

ON A POSITIVITY PRESERVATION PROPERTY FOR SCHRÖDINGER OPERATORS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We study a positivity preservation property for Schrödinger operators with singular potential on geodesically complete Riemannian manifolds with non-negative Ricci curvature. We apply this property to the question of self-adjointness of the maximal realization of the corresponding operator.

1. INTRODUCTION

In his landmark paper [17], Kato proved a powerful distributional inequality, today known as Kato's inequality, which has since found numerous applications in self-adjointness (and m -accretivity) problems in $L^2(\mathbb{R}^n)$ for Schrödinger operators with a singular potential. For example, consider the operator $-\Delta + q$ with $q^+ \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $q^- \in L^\infty(\mathbb{R}^n) + L^{n/2}(\mathbb{R}^n)$, where $n \geq 5$, $q^+ := \max(q, 0)$ and $q^- := \max(-q, 0)$. Under these conditions, the operator $-\Delta + q$ is semi-bounded from below on $C_c^\infty(\mathbb{R}^n)$; see Lemma 2.1 in [3]. By an abstract fact (see Theorem X.26 in [24]), to prove the essential self-adjointness of $-\Delta + q$ on $C_c^\infty(\mathbb{R}^n)$, it is enough to show that for any $v \in L^2(\mathbb{R}^n)$ such that $(-\Delta + q + \lambda)v = 0$ in distributional sense, where λ is a sufficiently large constant, we have $v = 0$. To this end, we apply Kato's inequality to $v \in L^2(\mathbb{R}^n)$ satisfying $(-\Delta + q + \lambda)v = 0$. This leads to the distributional inequality

$$-\Delta|v| + \lambda|v| - q^-|v| \leq 0.$$

The equality $v = 0$ will follow if q^- satisfies the positivity preservation property described below.

Positivity preservation property (PPP). Let $F \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a non-negative function. Then, there exists $\lambda_0 \geq 0$ so that if $\lambda > \lambda_0$, $u \in L^2(\mathbb{R}^n)$, $Fu \in L^1_{\text{loc}}(\mathbb{R}^n)$, and

$$-\Delta u + \lambda u - Fu \geq 0, \quad \text{in distributional sense,}$$

then $u \geq 0$.

Brézis and Kato showed in [3] that (PPP) holds for (non-negative) functions $F \in L^\infty(\mathbb{R}^n) + L^p(\mathbb{R}^n)$ with $p = \frac{n}{2}$ for $n \geq 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$,

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together with the assumption $F \in L_{\text{loc}}^{n/2+\epsilon}(\mathbb{R}^n)$, $\epsilon > 0$, in dimensions $n = 3$ and $n = 4$. The proof of (PPP) in [3] was based on elliptic equation theory and Sobolev space techniques.

Subsequently, using stochastic analysis techniques, Devinatz [6] showed that (PPP) holds for (non-negative) functions $F \in L_{\text{loc}}^1(\mathbb{R}^n)$ satisfying the property

$$(1.1) \quad \lim_{\alpha \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^n} \frac{1}{4\pi^{n/2}} \int_{\mathbb{R}^n} \frac{F(x-y)}{|y|^{n-2}} \left(\int_{\alpha y^2}^{\infty} \tau^{n/2-2} e^{-\tau} d\tau \right) dy \right) < 1.$$

We should note that the results of [6] include those of Jensen [15]. As an application of (PPP), the papers [3, 6, 15] studied the self-adjointness problem of the corresponding Schrödinger operator.

In the context of a Riemannian manifold M , a simpler variant of (PPP) with $F \equiv 0$, which we label as (PPP-0), was considered in Proposition B.3 of [2], where it was shown that (PPP-0) holds under C^∞ -bounded geometry assumption on M , that is, M has a positive injectivity radius and all Levi-Civita derivatives of the curvature tensor of M are bounded. The main point here is that the corresponding proof of [2] depends on the existence of a sequence of smooth compactly supported functions χ_k with the following properties:

- (C1) $0 \leq \chi_k(x) \leq 1$, $x \in M$, $k = 1, 2, \dots$;
- (C2) for every compact set $K \subset M$, there exists k_0 such that $\chi_k = 1$ on K , for $k \geq k_0$;
- (C3) $\sup_{x \in M} |d\chi_k(x)| \rightarrow 0$ as $k \rightarrow \infty$;
- (C4) $\sup_{x \in M} |\Delta\chi_k(x)| \rightarrow 0$ as $k \rightarrow \infty$.

While the existence of a sequence χ_k satisfying (C1), (C2), and (C3) on an arbitrary geodesically complete Riemannian manifold is well known (see [16]), a sequence satisfying all four properties has not yet been constructed (to our knowledge) in such a general context.

Very recently, Güneysu [13] has improved the (PPP-0) result considerably. In particular, in the context of a geodesically complete Riemannian manifold with non-negative Ricci curvature, the author of [13] has constructed a sequence χ_k satisfying (C1)–(C4) and proved (PPP-0). We should also note that the paper [13] contains, among other things, a study of (PPP-0) in the setting of L^p spaces with $p \in [1, \infty]$.

