

A TOPOLOGICAL CHARACTERIZATION FOR NON-WANDERING SURFACE FLOWS

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ABSTRACT. Let v be a continuous flow with arbitrary singularities on a compact surface. Then we show that if v is non-wandering, then v is topologically equivalent to a C^∞ flow such that $\text{Per}(v)$ is open, there are no exceptional orbits, and that $P \sqcup \text{Sing}(v) = \{x \in M \mid \omega(x) \cup \alpha(x) \subseteq \text{Sing}(v)\}$, where P is the union of non-closed proper orbits and \sqcup is the disjoint union symbol. Moreover, v is non-wandering if and only if $\overline{LD} \sqcup \text{Per}(v) \supseteq M - \text{Sing}(v)$, where LD is the union of locally dense orbits and \overline{A} is the closure of a subset $A \subseteq M$. On the other hand, v is topologically transitive if and only if v is non-wandering such that $\text{int}(\text{Per}(v) \sqcup \text{Sing}(v)) = \emptyset$ and $M - (P \sqcup \text{Sing}(v))$ is connected, where $\text{int}A$ is the interior of a subset $A \subseteq M$. In addition, we construct a smooth flow on \mathbb{T}^2 with $\overline{P} = \overline{LD} = \mathbb{T}^2$.

1. INTRODUCTION AND PRELIMINARIES

In [19], H. Poincaré has constructed a flow on a torus with an exceptional minimal set. A. Denjoy has constructed such a C^1 -flow [7]. On the other hand, the author has shown that there are no exceptional minimal sets of C^2 -flows on tori. This result is generalized to the compact surface cases by A. J. Schwartz [21]. On the other hand, in [18] and [6], they have shown the characterization of the non-wandering flows on compact surfaces with finitely many singularities. In [17], one has established a topological invariant of such flows. In [14], one has given a description near orbits of the non-wandering flow with the set of singularities which is totally disconnected. Moreover, in [12], it is shown that if a non-trivial recurrent point x of a surface flow belongs to the limit set of another non-trivial recurrent point y , then y belongs to the limit set of x . We sharpen these results to analysis surface flows. In particular, it is shown that the orbit class of each non-trivial weakly recurrent point is the orbit closure in the set of regular weakly recurrent points. In this paper, we show the non-existence of exceptional minimal sets of continuous non-wandering flows with arbitrary singularities on compact surfaces, and obtain a topological characterization of non-wandering flows and a topological characterization of non-periodic proper orbits of non-wandering flows. Moreover, we give a smoothability of non-wandering flows and a characterization of topological transitivity for continuous flows. In addition, we construct a smooth flow on \mathbb{T}^2

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and a foliation on an open manifold contained in \mathbb{R}^2 such that LD and P are dense, where LD is the union of locally dense orbits (resp. leaves) and P is the union of non-closed proper orbits (resp. leaves). On the other hand, if $\text{Sing}(v)$ consists of finitely many contractible connected components, then LD is open and P consists of finitely many orbits.

By flows, we mean continuous \mathbb{R} -actions on surfaces. Let $v : \mathbb{R} \times M \rightarrow M$ be a flow on a compact surface M . Put $v_t(\cdot) := v(t, \cdot)$ and $O_v(\cdot) := v(\mathbb{R}, \cdot)$. A subset of M is said to be saturated if it is a union of orbits. Recall that a point x of M is singular if $x = v_t(x)$ for any $t \in \mathbb{R}$, is regular if x is not singular, and is periodic if there is positive number $T > 0$ such that $x = v_T(x)$ and $x \neq v_t(x)$ for any $t \in (0, T)$. Denote by $\text{Sing}(v)$ (resp. $\text{Per}(v)$) the set of singular (resp. periodic) points. A point x is wandering if there are a neighbourhood U of x and a positive number N such that $\bigcup_{t > N} v_t(U) \cap U = \emptyset$, and is non-wandering if x is not wandering (i.e. for each neighbourhood U of x and each positive number N , there is $t \in \mathbb{R}$ with $|t| > N$ such that $v_t(U) \cap U \neq \emptyset$). An orbit is non-wandering if it consists of non-wandering points and the flow v is non-wandering if every point is non-wandering. For a point $x \in M$, define the omega limit set $\omega(x)$ and the alpha limit set $\alpha(x)$ of x as follows: $\omega(x) := \bigcap_{n \in \mathbb{R}} \overline{\{v_t(x) \mid t > n\}}$, $\alpha(x) := \bigcap_{n \in \mathbb{R}} \overline{\{v_t(x) \mid t < -n\}}$. A point x of M is positive recurrent (resp. negative recurrent) if $x \in \omega(x)$ (resp. $x \in \alpha(x)$), and x is recurrent (resp. weakly recurrent) if x is positive and (resp. or) negative recurrent. A (weakly) recurrent orbit is an orbit of such a point. An orbit is proper if it is embedded, locally dense if the closure of it has non-empty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if so is its orbit. Denote by LD (resp. E, P) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Note P is the complement of the set of weakly recurrent points. By the definitions, we have a decomposition $\text{Sing}(v) \sqcup \text{Per}(v) \sqcup P \sqcup \text{LD} \sqcup E = M$. A (weakly) recurrent orbit is non-trivial if it is not closed. Note that the union of non-trivial weakly recurrent orbits corresponds with $\text{LD} \sqcup E$. A quasi-minimal set of v is the closure of a non-trivial weakly recurrent orbit. It's known that the total number of quasi-minimal sets for v cannot exceed g if M is an orientable surface of genus g [12], and $\frac{p-1}{2}$ if M is a non-orientable surface of genus p [1], [13]. Therefore the closure $\overline{\text{LD} \sqcup E}$ is a finite union of quasi-minimal sets. By a limit cycle, we mean a periodic orbit of v which is the α -limit set or the ω -limit set of some point not on the periodic orbit.

