PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 12, December 2015, Pages 5181–5193
http://dx.doi.org/10.1090/proc/12670
Article electronically published on April 14, 2015

TRIVIALITY OF THE HIGHER FORMALITY THEOREM

DAMIEN CALAQUE AND THOMAS WILLWACHER

(Communicated by Lev Borisov)

ABSTRACT. It is noted that the higher version of M. Kontsevich's Formality Theorem is much easier than the original one. Namely, we prove that the higher Hochschild-Kostant-Rosenberg map taking values in the n-Hochschild complex already respects the natural E_{n+1} operad action whenever $n \geq 2$. To this end we introduce a higher version of the braces operad, which—analogously to the usual braces operad—acts naturally on the higher Hochschild complex, and which is a model of the E_{n+1} operad.

1. Introduction

Let A be any smooth commutative $\mathbb{K}$-algebra essentially of finite type. We may consider A as an associative $\mathbb{K}$-algebra only, say A_1. As such we may form its Hochschild cochain complex

$$C(A_1) = \bigoplus_{k \geq 0} Hom_{\mathbb{K}}(A^{\otimes k}, A)[-k]$$

endowed with the Hochschild differential. The cohomology of $C(A_1)$ is computed by the Hochschild-Kostant-Rosenberg Theorem, which states that the Hochschild-Kostant-Rosenberg (HKR) map

$$\Phi_{HKR}\colon S_A\Big(\mathsf{Der}(A)[-1]\Big) \longrightarrow C(A_1)$$

sending a k-multiderivation to the obvious map $A^{\otimes k} \to A$ is a quasi-isomorphism of complexes. Note that $S_A\Big(\mathsf{Der}(A)[-1]\Big)$ is endowed with the zero differential.

In fact, the degree shifted complexes $S_A\Big(\mathsf{Der}(A)[-1]\Big)[1]$ and $C(A_1)[1]$ are endowed with differential graded (dg) Lie algebra structures, with the Schouten bracket and the Gerstenhaber bracket, respectively. The central result of deformation quantization is M. Kontsevich's Formality Theorem [14], stating that there is an ∞-quasi-isomorphism of dg Lie algebras

$$S_A\Big(\mathsf{Der}(A)[-1]\Big)[1] \longrightarrow C(A_1)[1]$$

extending the HKR map.

Received by the editors November 23, 2013 and, in revised form, May 5, 2014 and October 31, 2014.

2010 *Mathematics Subject Classification.* Primary 18D50.

The first author acknowledges the support of the Swiss National Science Foundation (grant 200021_137778).

The second author acknowledges the support of the Swiss National Science Foundation (grants PDAMP2_137151 and 200021_150012).

Actually, $S_A\Big(\mathsf{Der}(A)[-1]\Big)$ also carries the structure of a Gerstenhaber algebra (or, e_2 algebra). Let Kontsevich's result has been strengthened by D. Tamarkin [18], who showed that there also exists an ∞-quasi-isomorphism of homotopy Gerstenhaber algebras

$$S_A\Big(\mathsf{Der}(A)[-1]\Big) \longrightarrow C(A_1)$$

extending the HKR map, for some choice of homotopy Gerstenhaber structure on the right hand side.

There is a natural generalization of the objects involved to the higher setting. Denote by $\mathsf{e}_n = H_{-\bullet}(E_n)$ the homology operad of the little n-disks operad, without zero-ary operations.[1] For $n \geq 2$ it is isomorphic to the n-Poisson operad Pois_n (see [11]) and hence generated by two binary operations, a commutative product of degree 0, and a compatible Lie bracket of degree $1-n$ [2,3]. We may consider the commutative algebra A as an e_n-algebra, say A_n, with trivial bracket. We assume that $n \geq 2$. The operad e_n is Koszul [11], and by the standard Koszul theory of operads (see [16, chapter 7], [12]) we may define an e_n-deformation complex which we denote by $C(A_n)$. There is a version of the Hochschild-Kostant-Rosenberg Theorem stating that the natural inclusion

$$\Phi^n_{HKR}\colon S_A\Big(\mathsf{Der}(A)[-n]\Big) \to C(A_n)$$

is a quasi-isomorphism of complexes.

There is a natural e_{n+1} algebra structure on $S_A\Big(\mathsf{Der}(A)[-n]\Big)$, with product being the symmetric product and bracket being a degree-shifted version of the Schouten bracket. Similarly, there is an explicit hoe_{n+1} structure on $C(A_n)$, constructed by D. Tamarkin [19]. Here $\mathsf{hoe}_{n+1} = \Omega(\mathsf{e}^{\mathsf{i}}_{n+1})$ is the minimal resolution of the operad e_{n+1}, i.e., the cobar construction of the Koszul dual cooperad $\mathsf{e}^{\mathsf{i}}_{n+1} \cong \mathsf{e}^*_{n+1}\{n+1\}$, cf. [16, sections 6.5, 13.3]. The higher formality conjecture states that the (quasi-iso)morphism Φ^n_{HKR} may be extended to an ∞-(quasi-iso)morphism of hoe_{n+1} algebras.

The content of the present paper points out that this conjecture is somehow trivial.

Theorem 1. *For $n \geq 2$ the HKR map $\Phi^n_{HKR}\colon S_A\Big(\mathsf{Der}(A)[-n]\Big) \to C(A_n)$ is already a quasi-isomorphism of hoe_{n+1} algebras.*

This result might be known to the experts, but the authors are unaware of any reference. The proof boils down to a straightforward direct calculation.