Let us point out that under C^∞ -bounded geometry assumptions on M , an earlier study [23] showed that (PPP) holds for (non-negative) functions F belonging to the Kato class (see Section 3.1 below) and satisfying the following additional assumption: $F \in L_{\text{loc}}^p(M)$ with $p = n/2 + \epsilon$, with some arbitrarily small $\epsilon > 0$, for the case $2 \leq n \leq 4$; $p = n/2$ for the case $n \geq 5$. We note that the paper [23] used the latter assumption for elliptic equation and Sobolev space arguments. Based on recent developments in path-integral representations for semi-groups of Schrödinger operators with singular potential on Riemannian manifolds and the construction of cut-off functions satisfying (C1)–(C4) above, as seen in Güneysu's works [10–13], we will study (PPP) for a class functions F that shares some properties with (1.1) and includes, in particular, Kato class. In this regard, within the class of non-negative Ricci curvature, our results include those in [23]. In particular, we eliminate the assumption $F \in L_{\text{loc}}^p(M)$ with p as described above. Finally, as an application of the corresponding (PPP), we give sufficient conditions for the self-adjointness of

the “maximal” realization of the Schrödinger operator with electric potential whose negative part satisfies the same assumptions as F in (PPP).

For reviews of results concerning the question of self-adjointness of Schrödinger operators in $L^2(\mathbb{R}^n)$ and $L^2(M)$, see, for instance, [5] and [2]. For more recent studies, see the papers [1, 4, 9, 13, 14].

Finally, we remark that it might be possible to obtain a variant of (PPP) for perturbations of Dirichlet forms by measures. For the background on Dirichlet forms and their perturbations by measures, see, for instance, the book [7], papers [20, 21, 25], and the references therein.

2. RESULTS

2.1. Notation. Let M be a connected smooth Riemannian n -manifold without boundary. Throughout the paper, by Δ we denote the corresponding *negative* Laplace–Beltrami operator on M , by $d\mu$ the volume measure of M , by $C^\infty(M)$ the space of complex-valued smooth functions on M , by $C_c^\infty(M)$ the space of complex-valued smooth compactly supported functions on M , by $\Omega^1(M)$ the space of smooth 1-forms on M , by $L^2(M)$ the space of square integrable complex-valued functions on M , and by (\cdot, \cdot) the usual inner product on $L^2(M)$. Additionally, $p(t, x, y)$ denotes the heat kernel of M as in Theorem 7.13 in [8]. We should emphasize that in this paper $p(t, x, y)$ corresponds to $e^{-t(-\Delta/2)}$, $t \geq 0$, instead of $e^{-t(-\Delta)}$.

2.2. Positivity preservation property. We are ready to formulate sufficient conditions for the positivity preservation property introduced in Section 1.

Theorem 2.1. *Assume that M is a geodesically complete connected Riemannian manifold with non-negative Ricci curvature. Let $F: M \rightarrow [0, \infty)$ be a measurable function satisfying the following property: there exists $t_0 > 0$ such that*

$$(2.1) \quad \sup_{x \in M} \left(\int_0^{t_0} \int_M p(s, x, y) F(y) d\mu(y) ds \right) < 1.$$

Then, there exists $\lambda_ \geq 0$ such that if $\lambda > \lambda_*$ and $u \in L^2(M)$ and $Fu \in L^1_{\text{loc}}(M)$ and u satisfies the distributional inequality*

$$(2.2) \quad (-\Delta/2 - F + \lambda)u \geq 0,$$

then $u \geq 0$ a.e. on M .

Remark 2.2. If F belongs to Kato class, then (2.1) is satisfied; see Section 3.1 below.

2.3. Hermitian vector bundles and Bochner Laplacian. We will formulate our self-adjointness result for Schrödinger operators acting on Hermitian vector bundles over M . Before doing so, we explain some additional notation. Let $E \rightarrow M$ be a smooth Hermitian vector bundle over M with underlying Hermitian structure $\langle \cdot, \cdot \rangle_x$ and the corresponding norms $|\cdot|_x$ on fibers E_x . Smooth sections of E will be denoted by $C^\infty(E)$ and compactly supported smooth sections by $C_c^\infty(E)$. With $d\mu$ as in Section 2.1, for all $1 \leq p < \infty$ we obtain the L_p -spaces of sections $L^p(E)$ with norms $\|\cdot\|_p$. The space of essentially bounded sections of E will be denoted by $L^\infty(E)$ with the corresponding norm $\|\cdot\|_\infty$. The notation $(\cdot, \cdot)_{L^2(E)}$ or just (\cdot, \cdot) , when there is no danger of confusion, stands for the usual inner product in $L^2(E)$.

Let ∇ be a Hermitian connection on E and let ∇^* be its formal adjoint with respect to $(\cdot, \cdot)_{L^2(E)}$. In what follows, we will consider the so-called Bochner Laplacian operator $\nabla^* \nabla: C^\infty(E) \rightarrow C^\infty(E)$. For example, if we take $\nabla = d$, where

$d: C^\infty(M) \rightarrow \Omega^1(M)$ is the standard differential, then $d^*d: C^\infty(M) \rightarrow C^\infty(M)$ is just the (non-negative) Laplace–Beltrami operator $-\Delta$.

We are interested in the Schrödinger-type differential expression

$$(2.3) \quad L_V = \nabla^* \nabla / 2 + V,$$

where V is a measurable section of $\text{End } E$ such that $V(x): E_x \rightarrow E_x$ is a self-adjoint operator for almost every $x \in M$.

