n-2}(A\otimes A)/\left([\Delta]\right) \twoheadrightarrow \{C(\xi)\colon \xi \in (A\otimes A)^{2n-2}\}/\simeq_{A\otimes A},$$

where the codomain of the map ψ is the set of equivalence classes under $A \otimes A$ of the CDGA $C(\xi)$.

We conjecture the following:

Conjecture 7.4. *The map ψ of the previous corollary is a bijection.*

The interest of this conjecture is to link it to the next proposition and then obtain a more intrinsic characterisation of the CDGA $C(\xi)$.

Proposition 7.5. *For A a 1-connected Poincaré duality CDGA of formal dimension n odd, we have a natural linear isomorphism*

$$\Phi\colon H^{n-2}(A) \xrightarrow{\cong} \frac{H^{2n-2}(A\otimes A)}{(\Delta)}\ ;\ [a] \mapsto [a\otimes\omega] \bmod ([\Delta]).$$

Proof. The morphism Φ is well defined since it is induced by the linear map, $A^{n-2} \to (A\otimes\ A)^{2n-2}$ defined by $a \mapsto a\otimes\omega$, that sends coboundaries to coboundaries. To prove the surjectivity of Φ, first of all, since A is a 1-connected CDGA, we suppose, without loss of generality, that $A^0 \cong \mathbb{Q}$, $A^1 = 0$ and $A^2 \cong H^2(A)$. Henceforth, by Poincaré duality we have that $A^{n-2} \cong H^{n-2}(A)$. Then we notice that

$$H^{2n-2}(A\otimes A) \cong (A\otimes A)^{2n-2} = \left(A^{n-2}\otimes \mathbb{Q}\cdot\omega\right) \oplus \left(\mathbb{Q}\cdot\omega\otimes A^{n-2}\right).$$

Thus, it suffices to prove that every element of the form $[\omega\otimes a] \bmod ([\Delta])$, with $a \in A^{n-2}$, is in the image of Φ. A simple computation shows that $(a\otimes 1)\cdot\Delta = a\otimes\omega + \omega\otimes a$, hence $[\omega\otimes a] = \Phi([a]) \bmod ([\Delta])$.

To prove injectivity suppose that $\Phi([a]) = 0$, in other words $a\otimes\omega$ is cohomologous to a multiple of Δ. The multiples of Δ in degree $2n-2$ are of the form $u\otimes\omega+\omega\otimes u$ with $u \in A^{n-2}$. For degree reasons and since $A^{n-1} = 0$ (because A is 1-connected),

$$(A\otimes A)^{2n-3} = (\mathbb{Q}\cdot\omega\otimes A^{n-3}) \oplus (A^{n-3}\otimes\mathbb{Q}\cdot\omega).$$

Therefore, there exists $x, y \in A^{n-3}$ such that

$$a\otimes\omega = u\otimes\omega + \omega\otimes u + dx\otimes\omega + \omega\otimes dy.$$

Hence, $a = u + dx$ and $0 = u + dy$, whence $a = d(x-y)$, namely $[a] = 0$ in $H^{n-2}(A)$. □

Let $x \in H^{n-2}(M;\mathbb{Q})$ and assume that A is a 1-connected Poincaré duality CDGA model of M. We can choose $\xi \in (A\otimes A)^{n-2}$ such that $\Phi(x) = \xi \pmod{\Delta}$. Moreover, Proposition 7.2 and Proposition 7.5 show that the CDGA map

$$A\otimes A \to C(\xi)$$

is independent of the choice of $\xi \in A\otimes A$. We define then the CDGA

$$C(x) := C(\xi).$$

Theorem 5.3 can be reformulated as follows:

Corollary 7.6. *Let M be a simply connected closed manifold of odd dimension n and let A be a 1-connected Poincaré duality model of M of formal dimension n. There exists $x \in H^{n-2}(M;\mathbb{Q})$, such that the map*

$$A\otimes A \to C(x)$$

is a CDGA model of the inclusion $F(M,2) \hookrightarrow M\times M$.

We conclude with an example that illustrates the fact that not all the CDGA $C(\xi)$ are weakly equivalent under $A\otimes A$.