Remark 1. As will be clear from the proof, the statement of Theorem 1 holds true for A the algebra of smooth functions on a smooth manifold, if one replaces the Hochschild complex by the continuous Hochschild complex, or by the complex of multi-differential operators.

Remark 2. Note that there is a choice in the precise definition of the "Hochschild" complex $C(A_n)$, essentially depending on a choice of cofibrant model for e_n. We choose here the minimal model hoe_n. For some other model $\mathcal{P}$, solving the higher formality conjecture will be "as complicated as" picking a morphism $\mathcal{P} \to \mathsf{e}_n$.

[1] Deviating from standard notation we will denote by E_n an operad quasi-isomorphic to the chains operad of the little n-cubes operad, *without zero-ary operations.*

The higher formality conjecture for that model can then be recovered by transfer, using Theorem 1.

Remark 3. Theorem 1 remains valid for any differential graded algebra A as soon as one replaces $\mathsf{Der}(A)$ by its right derived variant $\mathbb{D}\mathsf{er}(A)$. Moreover, functoriality of the HKR map allows us to freely sheafify and get in particular that, for a quasi-projective derived scheme X and $n \geq 2$, the HKR map

$$\Phi^n_{HKR} \colon S_{\mathcal{O}_X}\Big(\mathbb{T}_X[-n]\Big) \to C\big((\mathcal{O}_X)_n\big)$$

is a quasi-isomorphism of sheaves of hoe_{n+1} algebras (the only subtlety is to make $C\big((\mathcal{O}_X)_n\big)$ into a sheaf[2]). Note that this is slightly different from the main result of [20, section 5], also called higher formality, where it is proved that the e_n Hochschild complex of X is weakly equivalent to the E_n Hochschild complex of X as a Lie_{n+1} algebra (in our context this is more or less the content of Remark 2, but then the hard part would be to prove that the Lie_{n+1} structures on Hochschild complexes appearing in the present paper and the ones appearing in [20] are the same).

Remark 4. All our results and constructions remain valid for every $n \in \mathbb{Z}$ if one uses Pois_n in place of e_n.

Structure of the paper. In section 2 we recall some basic definitions and notation. Section 3 contains a rewording of D. Tamarkin's construction of the hoe_{n+1} algebra structure on $C(A_n)$. The proof of Theorem 1 is a small direct calculation which is presented in section 5.

2. Notation

We will work over a ground field $\mathbb{K}$ of characteristic 0; all algebraic structures should be understood over $\mathbb{K}$. We will use the language of operads throughout. A good introduction can be found in the textbook [16], from which we freely borrow some terminology.

For a (co)augmented (co)operad $\mathcal{O}$ we denote by $\mathcal{O}_\circ$ the (co)kernel of the (co)augmentation. It is a pseudo-(co)operad: i.e. it does not have a (co)unit.

2.1. **Our favorite operads.**

2.1.1. *The* e_n *operad.* We will denote by e_n the homology of the topological operad E_n, for every $n \geq 1$, cf. [3, 4]. Note that we work with cohomological gradings (i.e. our differentials have degree $+1$), so that homology sits in non-positive (cohomological) degree.

As an example, e_1 is the operad governing non-unital associative algebras. For $n \geq 2$, the operad e_n is isomorphic to an operad obtained by means of a distributive law: $\mathsf{e}_n \cong \mathsf{Com} \circ \mathsf{Lie}_n$, where $\mathsf{Lie}_n := \mathsf{Lie}\{n-1\} := \mathcal{S}^{1-n}\mathsf{Lie}$ is a degree shifted variant of the Lie operad.

Let $\mathcal{O}$ be an operad and let O_N be the free $\mathcal{O}$-algebra cogenerated by symbols $X_1, \ldots, X_N$. Then O_N carries a natural $\mathbb{Z}^N$ grading by counting the number of occurrences of $X_1, \ldots, X_N$. Furthermore $\mathcal{O}(N)$ may be identified with the subspace of homogeneous degree $(1, \ldots, 1)$, $\mathcal{O}(N) \cong O_N^{(1,\ldots,1)}$. This often provides convenient notation for elements of $\mathcal{O}(N)$.

[2] One shall use the quasi-isomorphic sub-complex of multi-differential operators in $C(A_n)$, which sheafifies well.

In particular, the space $\mathsf{e}_n(N)$ is spanned by formal linear combinations of "Gerstenhaber words", like

$$[X_1, X_2] \cdot X_4 \cdot [X_3, X_5]\,,$$

in N formal variable $X_1, \ldots, X_N$, each occurring once. We thus have an obvious map $\mathsf{Lie}_n \to \mathsf{e}_n$.

2.1.2. *The* hoe_n *dg operad.* The minimal resolution of e_n, resp. Lie_n, is denoted by hoe_n, resp. hoLie_n. In particular $\mathsf{hoe}_n = \Omega(\mathsf{e}_n^{¡})$ where $\Omega(\cdot)$ denotes the cobar construction and $\mathsf{e}_n^{¡} \cong \mathsf{e}_n^*\{n\}$ is the Koszul dual cooperad of e_n. Note that hoe_n and hoLie_n are dg operads.