For every $x \in M$ we have the following canonical decomposition:

$$(2.4) \quad V(x) = V^+(x) - V^-(x),$$

where

$$V^+(x) := P_+(x)V(x) \quad \text{and} \quad V^-(x) := -P_-(x)V(x).$$

Here, $P_+(x) := \chi_{[0,+\infty)}(V(x))$ and $P_-(x) := \chi_{(-\infty,0)}(V(x))$, and χ_G denotes the characteristic function of the set G .

Let $|V^-|$ denote the norm of the operator $V^-(x): E_x \rightarrow E_x$, where $x \in M$. Thus, $|V^-|$ is a (real-valued) measurable function on M .

2.4. Self-adjoint realization of L_V . Assume that $V \in L^1_{\text{loc}}(\text{End } E)$ and $|V^-| \in L^1_{\text{loc}}(M)$. We define S as an operator in $L^2(E)$ by $Su = L_V u$ with the following domain $\text{Dom}(S)$:

$$(2.5) \quad \{u \in L^2(E) : V^+u \in L^1_{\text{loc}}(E), |V^-|u \in L^1_{\text{loc}}(E), \text{ and } L_V u \in L^2(E)\}.$$

Here, the expression $L_V u$ is understood in distributional sense.

Theorem 2.3. *Assume that M is a geodesically complete connected Riemannian manifold with non-negative Ricci curvature. Assume that $V^+ \in L^1_{\text{loc}}(\text{End } E)$ and $|V^-|$ satisfies the property (2.1). Then S is a self-adjoint operator.*

Remark 2.4. The assumption (2.1) on $|V^-|$ enables us to define the operator L_V as a form sum; see Lemma 4.1 below.

Remark 2.5. If $|V^-|$ satisfies (2.1), then $|V^-| \in L^1_{\text{loc}}(M)$; see Lemma 3.3 below.

Remark 2.6. The condition $|V^-|u \in L^1_{\text{loc}}(E)$ in (2.5), which we need in order to apply Theorem 2.1, is stronger than the condition $V^-u \in L^1_{\text{loc}}(E)$. For the operator $L_V = -\Delta/2 + V$ acting on scalar functions, the requirement $V^+u \in L^1_{\text{loc}}(E)$ and $|V^-|u \in L^1_{\text{loc}}(E)$ is equivalent to $Vu \in L^1_{\text{loc}}(M)$. In this case, (2.5) describes the “maximal” realization of L_V in the sense of [18].

3. PROOF OF THEOREM 2.1

We first recall two definitions from [21].

3.1. Contractive Dynkin class and Kato class. Let $p(t, x, y)$ be as in Section 2.1. We say that a measurable function $f: M \rightarrow \mathbb{R}$ belongs to *contractive Dynkin class* relative to $p(t, x, y)$ and write $f \in S^0_{CD}$ if $|f|$ satisfies (2.1). We say that a measurable function $f: M \rightarrow \mathbb{R}$ belongs to *Kato class* relative to $p(t, x, y)$ and write $f \in S^0_K$ if

$$(3.1) \quad \lim_{t \rightarrow 0^+} \sup_{x \in M} \int_0^t \int_M p(s, x, y) |f(y)| d\mu(y) ds = 0.$$

Clearly, we have the inclusion $S^0_K \subset S^0_{CD}$.

Remark 3.1. The term *contractive Dynkin class* was suggested to the author of this paper by B. Güneysu. We remark that the authors of [20] and [21] use the term *Dynkin class* for the class of measurable functions $f: M \rightarrow \mathbb{R}$ such that $|f|$ satisfies (2.1) with 1 on the right hand side replaced by ∞ .

For $\alpha > 0$, set

$$r_\alpha(x, y) := \int_0^\infty e^{-\alpha t} p(t, x, y) dt.$$

For $\alpha > 0$ and $f \in S_{CD}^0$ define

$$(3.2) \quad c_\alpha(f) := \sup_{x \in M} \int_M r_\alpha(x, y) |f(y)| d\mu(y).$$

By Lemma 3.2 in [21], we have $c_\alpha(f) < \infty$ for all $\alpha > 0$. We now set

$$c(f) := \inf_{\alpha > 0} c_\alpha(f).$$

Lemma 3.2. *If $f \in S_{CD}^0$, then $c(f) < 1$.*

Proof. By Lemma 3.1 in [20] (or Proposition 2.7(a) in [12]), for any measurable function $f: M \rightarrow \mathbb{R}$ and all $\alpha, t > 0$ we have

$$\begin{aligned} & (1 - e^{-\alpha t}) \sup_{x \in M} \int_M r_\alpha(x, y) |f(y)| d\mu(y) \\ & \leq \sup_{x \in M} \int_0^t \int_M p(s, x, y) |f(y)| d\mu(y) ds. \end{aligned}$$

Since $f \in S_{CD}^0$, there exists $t = t_0 > 0$ such that the right hand side is less than 1. Consequently, we get

$$c_\alpha(f) < \frac{1}{1 - e^{-\alpha t_0}},$$

for all $\alpha > 0$, and from here $c(f) < 1$ follows easily. □

The following lemma follows from Proposition 2.7(b) in [12]:

Lemma 3.3. *If $f \in S_{CD}^0$, then $f \in L^1_{\text{loc}}(M)$.*

Remark 3.4. The proof of Proposition 2.7(b) in [12] uses strict positivity of $p(t, x, y)$, which requires connectedness of M .