2. A TOPOLOGICAL CHARACTERIZATION OF NON-WANDERING SURFACE FLOWS WITH ARBITRARY SINGULARITIES

Let v be a continuous flow on a compact surface M . We note that a collar A of a periodic orbit $O \subset A$ is an annulus A , one of whose connected component of ∂A is O , where $\partial A := \overline{A} - \text{int}A$ is the topological boundary of A . First, we state the following easy observation.

Lemma 2.1. *$O \subset \text{Per}(v) \sqcup P$ for an orbit O with $\overline{O} \cap \text{Per}(v) \neq \emptyset$. Moreover, each limit cycle is contained in $\partial(\text{int}P)$.*

Proof. Let O be a non-periodic orbit with $\overline{O} \cap \text{Per}(v) \neq \emptyset$. Then $\overline{O} - O$ contains a limit cycle γ . The flow box theorem (cf. Theorem 1.1, p.45, [2]) implies that a limit cycle γ is covered by finitely many flow boxes $\{U'_i\}$. Since $\overline{O} \cap \gamma \neq \emptyset$, the one-sided holonomy of γ is contracting or expanding. Fix a point z of O in a flow box U'_i .

Then the point z has an open neighbourhood $U \subset U'_i$ which is a flow box such that $O \cap U$ is one arc. Then $\bigcup_{t \in \mathbb{R}} v_t(U)$ is an open neighbourhood of O in which O is closed. Hence $O \subset P$. Let $U' \subset \bigcup_i U'_i$ be a sufficiently small collar of γ where the holonomy along γ is contracting or expanding. Then the orbit closure of each point y in $U' - \gamma$ contains γ but the orbit of y is not closed. Therefore $y \in P$ and so $V := \bigcup_{t \in \mathbb{R}} v_t(U') - \gamma \subseteq P$ is a saturated open subset with $\gamma \subseteq \partial V \subseteq \partial(\text{int}P)$. This implies the second assertion. \square

Recall that the orbit class \hat{O} of an orbit O is the union of points, each of whose orbit closure corresponds with \overline{O} (i.e. $\hat{O} := \{y \in M \mid \overline{O} = \overline{O_v(y)}\}$). Now we show that the orbit class of each non-trivial weakly recurrent point is the orbit closure in the set of regular weakly recurrent points, which refine a Maier-type result [12].

Proposition 2.2. *For an orbit $O \subset \text{LD} \sqcup \text{E}$, we have $\hat{O} = \overline{O} \setminus (\text{Sing}(v) \sqcup P) \subset \text{LD} \sqcup \text{E}$.*