Let $\mathcal{C}$ be a cooperad and let C_N be the cofree $\mathcal{C}$ coalgebra cogenerated by symbols $X_1, \ldots, X_N$. Then C_N carries a natural $\mathbb{Z}^N$ grading by counting the number of occurrences of $X_1, \ldots, X_N$. Furthermore $\mathcal{C}(N)$ may be identified with the subspace of homogeneous degree $(1, \ldots, 1)$, $\mathcal{C}(N) \cong C_N^{(1,\ldots,1)}$. This often provides convenient notation for elements of $\mathcal{C}(N)$. In particular, one may understand elements of $\mathsf{e}_n^*(N)$ by linear combinations of "co-Gerstenhaber words", like

$$\underline{X_1 X_2} \wedge X_4 \wedge \underline{X_3 X_5 X_6}\,,$$

in N formal variable $X_1, \ldots, X_N$, each occurring once. The underline shall indicate that one equates linear combinations that correspond to (signed) sums of shuffle permutations to zero.

We may also consider the extended e_n operad $\mathsf{ue}_n = \mathsf{uCom} \circ \mathsf{Lie}_n$, which contains one nullary operation, i.e., $\mathsf{ue}_n(0) = \mathsf{uCom}(0) = \mathbb{K}$. It governs unital e_n-algebras and can be obtained as the homology of the topological little disks operad, which has a nullary operation acting by deleting disks.

2.1.3. *The braces operad and the Kontsevich-Soibelman minimal operad.* Let V be a dg vector space, and consider the cofree coassociative coalgebra without counit $B(V)$ generated by $V[1]$. A B_∞-algebra structure on V (see [11, section 5.2]) is a dg bialgebra structure on $B(V)$ extending the coalgebra structure. Concretely, since $B(V)$ is cofree such a structure is determined by the projections of the product and differential onto the cogenerators

$$m_{j,k} : (V[1])^{\otimes j+k} \to V[1],\ j,k \geq 1 \qquad m_k : (V[1])^{\otimes k} \to V[1],\ k \geq 2.$$

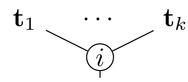
An algebra over the Kontsevich-Soibelman minimal operad $\widetilde{\mathsf{Br}}$ (see [15]) is a B_∞ algebra such that $m_{j,k} = 0$ for $j \geq 1$.[3] A braces algebra is a $\widetilde{\mathsf{Br}}$-algebra such that in addition $m_k = 0$ for $k \geq 3$. We denote the braces operad by Br. It is clearly a quotient of $\widetilde{\mathsf{Br}}$, and it is an easy exercise to check that the projection map $\widetilde{\mathsf{Br}} \to \mathsf{Br}$ is a quasi-isomorphism [13]. It is well known that $H(\mathsf{Br}) = H(\widetilde{\mathsf{Br}}) = \mathsf{e}_2$, see, e.g., [8–10, 17].

Furthermore, the Hochschild complex of an A_∞ algebra is a $\widetilde{\mathsf{Br}}$-algebra. If the A_∞ algebra is in fact an ordinary associative algebra, then its Hochschild complex is also a Br-algebra; see [9, 11].

[3]The operad $\widetilde{\mathsf{Br}}$ has originally been described by Kontsevich and Soibelman combinatorially as an operad of planar trees. However, it is not hard to check that their description agrees with ours; see for example [21, Corollary 1].

2.1.4. *The* preLie *operad.* We will denote by preLie the operad encoding pre-Lie algebras. Following [5], it admits the following combinatorial description. We first introduce the set $\mathcal{T}(I)$ of rooted trees with vertices labelled by a finite set I, which is constructed via the following inductive process:

- $\mathcal{T}(\emptyset)$ is empty.
- $\mathcal{T}(\{i\})$ consists of a single rooted tree having only one root-vertex labelled by i: $\overset{\textcircled{\scriptsize i}}{\scriptstyle |}$.
- Let I be a finite set, $i \in I$ and a partition $I_1 \sqcup \cdots \sqcup I_k = I - \{i\}$. Given rooted trees $\mathbf{t}_\alpha \in \mathcal{T}(I_\alpha)$, $\alpha = 1, \dots, k$ one can construct a new rooted tree $B_+(\mathbf{t}_1, \dots, \mathbf{t}_k) \in \mathcal{T}(I)$ by grafting the root of each $\mathbf{t}_\alpha$, $\alpha = 1, \dots, k$, on a common new root labelled by i:



Then $\mathsf{preLie}(I)$ is the vector space generated by $\mathcal{T}(I)$, and the operadic composition can be defined in the following way: if J is another finite set, $i \in I$, $\mathbf{t} \in \mathcal{T}(I)$ and $\mathbf{t}' \in \mathcal{T}(J)$, then $\mathbf{t} \circ_i \mathbf{t}'$ is described as a sum over the set of functions f from incoming edges at the vertex i of $\mathbf{t}$ to the vertices of $\mathbf{t}'$. For any such f, the corresponding term is obtained by removing the vertex i from $\mathbf{t}$, reconnecting the outgoing edge to the root of $\mathbf{t}'$ and reconnecting the incoming edges e to the vertex $f(e)$. The root of the result is taken to be the root of $\mathbf{t}$ if this is not i, or the root of $\mathbf{t}'$ otherwise.

Note that for any operad $\mathcal{O}$, the vector spaces $\prod_{n\geq 0} \mathcal{O}(n)$ and $\prod_{n\geq 0} \mathcal{O}(n)^{S_n}$ are naturally preLie algebras.