Remark 3.5. In the sequel, for any $x \in M$, the symbol \mathbb{P}^x stands for the law of a Brownian motion X_t on M starting at x , and \mathbb{E}^x denotes the expected value corresponding to \mathbb{P}^x . Our hypothesis on M ensure that M is stochastically complete (see [10]); hence, the lifetime of X_t is $\zeta = \infty$. We should emphasize that in this paper \mathbb{P}^x is $-\Delta/2$ diffusion, as opposed to $-\Delta$ diffusion.

Remark 3.6. We should note that the geodesic completeness and non-negative Ricci curvature assumptions are not used until Lemma 3.11 below. Also, in the absence of stochastic completeness, path-integral formulas below can be rewritten by taking into account the lifetime ζ of X_t .

Lemma 3.7. *If $0 \leq f \in S_{CD}^0$, then there exist constants $\beta > 0$ and $\gamma > 0$ such that*

$$(3.3) \quad \sup_{x \in M} \mathbb{E}^x \left[e^{\int_0^t f(X_s) ds} \right] \leq \beta e^{\gamma t},$$

for all $t > 0$.

Proof. First note that we can write

$$\int_0^t \int_M p(s, x, y) f(y) d\mu(y) ds = \mathbb{E}^x \left[\int_0^t f(X_s) ds \right].$$

By the definition of the class S_{CD}^0 , there exists $t^* > 0$ such that

$$\nu_t := \sup_{x \in M} \mathbb{E}^x \left[\int_0^t f(X_s) ds \right] < 1,$$

for all $0 < t \leq t^*$. By Khasminskii's Lemma (see Lemma 3.37 in [22]) we have

$$\sup_{x \in M} \mathbb{E}^x \left[e^{\int_0^t f(X_s) ds} \right] \leq \frac{1}{1 - \nu_t},$$

for all $0 < t \leq t^*$. From here on we may repeat the proof of Lemma 3.38 of [22] to conclude that

$$\sup_{x \in M} \mathbb{E}^x \left[e^{\int_0^t f(X_s) ds} \right] \leq \left(\frac{1}{1 - \nu_{t^*}} \right)^{\lfloor t/t^* \rfloor + 1},$$

for all $t > 0$, where $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \leq a\}$.

Setting $\beta = \frac{1}{1 - \nu_{t^*}}$ and $\gamma = \frac{1}{t^*} \log \left(\frac{1}{1 - \nu_{t^*}} \right)$, we obtain (3.3). □

3.2. Quadratic forms. In what follows, all quadratic forms are considered in the space $L^2(M)$. Let $w \in L^1_{\text{loc}}(M)$. Set $w^+ := \max(w, 0)$ and $w^- := \max(-w, 0)$, so that $w = w^+ - w^-$. Define

$$Q_0(u) := \frac{1}{2} \int_M |du|^2 d\mu,$$

with the domain $D(Q_0) = \{u \in L^2(M) : Q_0(u) < \infty\}$. The form Q_0 is non-negative, densely defined (since $C_c^\infty(M) \subset D(Q_0)$), and closed. Define $Q_{w^\pm}(u) := \pm(w^\pm u, u)$ with the domain $D(Q_{w^\pm}) = \{u \in L^2(M) : w^\pm |u|^2 \in L^1(M)\}$. The forms Q_{w^\pm} are symmetric and densely defined (since $C_c^\infty(M) \subset D(Q_{w^\pm})$). Note that the form Q_{w^+} is non-negative.

Lemma 3.8. *Assume that $w^- \in S_{CD}^0$. Then there exist $a \in [0, 1)$ and $b \geq 0$ such that*

$$(3.4) \quad |Q_{w^-}(u)| \leq a|Q_0(u)| + b\|u\|^2, \quad \text{for all } u \in D(Q_0).$$

Proof. Let $c_\alpha(w^-)$ be as in (3.2). We have already observed that $w^- \in S_{CD}^0$ implies $c_\alpha(w^-) < \infty$ for all $\alpha > 0$. By Theorem 3.1 in [25] we have

$$(w^- u, u) \leq \frac{c_\alpha(w^-)}{2} \int_M |du|^2 d\mu + \alpha c_\alpha(w^-) \|u\|^2,$$

for all $u \in D(Q_0)$ and all $\alpha > 0$. By Lemma 3.2 we have $c(w^-) < 1$. Hence, there exists α^* such that $c_{\alpha^*}(w^-) < 1$, which shows (3.4). □

By Theorem VI.1.11 in [19] and Example VI.1.15 in [19], the form Q_{w^+} is closed. By Lemma 3.8 above and Theorem VI.1.33 in [19], the form $Q_{0,w} := (Q_0 + Q_{w^+}) + Q_{w^-}$ is densely defined, closed and semi-bounded from below with $D(Q_w) = D(Q_0) \cap D(Q_{w^+}) \subset D(Q_{w^-})$. Let $H(w)$ denote the semi-bounded from below self-adjoint operator in $L^2(M)$ associated to $Q_{0,w}$ by Theorem VI.2.1 of [19].