The example $A = H^*(S^2\times S^3;\mathbb{Q})$. Take the CDGA

$$(A,d) = \big(H^*(S^2\times S^3;\mathbb{Q}),0\big) = (\{1, x_2, y_3, xy\}\cdot\mathbb{Q}, 0)\,.$$

It is easily verified that A is a Poincaré duality CDGA and that the diagonal class in $A\otimes A$ is given by $\Delta = 1\otimes xy + x\otimes y - y\otimes x - xy\otimes 1$. Let us define the morphism $\Delta^!\colon s^{-n}A \to A\otimes A\ ;\ s^{-n}a \mapsto \Delta(1\otimes a)$. We will study the mapping cone $(C(\Delta^!),\delta)$ and for the sake of conciseness we will use the notation $S = ss^{-5}$.

We can compute that $\delta(S1) = \Delta$, $\delta(Sx) = x\otimes xy - xy\otimes x$, $\delta(Sy) = -y\otimes xy - xy\otimes y$, $\delta(Sxy) = -xy\otimes xy$.

Let us quotient the cone $C(\Delta^!)$ by the differential acyclic submodule $\langle Sxy, xy\otimes xy\rangle$ and endow $C(\Delta^!)/\langle Sxy, xy\otimes xy\rangle$ with one of the CDGA structures described in Section 5. For this, we have to define the product $S1\cdot S1$ which for degree reasons have to be an element in $(A\otimes A)^8$. For $q,r\in\mathbb{Q}$, set

$$S1\cdot S1 = q(y\otimes xy) + r(xy\otimes y) = \xi.$$

We denote by $C(q,r) = C(\xi)$ the obtained CDGA.

Proposition 7.7. *Set $A = (H^*(S^2\times S^3;\mathbb{Q}),0)$ which is a Poincaré duality CDGA model of $M = S^2\times S^3$ of formal dimesion $n=5$. Let $x, x' \in H^{n-2}(A)\cong\mathbb{Q}$; then*

$$C(x) \simeq_{A\otimes A} C(x') \iff x = x'.$$

To prove Proposition 7.7 let us construct the relative CDGA model of the morphism

$$A\otimes A \xrightarrow{\mathrm{id}\oplus 0} C(\Delta^!) \xrightarrow{\ \pi\ } C(q,r)$$

where $\pi : C(\Delta^!) \to C(q,r)$ is the canonical projection. We will construct a graded vector space Z, a differential D and a quasi-isomorphism $m : (A\otimes A\otimes \Lambda Z_{q,r}, D) \to C(q,r)$, such that the following diagram commutes:

$$\begin{tikzcd}
A\otimes A \arrow[r, "\mathrm{id}\oplus 0"] \arrow[dr] & C(\Delta^!) \arrow[r, "\pi"] & C(q,r) \\
& (A\otimes A\otimes \Lambda Z, D_{q,r}). \arrow[ur, "m"'] &
\end{tikzcd}$$

Constructing Z degree by degree up to degree seven, we obtain the generators described below:

Degree	Z_0	D	m
4	u	$D(u) = \Delta := 1 \otimes xy + x \otimes y - y \otimes x - xy \otimes 1$	$m(u) = s1$
5	z_5	$D(z_5) = u.(1 \otimes x) - u.(x \otimes 1)$	$m(z_5) = 0$
6	z_{61}	$D(z_{61}) = u.(1 \otimes y) - u.(y \otimes 1)$	$m(z_{61}) = 0$
6	z_{62}	$D(z_{62}) = z_5.(1 \otimes x) + z_5.(x \otimes 1)$	$m(z_{62}) = 0$
7	z_{71}	$D(z_{71}) = z_{62}(1 \otimes x) - z_{62}(x \otimes 1)$	$m(z_{71}) = 0$
7	z_{72}	$D(z_{72}) = z_{61}(1 \otimes x) + z_5(y \otimes 1) - z_5(1 \otimes y)$ $-z_{61}(x \otimes 1)$	$m(z_{72}) = 0$
7	h	$D(h) = u^2 - 2(z_{61}(1 \otimes x) + z_{61}(x \otimes 1))$ $+q(y \otimes xy) + r(xy \otimes y)$	$m(h) = 0$

Since for all $C(q,r)$ the relative CDGA model is the same excepting the differential, we will denote by $(A \otimes A \otimes \Lambda Z, D_{q,r})$ the relative CDGA model associated to $C(q,r)$.

The following result is useful to determine when two of these algebras are weakly equivalent under $A \otimes A$. The proof is left to the reader.