Recall also that there is a morphism of operads $\mathsf{Lie} \to \mathsf{preLie}$ which sends the generator of Lie to $\begin{smallmatrix} 2 \\ | \\ 1 \end{smallmatrix} - \begin{smallmatrix} 1 \\ | \\ 2 \end{smallmatrix}$. Hence any pre-Lie algebra is also a Lie algebra (obtained by skew-symmetrizing the pre-Lie product).

2.2. **The Hochschild complex of a hoe_n algebra.** For a hoe_n algebra B, we define the "Hochschild" complex as the degree shifted convolution dg Lie algebra[4]

$$C(B) = \mathrm{Conv}\big(\mathsf{ue}_n^*\{n\}, \mathrm{End}_B\big)[-n]$$

where End_B is the endomorphism operad of B and the differential is the Lie bracket with the element of $C(B)$ corresponding to the hoe_n structure. In particular, if $B = A_n$ is as in the introduction, then

$$C(B) \cong A \oplus \mathrm{Conv}\big(\mathsf{e}_n^*\{n\}, \mathrm{End}_A\big)[-n]$$

as complexes.

Remark 5. For $n = 1$ our definition of the Hochschild complex clearly agrees with the standard definition. More generally, if B is a $\mathcal{P}_\circ$-algebra, then the definition $C(B, B) := \mathbb{R}Hom_{\mathcal{P}_\circ - B - mod}(B, B)$ is often used. One can show that there is a quasi-isomorphism of complexes $C(B) \to C(B, B)$. Very shortly, following the notation and convention in [16], we have $\mathrm{Conv}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_B) = Hom\big(\mathcal{P}^{\mathsf{i}}(B), B\big)$, which computes $\mathbb{R}Hom_{\mathcal{P}_\circ - B - mod}(B, B)$.

[4]For the definition of the convolution dg Lie algebra of maps from a cooperad $\mathcal{C}$ to an operad $\mathcal{O}$ we refer the reader to [16, section 6.4], where the notation $\mathsf{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})$, or $\mathsf{Hom}_{\mathbb{S}}^{\alpha}(\mathcal{C}, \mathcal{O})$ for the twisted version is used instead.

Note that there is a natural inclusion

$$\Phi^n_{HKR}\colon S_A\Big(\mathsf{Der}(A)[-n]\Big) \longrightarrow C(A_n)$$

whose image consists of the elements in

$$\mathrm{Conv}\big(\mathsf{uCom}^*\{n\}, \mathrm{End}_A\big)$$

that furthermore are (i.e. take values in) derivations in each slot. Analogously to the usual HKR Theorem one may check the following result.

Theorem 2 (Higher Hochschild-Kostant-Rosenberg Theorem)**.** *If A is a smooth commutative $\mathbb{K}$-algebra essentially of finite type, then the map Φ^n_{HKR} is a quasi-isomorphism of complexes for each $n \geq 2$.*

Proof. One simply observes that, since the bracket on A is zero,

$$\mathrm{Conv}\big(\mathsf{ue}^*_n\{n\}, \mathrm{End}_A\big)[-n] = S_A\Big(\mathrm{Conv}\big(\mathsf{Lie}^*\{1\}, \mathrm{End}_A\big)[-n]\Big) = S_A\big(\mathbb{D}\mathsf{er}(A)[-n]\big)\,.$$

Here $\mathbb{D}\mathsf{er}$ is the right derived functor of the derivations functor Der. If A is smooth and essentially of finite type, then the canonical map $\mathsf{Der}(A) \to \mathbb{D}\mathsf{er}(A)$ is a quasi-isomorphism. □

3. A version of D. Tamarkin's proof of the higher Deligne conjecture

The goal of this section is to recall D. Tamarkin's proof of the following result.

Theorem 3 (Higher Deligne conjecture; see [19])**.** *For any hoe_n algebra B, the complex $C(B)$ carries a natural hoe_{n+1} action, given by explicit formulas, for $n \geq 2$.*

3.1. **Braces for a Hopf cooperad.** A Hopf operad is an operad in the category of counital coalgebras, cf. [16, section 5.3.5]. For a Hopf operad $\mathcal{O}$, the tensor product of two $\mathcal{O}$-algebras is naturally endowed with an $\mathcal{O}$-algebra structure. Dually, a Hopf cooperad is a cooperad in unital algebras.

For any coaugmented cooperad $\mathcal{C}$ we may define its bar construction $\Omega(\mathcal{C})$, which is an operad. For example $\mathsf{hoe}_n := \Omega(e_n^{¡})$. Here we define a similar construction, the brace construction, which takes a Hopf cooperad $\mathcal{C}$ and returns an operad $\mathsf{Br}_{\mathcal{C}}$.

3.1.1. *$\mathcal{C}$-operads.* In this paragraph we introduce the notion of a $\mathcal{C}$-operad, for $\mathcal{C}$ a Hopf cooperad as above. A $\mathcal{C}$-operad is an operad $\mathcal{O}$ such that each $\mathcal{O}(n)$ carries an S_n equivariant right $\mathcal{C}(n)$ module structure. We require that furthermore the right module structures are compatible with the operadic compositions, by which we mean that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(k)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)\otimes\mathcal{C}(\sum_j n_j) \arrow[rr] \arrow[d] & & \mathcal{O}(\sum_j n_j)\otimes\mathcal{C}(\sum_j n_j) \arrow[dd] \\
\mathcal{O}(k)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)\otimes\mathcal{C}(k)\otimes\mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_k) \arrow[d] & & \\
\mathcal{O}(k)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k) \arrow[rr] & & \mathcal{O}(\sum_j n_j)
\end{tikzcd}.$$

Here the two horizontal arrows are the operadic compositions. The upper left vertical arrow is defined using the cooperad structure on $\mathcal{C}$. The remaining two arrows are defined by using the right action of $\mathcal{C}(n)$ on $\mathcal{O}(n)$. Note that for $\mathcal{C} = \mathsf{uCom}^*$ a $\mathcal{C}$-operad is just an ordinary operad.