3.3. Semigroup associated to $H(-w^-)$. As seen from the proof of Lemma 3.8, for $w^- \in S_{CD}^0$, there exists α_* such that $c_{\alpha_*}(w^-) < 1$, and the form $Q_{0,-w^-} := Q_0 + Q_{w^-}$ is semi-bounded from below by $-\alpha_* c_{\alpha_*}(w^-)$. Let $H(-w^-)$ be the corresponding self-adjoint (semi-bounded from below) operator and let $U_{2,-w^-}(t) := e^{-tH(-w^-)}$, $t \geq 0$, be the corresponding C_0 -semigroup in $L^2(M)$. The following lemma was proven in Theorem 3.3 of [25]:

Lemma 3.9. *Assume that $w^- \in S_{CD}^0$. Then, the operators $U_{2,-w^-}(t)$ act as C_0 -semigroups in $L^p(M)$, for all $p \in [1, \infty)$, and we label those semigroups as $U_{p,-w^-}(t)$. Moreover, there exist $C \geq 0$ and $\omega \in \mathbb{R}$ (depending only on α_* and $c_{\alpha_*}(w^-)$) such that*

$$(3.5) \quad \|U_{p,-w^-}(t)\|_{L^p \rightarrow L^p} \leq Ce^{\omega t},$$

for all $p \in [1, \infty)$ and $t \geq 0$.

3.4. Path integral representation of $U_{2,-w^-}(t)$. Let X_t be as in Remark 3.5. For $w^- \in S_{CD}^0$ we have the Feynman–Kac formula

$$(3.6) \quad (U_{2,-w^-}(t)g)(x) = \mathbb{E}^x \left[e^{\int_0^t w^-(X_s) ds} g(X_t) \right],$$

for all $g \in L^2(M)$, all $t \geq 0$, and a.e. $x \in M$. In the Kato-class case $w^- \in S_K^0$, the formula (3.6) was proven in Theorem 2.9 of [11]. The same proof works for $w^- \in S_{CD}^0$ thanks to (3.4) and the following property: $w^- \in S_{CD}^0$ implies

$$\mathbb{P}^x[w^-(X_\bullet) \in L_{\text{loc}}^1[0, \infty)] = 1, \quad \text{a.e. } x \in M.$$

For the latter property see the proof of Lemma 2.4(b) in [11], which works without change for the class S_{CD}^0 instead of S_K^0 .

Lemma 3.10. *If $w^- \in S_{CD}^0$, then for all $g \in L^2(M) \cap L^\infty(M)$ and all $t \geq 0$ we have*

$$\|U_{2,-w^-}(t)g\|_\infty \leq \beta e^{\gamma t} \|g\|_\infty,$$

where $\beta > 0$ and $\gamma > 0$ are some constants.

Proof. The lemma follows by combining (3.6) and (3.3). □

3.5. Cut-off functions. The following lemma was proven in Theorem 2.2 of [13].

Lemma 3.11. *Assume that M is a geodesically complete Riemannian manifold with non-negative Ricci curvature. Then there exists a sequence of functions $\chi_k \in C_c^\infty(M)$ satisfying the properties (C1)–(C4) from Section 1.*

3.6. Sobolev space. Let $\tilde{H}^2(M)$ denote the space of measurable functions $u: M \rightarrow \mathbb{C}$ such that

$$\|u\|_{\tilde{H}^2} := \|u\| + \|du\| + \|\Delta u\| < \infty,$$

where $\|du\|$ denotes the norm in $L^2(\Lambda^1 T^*M)$.

Lemma 3.12. *Assume that M is a geodesically complete Riemannian manifold with non-negative Ricci curvature. Let $0 \leq u \in \tilde{H}^2(M)$. Then there exists a sequence of functions $0 \leq u_k \in C_c^\infty(M)$ such that $\|u_k - u\|_{\tilde{H}^2} \rightarrow 0$, as $k \rightarrow \infty$.*

Proof. In this proof, $(\tilde{H}^2(M))^+$ and $(C_c^\infty(M))^+$ denote the sets of non-negative elements of $\tilde{H}^2(M)$ and $C_c^\infty(M)$ respectively. Let $u \in (\tilde{H}^2(M))^+$ and let χ_k be the sequence of cut-off functions as in Lemma 3.11. We will first show that the set of compactly supported elements of $(\tilde{H}^2(M))^+$ is dense in $(\tilde{H}^2(M))^+$. To do this, first note that

$$(3.7) \quad d(\chi_k u) = u d\chi_k + \chi_k du$$

and

$$(3.8) \quad \Delta(\chi_k u) = \chi_k(\Delta u) + 2\langle d\chi_k, du \rangle + u(\Delta\chi_k).$$

If we denote the Riemannian metric of M by $r = (r_{jk})$, the notation $\langle \kappa, \psi \rangle$ in (3.8) for 1-forms $\kappa = \kappa_j dx^j$ and $\psi = \psi_k dx^k$ means

$$\langle \kappa, \psi \rangle := r^{jk} \kappa_j \psi_k,$$

where (r^{jk}) is the inverse matrix to (r_{jk}) , and the standard Einstein summation convention is understood. Now the property $\|\chi_k u - u\|_{\tilde{H}^2} \rightarrow 0$, as $k \rightarrow \infty$, easily follows from (3.7), (3.8), and (C1)–(C4). This shows that the set of compactly supported elements of $(\tilde{H}^2(M))^+$ is dense in $(\tilde{H}^2(M))^+$. It remains to show that $(C_c^\infty(M))^+$ is dense in the set of compactly supported elements of $(\tilde{H}^2(M))^+$. To see this, we start with a compactly supported element $u \in (\tilde{H}^2(M))^+$. Since the support of u is compact, using a partition of unity, we may assume that u is supported in a coordinate chart (G, ϕ) of M such that $\phi(G) = K_1$, where K_1 is an open ball of radius 1 in \mathbb{R}^n . Applying the Friedrichs mollification procedure to $u \circ \phi^{-1}$, we obtain a sequence of non-negative smooth functions v_j with support in K_1 converging to $u \circ \phi^{-1}$ with respect to $\|\cdot\|_{W^{2,2}}$, as $j \rightarrow \infty$, where $\|\cdot\|_{W^{k,p}}$ stands for the usual Sobolev norm in \mathbb{R}^n , with k indicating the highest derivative and p the corresponding L^p -space. Then $v_j \circ \phi$ converges to u in the norm $\|\cdot\|_{\tilde{H}^2}$, as $j \rightarrow \infty$. \square

With the above preparations, the proof of Theorem 2.1 proceeds as that of (PPP) in [6].