Lemma 7.8. *Let A be a Poincaré duality CDGA of formal dimension n odd and $\xi, \xi' \in (A \otimes A)^{2n-2}$. Let*

$$A \otimes A \rightarrowtail (A \otimes A \otimes \Lambda Z_\xi, D_\xi) \quad \text{and} \quad A \otimes A \rightarrowtail (A \otimes A \otimes \Lambda Z_{\xi'}, D_{\xi'})$$

be relative CDGA models of $C(\xi)$ and $C(\xi')$ respectively. We have that $C(\xi) \simeq_{A \otimes A} C(\xi')$ if and only if there exists an isomorphism

$$\psi : (A \otimes A \otimes \Lambda Z_\xi, D_\xi) \xrightarrow{\cong} (A \otimes A \otimes \Lambda Z_{\xi'}, D_{\xi'}),$$

such that $\psi(a \otimes b) = a \otimes b$, for all $a, b \in A$.

Let us prove Proposition 7.7.

Proof of Proposition 7.7. If $x = x' \in H^{n-2}(A) = H^3(A)$ it is clear by Proposition 7.2 and Proposition 7.5 that $C(x) \simeq_{A \otimes A} C(x')$.

For the other implication, first notice that $H^3(A) = \langle [y] \rangle$. By the isomorphism Φ defined in Proposition 7.5, for $q \in \mathbb{Q}$, $q[y] \in H^3(A)$ is sent to $[[qy \otimes xy]]$ in the quotient $H^8(A \otimes A)/(\Delta)$. A representative of the class $[[y \otimes xy]] \in H^8(A \otimes A)/(\Delta)$ is given by $[y \otimes xy]$ in $H^8(A \otimes A)$. Hence, by definition $C(q[y]) = C(qy \otimes xy) = C(q, 0)$.

Let $r, q \in \mathbb{Q}$. If $C(q,0) \simeq_{A \otimes A} C(r,0)$, by Lemma 7.8 we have an isomorphism of CDGA

$$\psi : (A \otimes A \otimes \Lambda Z, D_{q,0}) \xrightarrow{\simeq} (A \otimes A \otimes \Lambda Z, D_{r,0})$$

such that $\psi(a \otimes b \otimes 1) = a \otimes b$.

For the sake of conciseness, when there is no ambiguity, both differentials ($D_{q,0}$ and $D_{r,0}$) will be denoted by D. For degree reasons $\psi(u) = \alpha_1 u + \alpha_2(x \otimes x)$ for some $\alpha_1, \alpha_2 \in \mathbb{Q}$. We then have $D\psi(u) = D(\alpha_1 u + \alpha_2(x \otimes x)) = \alpha_1 \Delta$ and $\psi(Du) = \psi(\Delta) = \Delta$. Since ψ commutes with differentials, we deduce that $\alpha_1 = 1$. Hence, $\psi(u) = u + \alpha_2(x \otimes x)$. Analogously we have that since z_{61} is of degree 6,

$$\psi(z_{61}) = \beta z_{61} + \beta_1 y \otimes y + \beta_2 u(1 \otimes x) + \beta_3 u(x \otimes 1) + \beta_4 z_{62},$$

for some β_1, β_2, β_3, β_4 and $\beta_5 \in \mathbb{Q}$. On one side

$$\begin{aligned} D\psi(z_{61}) &= D(\beta_1 z_{61} + \beta_2 z_{62} + \beta_3 y \otimes y + \beta_4 u(1\otimes x) + \beta_5 u(x\otimes 1)) \\ &= \beta u(1\otimes y - y\otimes 1) + (\beta_2+\beta_3)(x\otimes xy - xy\otimes x) \\ &\quad +\beta_4 z_5(1\otimes x + x\otimes 1), \end{aligned}$$

and on the other

$$\begin{aligned} \psi(Dz_{61}) &= \psi(u(1\otimes y - y\otimes 1)) \\ &= u(1\otimes y - y\otimes 1) + \alpha_2(x\otimes xy - xy\otimes x). \end{aligned}$$

Then, for ψ to commute with the differentials, we have that $\beta = 1$, $\beta_4 = 0$ and therefore

$$\psi(z_{61}) = z_{61} + \beta_1(y\otimes y) + \beta_2 u(1\otimes x) + \beta_3 u(x\otimes 1)$$

with $\beta_2 + \beta_3 = \alpha_2$.