Example 1. One can check that for an operad $\mathcal{P}$ and a Hopf cooperad $\mathcal{C}$, the convolution operad $\mathsf{Hom}(\mathcal{C}\{k\}, \mathcal{P})$ (see [16, section 6.4.2]) is naturally a $\mathcal{C}$-operad for any k. Here the right action is obtained by composition with the multiplication on $\mathcal{C}(n)$ from the right, i.e.,

$$(f \cdot c)(x) = f(cx)$$

for $f \in \mathsf{Hom}(\mathcal{C}\{k\}, \mathcal{P})(n)$, $c \in \mathcal{C}(n)$ and $x \in \mathcal{C}\{k\}(n)$.

3.1.2. *The* $\mathsf{preLie}_{\mathcal{C}}$ *operad.* In this section we introduce an operad encoding $\mathcal{C}$-pre-Lie algebras, which are to $\mathcal{C}$-operads what pre-Lie algebras are to operads. For simplicity, we will assume that the Hopf cooperad $\mathcal{C}$ satisfies: $\mathcal{C}(0) \cong \mathcal{C}(1) \cong \mathbb{K}$. Then one has natural maps

$$\mathcal{C}(j) \to \mathcal{C}(j+k) \otimes \underbrace{\mathcal{C}(1) \otimes \cdots \mathcal{C}(1)}_{j\times} \otimes \underbrace{\mathcal{C}(0) \otimes \cdots \mathcal{C}(0)}_{k\times} \cong \mathcal{C}(j+k) \tag{1}$$

where the arrow is a cocomposition and the right hand identification uses the canonical identifications $\mathcal{C}(0) \cong \mathcal{C}(1) \cong \mathbb{K}$ as algebras.

Example 2. The most interesting example for us is the Hopf cooperad $\mathcal{C} = \mathsf{ue}_n^*$, whose j-ary cooperations may be interpreted as the cohomology of the configuration space of j points in $\mathbb{R}^n$ for $j \geq 1$, and as $\mathbb{K}$ for $j = 0$. There are forgetful maps from the configuration space of $k+j$ points to that of j points, and in this case the extension map (1) above is just the pull-back of the forgetful map, forgetting the location of the last k points.

The operad $\mathsf{preLie}_{\mathcal{C}}$ consists of rooted trees decorated by a Hopf cooperad $\mathcal{C}$. Namely, for every finite set I,

$$\mathsf{preLie}_{\mathcal{C}}(I) := \bigoplus_{\mathbf{t} \in \mathcal{T}(I)} \left(\bigotimes_{i \in I} \mathcal{C}(\mathbf{t}_i) \right),$$

where $\mathbf{t}_i$ is the set/number of incoming edges at the vertex labelled by i.

The operadic structure on the underlying trees is the one described in section 2.1.4. Let us now explain what happens to the decoration when doing the partial composition $\circ_i$. Borrowing the notation from section 2.1.4, for every f we apply a cooperation:[5]

$$\mathcal{C}(\mathbf{t}_i) \longrightarrow \bigotimes_{j \in J} \mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big).$$

Then observe that we have natural maps

$$\mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big) \otimes \mathcal{C}\big(\mathbf{t}'_j\big) \to \mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big) \otimes \mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big) \to \mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big) = \mathcal{C}\big((\mathbf{t} \circ_i \mathbf{t}')_j\big)$$

where the first map uses the extension map (1) on the second factor and the second map uses the Hopf structure, i.e., it is the multiplication of the algebra $\mathcal{C}\big(f^{-1}(j) \cup \mathbf{t}'_j\big)$.

[5] Note that possible cooperations that we may apply to elements of $\mathcal{C}(\mathbf{t}_i)$ are naturally labelled by rooted trees with leaves labelled by $\mathbf{t}_i$. The cooperation we apply here is the one labelled by the tree $\mathbf{t}'$, with labelled leaves attached according to f and with the labelling of the vertices of $\mathbf{t}'$ disregarded.

The definition is made such that $\mathsf{preLie}_{\mathcal{C}}$ naturally acts on the convolution "algebra"

$$\mathrm{Conv}_0(\mathcal{C}\{k\},\mathcal{P}) := \prod_{n\geq 0} \mathsf{Hom}(\mathcal{C}\{k\},\mathcal{P})(n)^{S_n}.$$

More generally, for any $\mathcal{C}$-operad $\mathcal{O}$, $\prod_n \mathcal{O}(n)^{S_n}$ is a $\mathsf{preLie}_{\mathcal{C}}$ algebra (and we have already seen that $\mathsf{Hom}(\mathcal{C}\{k\},\mathcal{P})$ is a $\mathcal{C}$-operad).