Proof of Theorem 2.1. Let F be as in hypotheses of the theorem. Define $F_k := \min(F, k)$, $k \in \mathbb{Z}_+$, and consider the semigroup $U_{2,-F_k}(t)$ as in Section 3.3. Denote the generator of this semigroup by $H(-F_k)$. As $F_k \in L^\infty(M)$ and M is geodesically complete, it is well known that $(-\Delta/2 - F_k)|_{C_c^\infty(M)}$ is essentially self-adjoint and its (self-adjoint) closure is $\overline{(-\Delta/2 - F_k)|_{C_c^\infty(M)}} u = (-\Delta/2 - F_k)u$, for all

$$u \in \text{Dom}(\overline{(-\Delta/2 - F_k)|_{C_c^\infty(M)}}) = \{u \in L^2(M) : \Delta u \in L^2(M)\}.$$

Furthermore, by Theorem VI.2.9 in [19], the operator $\overline{(-\Delta/2 - F_k)|_{C_c^\infty(M)}}$ coincides with $H(-F_k)$, which, in turn, coincides with the operator $\text{sum } H(0) - F_k$, where F_k stands for the corresponding multiplication operator by the function F_k .

Noting $-F_k \geq -F$ and using the representation (3.6) together with (3.5) we have

$$(3.9) \quad \|U_{2,-F_k}(t)\|_{L^2 \rightarrow L^2} \leq \|U_{2,-F}(t)\|_{L^2 \rightarrow L^2} \leq C e^{\omega t},$$

where $U_{2,-F}(t)$ is the semigroup corresponding to $H(-F)$ as in Section 3.3.

Let $\lambda_* := \max\{\omega, \gamma, \alpha_* c_{\alpha_*}(F)\}$, where γ is as in Lemma 3.7 and $\alpha_* c_{\alpha_*}(F)$ is as in Section 3.3. For $\lambda > \lambda_*$ the (linear) operator $(\lambda + H(-F_k))^{-1} : L^2(M) \rightarrow L^2(M)$ is bounded. Let $g \in L^2(M) \cap L^\infty(M)$ and $g \geq 0$. For $k \in \mathbb{Z}_+$, define

$$(3.10) \quad u_k := (\lambda + H(-F_k))^{-1}g.$$

Using the representation

$$(3.11) \quad (\lambda + H(-F_k))^{-1}g = \int_0^\infty e^{-\lambda t} U_{2,-F_k}(t)g \, dt,$$

the estimate (3.9) and the inequality $\lambda > \lambda_*$, we obtain

$$(3.12) \quad \|u_k\| \leq C \int_0^\infty e^{-(\lambda-\omega)t} \|g\| \, dt \leq C_1 \|g\|,$$

for all $k \in \mathbb{Z}_+$, with some constant $C_1 \geq 0$.

Note that $u_k \geq 0$ by (3.11), (3.6) and the assumption $g \geq 0$. By Lemma 3.10 we have

$$(3.13) \quad \begin{aligned} 0 \leq u_k(x) &= \int_0^\infty e^{-\lambda t} U_{2,-F_k}(t)g \, dt \\ &\leq \int_0^\infty e^{-\lambda t} \|U_{2,-F_k}(t)g\|_\infty \, dt \\ &\leq \beta \int_0^\infty e^{-(\lambda-\gamma)t} \|g\|_\infty \, dt \leq C_2 \|g\|_\infty, \end{aligned}$$

where $C_2 \geq 0$ is a constant, and in the last inequality we used $\lambda > \lambda_* \geq \gamma$.

By the definition of u_k we have

$$(\lambda + H(-F_k))u_k = g.$$

Taking the inner product in $L^2(M)$ with u_k , using the fact that $H(0)$ is the operator associated to the form Q_0 , and recalling the inequality $-F_k \geq -F$, we obtain

$$\begin{aligned} (g, u_k) &= ((\lambda + H(0) - F_k)u_k, u_k) = \lambda \|u_k\|^2 + \frac{1}{2} \int_M |du_k|^2 \, d\mu - (F_k u_k, u_k) \\ &\geq \lambda \|u_k\|^2 + \frac{1}{2} \int_M |du_k|^2 \, d\mu - (F u_k, u_k), \end{aligned}$$

which, upon combining with (3.4) and rearranging, leads to

$$(g, u_k) \geq \frac{1-a}{2} \int_M |du_k|^2 \, d\mu + (\lambda - b) \|u_k\|^2.$$

From the last inequality we get

$$(3.14) \quad \begin{aligned} \frac{1-a}{2} \int_M |du_k|^2 \, d\mu &\leq (g, u_k) + (b - \lambda) \|u_k\|^2 \\ &\leq |b - \lambda| (C_1)^2 \|g\|^2 + C_1 \|g\|^2, \end{aligned}$$

where in the last estimate we used Cauchy–Schwarz inequality and (3.12).