Let's study what happens with the element $h \in Z^7$. For degree reasons

$$\begin{aligned} \psi(h) &= \gamma_1 h + \gamma_2(xy\otimes y) + \gamma_3(y\otimes xy) + \gamma_4 u(1\otimes y) + \gamma_5 u(y\otimes 1) \\ &\quad +\gamma_6 z_5(1\otimes x) + \gamma_7 z_5(x\otimes 1) + \gamma_8 z_{71} + \gamma_9 z_{72}. \end{aligned}$$

Simple calculations yield

$$\begin{aligned} D_{q,0}\psi(h) &= \gamma_1 u^2 - 2\gamma_1 z_{61}(1\otimes x + x\otimes 1) + (\gamma_1 q - (\gamma_4+\gamma_5))(y\otimes xy) \\ &\quad -(\gamma_4+\gamma_5)(xy\otimes y) + (\gamma_6+\gamma_7)u(x\otimes x) \\ &\quad +\gamma_8 z_{62}(1\otimes x - x\otimes 1) \\ &\quad +\gamma_9(z_{61}(1\otimes x - x\otimes 1) + z_5(y\otimes 1 - 1\otimes y)) \end{aligned}$$

and on the other side

$$\begin{aligned} \psi(D_{r,0}(h)) &= u^2 + 2(\alpha_2 - (\beta_2+\beta_3))u(x\otimes x) - 2z_{61}(1\otimes x + x\otimes 1) \\ &\quad +(r - 2\beta_1)(y\otimes xy) - \beta_1(xy\otimes y). \end{aligned}$$

For ψ to commute with the differentials we must have that $\gamma_1 = 1$, $q - (\gamma_4 + \gamma_5) = r - 2\beta_1$ and $-(\gamma_4+\gamma_5) = -2\beta_1$. From these last equations we deduce that $q - r = (\gamma_4+\gamma_5) - 2\beta_1 = 0$, hence $q = r$. □

Remark 7.9. As a consequence of this last proposition we see that there are as many different (under $A\otimes A$) CDGAs as elements in $H^3(A)$. An interesting fact to notice is that if we take out the condition "under $A\otimes A$" we obtain, using the same techniques as in the proof of Proposition 7.7 (but this time without imposing that $\psi|_{A\otimes A} = id$), *only two* not weakly equivalent CDGAs: the CDGA $C(0)$ and $C(x)$ for $x \neq 0$. Indeed the CDGA $C(0)$ and the CDGA $C(x)$ are not weakly equivalent when $x \neq 0$. But, as CDGAs (not under $A\otimes A$), $C(x) \simeq C(x')$ if $x \neq 0 \neq x'$.

8. Further questions

The next step to understand the rational model of $F(M,2)$ for a simply connected manifold M of odd dimension would be to prove Conjecture 7.4. The fact of having a bijection between equivalence classes of CDGAs $C(\xi)$ and the cohomology group $H^{n-2}(M;\mathbb{Q})$ would suggest (inspired by N. Habegger's thesis [3]) that the different CDGA structures $C(\xi)$ would be in one-to-one correspondance with the isotopy classes of embeddings $M \hookrightarrow M\times M$ homotopic to the diagonal embedding. The interest of this would be to find a geometrical argument that will give us information about the choice of $x \in H^{n-2}(M;\mathbb{Q})$ such that $C(x)$ is a CDGA model of $F(M,2)$.

An example of another geometrical argument that could give information about the choice of $x \in H^{n-2}(M;\mathbb{Q})$ is the fact that the inclusion

$$F(M,2) \hookrightarrow M \times M$$

is Σ_2-equivariant (by the action of the symmetric group of two letters which acts by permutation of the coordinates). In [1, Section 6.4] it is shown that this fact doesn't allow us to restrict the choice of $x \in H^{n-2}(M;\mathbb{Q})$.

To conclude, recall that in Section 6 we have introduced the notion of an *untwisted* manifold for which $C(0)$ is a model of the configuration space of two points. We also conjectured in that section that every simply connected closed manifold is untwisted. Oppositely, a very interesting fact will be to construct a *twisted* closed simply connected manifold M of odd dimension (i.e. a manifold such that $C(0)$ is *not* a rational model of $F(M,2)$).

Acknowledgements

The author would like to thank Yves Felix for suggesting that Proposition 7.7 could be obtained from the example $A = (H^*(S^2 \times S^3;\mathbb{Q}), 0)$. The author also thanks his advisor, Pascal Lambrechts, for making this work possible through his advice and encouragement.

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