Remark 6. Note that for $\mathcal{C} = \mathsf{uCom}^*$ we recover the usual preLie operad, i.e., $\mathsf{preLie}_{\mathsf{uCom}^*} = \mathsf{preLie}$. Furthermore the construction $\mathsf{preLie}_{\mathcal{C}}$ is functorial in $\mathcal{C}$. Hence from the unit map $\mathsf{uCom}^* \to \mathcal{C}$ we obtain a map of operads $\mathsf{preLie} \to \mathsf{preLie}_{\mathcal{C}}$ for any Hopf cooperad $\mathcal{C}$. In particular, any $\mathsf{preLie}_{\mathcal{C}}$ algebra is a Lie algebra, and we recover the usual Lie algebra structure on $\mathrm{Conv}_0(\mathcal{C}\{k\},\mathcal{P})$.

Remark 7. Note that the operad $\mathsf{preLie}_{\mathcal{C}}$ is not the same as the cobar construction $\Omega(\mathcal{C})$ of $\mathcal{C}$, not even up to degree shifts. A basis of $\Omega(\mathcal{C})(I)$ is given by rooted trees whose leafs are decorated by elements of the finite set I, and whose internal nodes are decorated by elements of $\mathcal{C}[1]$. The operadic composition is obtained by gluing the root of one tree to a leaf of another. In contrast, in the trees giving rise to $\mathsf{preLie}_{\mathcal{C}}(I)$ each node, also each internal node, is decorated by an element of I. The operadic composition allows for inserting one tree at an internal node of another, not just for grafting the root to a leaf.

Next, if we have a morphism $f : \Omega(\mathcal{C}\{k\}) \to \mathcal{P}$ of dg operads, then it determines a Maurer-Cartan element γ_f in $\mathrm{Conv}_0(\mathcal{C}\{k\},\mathcal{P})$, and a new convolution dg Lie algebra $\mathrm{Conv}_f(\mathcal{C}\{k\},\mathcal{P})$ is obtained from the original one by twisting with γ_f. Note that we may drop the "f" from the notation when there is no ambiguity. In general, the action of the operad $\mathsf{preLie}_{\mathcal{C}}$ will unfortunately not lift to an action on the twisted convolution Lie algebra $\mathrm{Conv}_f(\mathcal{C}\{k\},\mathcal{P})$. However, we now may invoke the formalism of operadic twisting [6]. Given an operad $\mathcal{P}$ together with a map $\mathsf{Lie} \to \mathcal{P}$, operadic twisting produces:

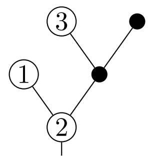
- A dg operad $Tw\mathcal{P}$, the *twisted operad*.[6]
- Dg operad maps $\mathsf{Lie} \to Tw\mathcal{P} \to \mathcal{P}$ whose composition is the given map $\mathsf{Lie} \to \mathcal{P}$.
- The dg operad $Tw\mathcal{P}$ has the property that if we are given a $\mathcal{P}$ algebra A, together with a Maurer-Cartan element m of the Lie algebra A, then the action of the Lie operad on the twisted Lie algebra A^m lifts naturally to an action of the dg operad $Tw\mathcal{P}$. Moreover, $Tw\mathcal{P}$ is universal among such dg operads

In our case we obtain a dg operad[7] $Tw\mathsf{preLie}_{\mathcal{C}}$, acting naturally on the twisted convolution algebra $\mathrm{Conv}_f(\mathcal{C}\{k\},\mathcal{P})$. Concretely the dg operad $Tw\mathsf{preLie}_{\mathcal{C}}$ is a completed version of the dg operad generated by $\mathsf{preLie}_{\mathcal{C}}$ and one formal nullary element. The differential on $Tw\mathsf{preLie}_{\mathcal{C}}$ is defined so that upon replacing the formal nullary element by the Maurer-Cartan element γ_f we obtain an action of $Tw\mathsf{preLie}_{\mathcal{C}}$ on $\mathrm{Conv}_f(\mathcal{C}\{k\},\mathcal{P})$. The formal nullary element we denote in pictures by coloring

[6] $Tw\mathcal{P}$ is defined as follows. First observe that there is a functor from dg operads to 2-colored dg operads which send $\mathcal{P}$ to the 2-colored operad $Lie-mod(\mathcal{P})$ encoding a $\mathcal{P}$ algebra together with a Lie algebra acting on it by derivations. It has a right adjoint denoted tw. Tw is then obtained by applying tw and identifying the two colors. We refer to [1,6] for more details.

[7] Which does NOT depend on f.

the appropriate vertices of the tree black. We call these vertices the internal vertices (as opposed to external ones).



Combinatorially, the differential on $Tw\mathsf{preLie}_{\mathcal{C}}$ splits vertices, either an internal vertex into two internal vertices, or an external vertex into an external and an internal vertex.

We define the brace construction $\mathsf{Br}(\mathcal{C}) = Tw\mathsf{preLie}_{\mathcal{C}}$ as a synonym for the twisted pre-Lie operad. By construction $\mathsf{Br}(\mathsf{ue}_n^*)$ acts on the convolution dg Lie algebra

$$C(B)[n] = \mathrm{Conv}\big(\mathsf{ue}_n^*\{n\}, \mathrm{End}_B\big)$$

for any hoe_n algebra B. For cosmetic reasons and consistency with the literature we make the following definition.