Let $0 \leq \psi \in C_c^\infty(M)$, let u be as in the hypothesis of the theorem, and let $0 \leq g \in L^2(M) \cap L^\infty(M)$ and u_k be as in (3.10). We have the following equality:

$$(3.15) \quad (\psi u, g) = (\psi u, (-\Delta/2 + \lambda - F_k)u_k).$$

Using (3.14) and the property

$$0 \leq u_k \in \text{Dom}(H(-F_k)) = \{v \in L^2(M) : \Delta v \in L^2(M)\},$$

we have $0 \leq u_k \in \tilde{H}^2(M)$. Thus, by Lemma 3.12, without loss of generality, we may assume that $0 \leq u_k \in C_c^\infty(M)$ in (3.15), which we will do from now on.

Using (3.7) and (3.8) we have

$$\begin{aligned} (\psi u, (-\Delta/2 + \lambda - F_k)u_k) &= ((-\Delta/2)(\psi u), u_k) + \lambda(u, \psi u_k) \\ &+ ((F - F_k)\psi u, u_k) - (F\psi u, u_k) \\ &= ((-\Delta/2 + \lambda - F)u, \psi u_k) + (((-\Delta/2)\psi)u, u_k) \\ &- (du, (d\psi)u_k) + ((F - F_k)\psi u, u_k) \\ &\geq (((-\Delta/2)\psi)u, u_k) - (du, (d\psi)u_k) + ((F - F_k)\psi u, u_k), \end{aligned}$$

where in the last inequality we used $0 \leq \psi u_k \in C_c^\infty(M)$ and the assumption (2.2). Using the fact that $-\Delta = d^*d$ we have

$$\begin{aligned} (3.16) \quad (du, (d\psi)u_k) &= (u, d^*((d\psi)u_k)) = (u, (d^*d\psi)u_k) - (ud\psi, du_k) \\ &= ((-\Delta\psi)u, u_k) - (ud\psi, du_k), \end{aligned}$$

which upon combining with the preceding estimate and (3.15) leads to

$$(3.17) \quad (\psi u, g) \geq ((F - F_k)\psi u, u_k) + ((\Delta/2)\psi)u, u_k) + (ud\psi, du_k).$$

Let us replace $0 \leq \psi \in C_c^\infty(M)$ by a sequence χ_m of cut-off functions from Lemma 3.11. Using (3.12), (3.14), and the properties of χ_m , it is easy to see that the last two terms on the right hand side of (3.17) converge to 0 as $m \rightarrow \infty$. We now consider the term $((F - F_k)\chi_m u, u_k)$. For a fixed $m \in \mathbb{Z}_+$, using the property (3.13) we have

$$(F - F_k)\chi_m u u_k \rightarrow 0, \quad \text{a.e. } x \in M, \quad \text{as } k \rightarrow \infty$$

and

$$|(F - F_k)\chi_m u u_k| \leq C_3 \chi_m F |u| \in L^1(M),$$

where $C_3 \geq 0$ is some constant. Thus, by dominated convergence theorem we have

$$((F - F_k)\chi_m u, u_k) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Returning to (3.17) and using our convergence observations, together with property (C2) of χ_m , we obtain $(u, g) \geq 0$. Since $0 \leq g \in L^2(M) \cap L^\infty(M)$ is arbitrary, we get $u \geq 0$ a.e. \square

4. PROOF OF THEOREM 2.3

Let E, ∇ , and V be as in hypotheses of Theorem 2.3. We begin by describing $L^2(E)$ analogues of quadratic forms from Section 3.2.

4.1. Quadratic forms in vector-bundle setting. Define

$$Q_{\nabla,0}(u) := \frac{1}{2} \int_M |\nabla u|^2 d\mu$$

with the domain $D(Q_{\nabla,0}) = \{u \in L^2(E) : \nabla u \in L^2(T^*M \otimes E)\}$. Note that $Q_{\nabla,0}$ is non-negative, densely defined, and closed. Next we define $Q_{V^\pm}(u) = \pm(V^\pm u, u)$ with the domain $D(Q_{V^\pm}) = \{u \in L^2(E) : \langle V^\pm u, u \rangle \in L^1(M)\}$. The forms Q_{V^\pm} are densely defined and symmetric. Note that the form Q_{V^+} is non-negative.

Lemma 4.1. *Let V^- be as in hypotheses of Theorem 2.3. Then there exist $a \in [0, 1)$ and $b \geq 0$ such that*

$$(4.1) \quad \int_M \langle V^- u, u \rangle d\mu \leq (a/2) \|\nabla u\|_{L^2(T^*M \otimes E)}^2 + b \|u\|_{L^2(E)}^2,$$

for all $u \in D(Q_{\nabla,0})$.