Definition 1. We define the higher braces dg operad Br_{n+1} to be the suboperad

$$\mathsf{Br}_{n+1} \subset \mathsf{Br}(\mathsf{ue}_n^*)\{n\}$$

formed by operations whose underlying trees contain no internal vertices with less than 2 children.

By very definition, the operad Br_{n+1} acts naturally on the Hochschild complex $C(B)$.

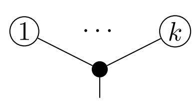
Example 3. The higher braces dg operad Br_2 is the same as the Kontsevich-Soibelman minimal operad $\widetilde{\mathsf{Br}}$, cf. section 2.1.3 or [15].

3.2. **Tamarkin's morphism.** D. Tamarkin proved Theorem 3 by noting that for $n \geq 2$ there is a quite simple but very remarkable explicit map

$$T\colon \mathsf{hoe}_{n+1} \to \mathsf{Br}_{n+1}.$$

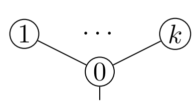
It is defined on generators by the following prescription:

- Generators of the form $\underline{X_1 \cdots X_k} \in e_{n+1}^{¡}(k)$ are mapped to a corolla of the form



(2)

  decorated by $\underline{X_1 \cdots X_k} \in e_n^{¡}(k)$.
- Generators of the form $X_0 \wedge \underline{X_1 \cdots X_k} \in e_{n+1}^{¡}(k+1)$ are mapped to a corolla of the form



(3)

  decorated by $\underline{X_1 \cdots X_k} \in e_n^{¡}(k)$. In the special case $k = 1$ one takes the (anti-)symmetric combination of the two possible choices.
- All other generators are mapped to zero.

This prescription indeed gives a morphism $\mathsf{hoe}_{n+1} \to \mathsf{Br}_{n+1}$ of the underlying graded operads (because, as such, hoe_{n+1} is free). This leaves us with the task of verifying that the map $T : \mathsf{hoe}_{n+1} \to \mathsf{Br}_{n+1}$ commutes with the differentials. It suffices to check this on the generators. Furthermore it suffices to check the statement on generators of one of the forms

$$\underline{X_1 \cdots X_k} \qquad \underline{X_1 \cdots X_k} \wedge \underline{X_{k+1} \cdots X_{k+l}} \qquad X_1 \wedge \underline{X_2 \cdots X_{k+1}} \wedge \underline{X_{k+2} \cdots X_{k+l+1}} \tag{4}$$

since in all other cases the differential of the generator and the generator itself are mapped to zero, so that the map T trivially commutes with the differentials. One has to check each of the three types of generators above in turn. The calculation is a bit lengthy, due to several special cases that need to be considered. Since the construction of T is essentially the result of D. Tamarkin [19] we will only show how to handle a few cases in Appendix A as an illustration.

4. Br_n IS AN E_n OPERAD

Theorem 4. *The above map $T : \mathsf{hoe}_{n+1} \to \mathsf{Br}_{n+1}$ is a quasi-isomorphism of operads for all $n = 2, 3, 4, \ldots$, so in particular $H(\mathsf{Br}_{n+1}) \cong \mathsf{e}_{n+1}$.*

In the case $n = 1$ it is still true that there is a quasi-isomorphism $\mathsf{hoe}_2 \to \mathsf{Br}_2$, but this morphism is much more complicated to construct than the Tamarkin quasi-isomorphism T we described above. It can be obtained by combining a quasi-isomorphism from Br_2 to the chains of the little disks operad [15] with a choice of formality morphism of the little disks operad.

Theorem 4 is not used in this note, so we only sketch the proof.

Sketch of proof. First one checks that $H(\mathsf{Br}_{n+1}) \cong \mathsf{e}_{n+1}$. The proof of this statement follows along the lines of the proof of the $n = 1$ case in [7]. The only point where the proof in loc. cit. has to be adapted is that in [7, section 4.2] one has to compute the Hochschild cohomology of a free e_{n+1} algebra, considered as an e_n algebra, instead of computing the Hochschild cohomology of a free e_2 algebra, considered as an e_1 algebra. The answer is provided by the higher Hochschild-Kostant-Rosenberg Theorem, i.e., Theorem 2 above, instead of the usual one.

Once one knows that $H(\mathsf{Br}_{n+1}) \cong \mathsf{e}_{n+1}$, the statement of the theorem is shown by checking that the induced map in cohomology

$$\mathsf{e}_{n+1} \cong H(\mathsf{hoe}_{n+1}) \to H(\mathsf{Br}_{n+1}) \cong \mathsf{e}_{n+1}$$

is the identity, which amounts to checking that it is the identity on the two generators. $\square$

The brace construction Br_{n+1} is intuitively similar to taking a product of $\Omega(\mathsf{e}_n^{\mathsf{i}})$ with an E_1 operad. So the above theorem shall be understood as a version of the statement that the product of an E_1 operad with an E_n operad is an E_{n+1} operad.

5. PROOF OF THEOREM 1

One shall verify that the action of all generators of hoe_{n+1} commutes with the HKR map Φ^n_{HKR}. Namely, for any $g \in e_n^{\mathsf{i}}(N)$ we shall prove that $(g\cdot) \circ (\Phi^n_{HKR})^{\otimes N} = \Phi^n_{HKR} \circ g\cdot$.