Proof. Let $u \in D(Q_{\nabla,0})$ and let Q_0 be as in Section 3.2. By Corollary 2.5 in [12] we have $|u| \in D(Q_0)$, and

$$(4.2) \quad \|d|u|\|_{L^2(\Lambda^1 T^*M)}^2 \leq \|\nabla u\|_{L^2(T^*M \otimes E)}^2.$$

Using (3.4) and (4.2) we obtain

$$\begin{aligned} \int_M \langle V^- u, u \rangle d\mu &\leq \int_M |V^-| |u|^2 d\mu \leq (a/2) \|d|u|\|_{L^2(\Lambda^1 T^*M)}^2 + b \|u\|_{L^2(M)}^2 \\ &\leq (a/2) \|\nabla u\|_{L^2(T^*M \otimes E)}^2 + b \|u\|_{L^2(E)}^2, \quad \text{for all } u \in D(Q_{\nabla,0}), \end{aligned}$$

where a and b are as in (3.4). □

By Theorem VI.1.11 in [19] and Example VI.1.15 in [19], the form Q_{V^+} is closed. As a consequence of Lemma 4.1, analogously as in Section 3.2, the form $Q_{\nabla,V} := Q_0 + Q_{V^+} + Q_{V^-}$ is densely defined, closed and semi-bounded from below with $D(Q_{\nabla,V}) = D(Q_{\nabla,0}) \cap D(Q_{V^+}) \subset D(Q_{V^-})$. Let $H_{\nabla}(V)$ denote the semi-bounded from below self-adjoint operator in $L^2(E)$ associated to $Q_{\nabla,V}$.

4.2. Description of $H_{\nabla}(V)$. By Lemma 3.3 we have $|V^-| \in L^1_{\text{loc}}(M)$, which together with (4.1) and geodesic completeness of M , means that the hypotheses of Theorem 1.2 in [23] are satisfied. The latter theorem gives the following description of $H_{\nabla}(V)$:

$$\text{Dom}(H_{\nabla}(V)) = \{u \in D(Q_{\nabla,0}) : \langle V^+ u, u \rangle \in L^1(M) \text{ and } L_V u \in L^2(E)\}$$

and $H_{\nabla}(V)u = L_V u$, for all $u \in \text{Dom}(H_{\nabla}(V))$, where L_V is as in (2.3).

In the proof of Theorem 2.3 we will use Kato’s inequality for Bochner Laplacian, whose proof is given in Theorem 5.7 of [2].

Lemma 4.2. *Let M be a connected Riemannian manifold (not necessarily geodesically complete). Let E be a Hermitian vector bundle over M , and let ∇ be a Hermitian connection on E . Assume that $w \in L^1_{\text{loc}}(E)$ and $\nabla^* \nabla w \in L^1_{\text{loc}}(E)$. Then*

$$(4.3) \quad -\Delta|w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign } w \rangle_{E_x},$$

where

$$\text{sign } w(x) = \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3. The original version of Kato’s inequality was proven in [17].

Proof of Theorem 2.3. Note that for all $u \in \text{Dom}(H_{\nabla}(V))$ we have $\langle V^+ u, u \rangle \in L^1(M)$ and $\langle V^- u, u \rangle \in L^1(M)$, where the latter inclusion follows by (4.1). Thus, as observed in (4.3) of [23], the mentioned two inclusions and hypotheses on V imply $V^+ u \in L^1_{\text{loc}}(E)$ and $|V^-|u \in L^1_{\text{loc}}(E)$. Now we just compare the descriptions of $H_{\nabla}(V)$ and S to conclude that the operator relation $H_{\nabla}(V) \subset S$ holds.

It remains to prove that $\text{Dom}(S) \subset \text{Dom}(H_{\nabla}(V))$. Let $u \in \text{Dom}(S)$. Let λ_* be as in Theorem 2.1. Since $H_{\nabla}(V)$ is a semi-bounded from below (self-adjoint) operator,

we can select $\lambda > \lambda_*$ large enough so that $H_{\nabla}(V) + \lambda$ is a positive self-adjoint operator. With this selection of λ , the operator $(H_{\nabla}(V) + \lambda)^{-1}: L^2(E) \rightarrow L^2(E)$ is bounded. Since $u \in \text{Dom}(S)$, we may define

$$v := (H_{\nabla}(V) + \lambda)^{-1}(S + \lambda)u$$

and write

$$(H_{\nabla}(V) + \lambda)v = (S + \lambda)u.$$

Since $H_{\nabla}(V) \subset S$, we can rewrite the last equality as

$$(4.4) \quad (S + \lambda)w = 0,$$

where $w := u - v$.

Since $w \in \text{Dom}(S)$, we have $V^+w \in L^1_{\text{loc}}(E)$ and $|V^-|w \in L^1_{\text{loc}}(E)$. Furthermore, from (4.4) we get

$$(\nabla^*\nabla/2)w = -Vw - \lambda w \in L^1_{\text{loc}}(E).$$

By Lemma 4.2 we have

$$\begin{aligned} -(\Delta/2)|w| &\leq \text{Re}\langle(\nabla^*\nabla/2)w, \text{sign } w\rangle_{E_x} = \text{Re}\langle-(V + \lambda)w, \text{sign } w\rangle_{E_x} \\ &\leq (|V^-| - \lambda)|w|, \end{aligned}$$

which leads to

$$(4.5) \quad (-\Delta/2 - |V^-| + \lambda)|w| \leq 0.$$

Since $|V^-||w| \in L^1_{\text{loc}}(M)$, we may use Theorem 2.1 with $F = |V^-|$ to conclude $|w| \leq 0$ a.e. on M . This shows that $w = 0$ a.e. on M , i.e., $u = v$ a.e. on M ; therefore, $u \in \text{Dom}(H_{\nabla}(V))$. \square

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