First of all let $g = X_1 \wedge X_2$ (i.e., the Lie bracket). Then it is a simple verification that

$$g\cdot\big(\Phi^n_{HKR}(u), \Phi^n_{HKR}(v)\big) = \big[\Phi^n_{HKR}(u), \Phi^n_{HKR}(v)\big] = \Phi^n_{HKR}([u,v]) = \Phi^n_{HKR}\big(g\cdot(u,v)\big).$$

Second of all let $g = \underline{X_1 X_2}$ (i.e., the product). Then we obviously have

$$g \cdot \big(\Phi^n_{HKR}(u), \Phi^n_{HKR}(v)\big) = \Phi^n_{HKR}(u)\Phi^n_{HKR}(v) = \Phi^n_{HKR}(uv) = \Phi^n_{HKR}\big(g \cdot (u, v)\big).$$

Finally, all other generators g act trivially on the domain of Φ^n_{HKR}. Thus it suffices to check that the action of the generators $\underline{X_1 \cdots X_k}$ ($k \geq 3$) and $X_0 \wedge \underline{X_1 \cdots X_k}$ ($k \geq 2$) on the image of Φ^n_{HKR} is trivial, which we do now.

Generators $\underline{X_1 \cdots X_k}$ act using the corresponding components of the hoe_n structure on A_n, which vanish. Hence they act trivially (as long as $k \geq 3$).

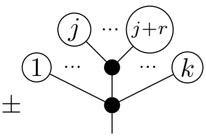
Note also that the image of Φ^n_{HKR} has only "hoLie_n components", i.e., the corresponding maps $\mathsf{e}^{\mathsf{i}}_n(N) \to \mathrm{End}(V)(N)$ factor through $\mathsf{e}^{\mathsf{i}}_n(N) \to \mathsf{Lie}^{\mathsf{i}}_n(N)$. However, the prescription for the action of the component $X_0 \wedge \underline{X_1 \cdots X_k}$ advises us to evaluate the arguments on components $\underline{X_1 \cdots X_k}$, which are sent to zero under the projection $\mathsf{e}^{\mathsf{i}}_n(k) \to \mathsf{Lie}^{\mathsf{i}}_n(k)$. Hence the action of the components $X_0 \wedge \underline{X_1 \cdots X_k}$ vanishes (as long as $k \geq 2$) on the image of the HKR map. $\square$

Appendix A. The map T commutes with the differentials

A.1. The generator $\underline{X_1 \cdots X_k}$. In this case the differential of the generator consists of

$$\sum_{j,r} \pm \underline{X_1 \ldots X_{j-1} * X_{j+r+1} \ldots X_k} \circ_* \underline{X_j \ldots X_{j+r}}\,,$$

where the notation $A \circ_* B$ shall mean the operadic composition in hoe_{n+1} of the operations A and B in hoe_{n+1}, with B being "inserted in the slot" of A labelled by $*$. The map T sends the above to

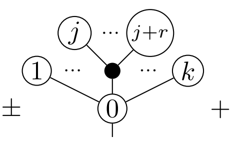
$$\sum_{j,r} \pm$$ 

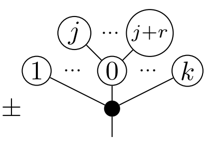
This is precisely the differential of the tree (2), which is the image of $\underline{X_1 \cdots X_k}$ by T. Decorations are obvious.

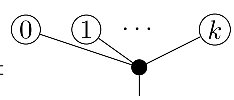
A.2. The generator $X_0 \wedge \underline{X_1 \cdots X_k}$. In this case the differential of the generator consists of the following terms:

$$\begin{aligned}(5)\quad & \sum_j \sum_{r \geq 1} \pm X_0 \wedge \underline{X_1 \cdots X_{j-1} * X_{j+r+1} \cdots X_k} \circ_* \underline{X_j \cdots X_{j+r}} + \sum_j \sum_{r \geq 1} \\ & \qquad \pm \underline{X_1 \cdots X_{j-1} * X_{j+r+1} \cdots X_k} \circ_* (X_0 \wedge \underline{X_j \cdots X_{j+r}}) \\ & \pm (X_0 \wedge *) \circ_* \underline{X_1 \cdots X_k} + \sum_j \pm \underline{X_1 \cdots X_{j-1} * X_{j+2} \cdots X_k} \circ_* (X_0 \wedge X_j)\,.\end{aligned}$$

This is mapped under T to a linear combination of trees of the following form:

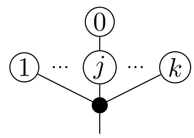
$$\sum_j \sum_{r \geq 1} \pm$$ 

$$+ \sum_j \sum_{r \geq 0} \pm$$ 

$$\pm$$ 

Here all corollas are decorated by the top degree elements of ue^*_n, except for the last tree, where the decoration is by the element $X_0 \wedge \underline{X_1 \cdots X_k}$.

One checks that this linear combination of trees is exactly the differential of (3), which is the image of the generator we considered by T. Note also that the trees of the form



occur twice, with the two contributions from the third and fourth term of (5) cancelling each other.

Acknowledgements

The first author thanks Bertrand Toën for explaining to him that HKR is true *without any assumption* in the derived context and for many discussions about the formality for derived schemes and stacks. We are grateful to the anonymous referee for his careful reading and suggestions improving our manuscript.

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I3M, Université Montpellier 2, Case courrier 051, 34095 Montpellier cedex 5, France
E-mail address: `damien.calaque@univ-montp2.fr`

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
E-mail address: `thomas.willwacher@math.uzh.ch`