Proof. Let $Q := \overline{O}$ be a quasi-minimal set. By Lemma 2.1, we have $Q \cap \text{Per}(v) = \emptyset$. Note that the inverse image of proper (resp. locally dense, exceptional) orbits by any finite covering are also proper (resp. locally dense, exceptional). By taking a double covering of M and the doubling of M if necessary, we may assume that v is transversally orientable and M is closed and orientable. For any point $y \in \hat{O} - O$, we have $\overline{O_v(y)} = \overline{O}$ and so $O \subseteq \omega(y) \cup \alpha(y)$. Since $y \in \overline{O} - O$, we obtain $y \in \omega(y) \cup \alpha(y)$. Then y is not proper. The regularity of y implies $y \notin \text{Sing}(v)$ and so $y \in Q \setminus (\text{Sing}(v) \sqcup P) \subseteq \text{LD} \sqcup \text{E}$. Thus $\hat{O} \subseteq Q \setminus (\text{Sing}(v) \sqcup P)$. On the other hand, we show that $\hat{O} \supseteq Q \setminus (\text{Sing}(v) \sqcup P)$. Indeed, if there is exactly one quasi-minimal set Q , then $\overline{O_v(x)} = Q$ for any $x \in Q \setminus (\text{Sing}(v) \sqcup P) \subseteq \text{LD} \sqcup \text{E}$. Thus we may assume that there are at least two quasi-minimal sets. The above Maier work [12] (cf. Remark 2 [3]) implies that the genus of M is at least two. Note Cherry has proved that a quasi-minimal set contains a continuum of non-trivially recurrent orbits, each of which is dense in the quasi-minimal set (Theorem VI [5]). Therefore Q contains non-trivially recurrent orbits. Fix a recurrent point $x \in (\text{LD} \sqcup \text{E}) \cap Q$ whose orbit closure is Q . For any point $y \in Q \setminus (\text{Sing}(v) \sqcup P)$, we have $y \in \text{LD} \sqcup \text{E}$ and so y is weakly recurrent. By the above Cherry result, there is a recurrent point $z \in \overline{O_v(y)}$ whose orbit closure is $\overline{O_v(y)}$. Since $z \in Q = \omega(x) = \alpha(x)$, by another Maier theorem (cf. Theorem 4.2 [3]) and its dual, we obtain $\omega(x) = \omega(z)$ and $\alpha(x) = \alpha(z)$. Thus $\overline{O} = Q = \overline{O_v(x)} = \overline{O_v(z)} = \overline{O_v(y)}$. This means $\hat{O} = Q \setminus (\text{Sing}(v) \sqcup P)$. \square

We state a key lemma which is a relation between exceptional and proper orbits. Recall that a subset S of a surface M is essential if S is not contractible in M .

Lemma 2.3. $\overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup \text{LD}} \cap \text{E} = \emptyset$ and $\overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{E}}} \cap \text{LD} = \emptyset$. Moreover, $\text{E} \subset \text{int}P$.

Proof. By taking a double covering of M and the doubling of M if necessary, we may assume that v is transversally orientable and M is closed and orientable. By the Maier theorem, $\overline{\text{E}}$ (resp. $\overline{\text{LD}}$) is a finite union of closures of exceptional (resp. locally dense) orbits. By Proposition 2.2, we have $\overline{\text{LD}} \cap \text{E} = \emptyset$ and $\text{LD} \cap \overline{\text{E}} = \emptyset$. Recall that the flow box theorem implies that the orbits of v on a surface $M - \text{Sing}(v)$ form a foliation \mathcal{F} . Moreover, there is a transverse foliation \mathcal{L} for this foliation \mathcal{F} by Proposition 2.3.8 (p.18 [10]). We show $\overline{\text{Per}(v)} \cap (\text{E} \sqcup \text{LD}) = \emptyset$. Otherwise there is a sequence of periodic orbits O_i such that the closure $\overline{\bigcup_i O_i}$ contains a point $x \in \text{E} \sqcup \text{LD}$. By Lemma 2.1, we have $\overline{O_v(x)} \cap \text{Per}(v) = \emptyset$. We show that there

is $K \in \mathbb{Z}_{>0}$ such that O_k is contractible in M_K for any $k > K$ where M_K is the resulting closed surface of adding $2K$ disks to $M - (O_1 \sqcup \dots \sqcup O_K)$. Indeed, we may assume that O_1 is essential by renumbering. Let M_1 be the resulting closed surface of adding two center disks to $M - O_1$. Then $g(M_1) < g(M)$, where $g(N)$ is the genus of a surface N . Since M and M_1 are closed surfaces, by induction for essential closed curves at most $g(M)$ times, the assertion is followed. Since M is normal, there are open disjoint neighbourhoods U_x and V of $\overline{O_v(x)}$ and $\bigcup_{i \leq K} O_i$ respectively. Then there is a transverse arc $\gamma \subset U_x \cap l_x$ through x which does not intersect $\bigcup_{i \leq K} O_i$, where $l_x \in \mathcal{L}$. Since $O_v(x)$ is exceptional or locally dense, we have x is a weakly recurrent point and so there is an arc γ' in $O_v(x)$ whose boundaries are contained in γ . Since M is normal, there is a neighbourhood of $\gamma \cup \gamma'$ which does not intersect $\bigcup_{i \leq K} O_i$. By the waterfall construction (cf. Lemma 1.2, p.46, [2]), we can construct a closed transversal T for v through x which does not intersect $\bigcup_{i \leq K} O_i$. Let v_K be a resulting flow on M_K by adding center disks. Then T is also a closed transversal for v_K . However there is $k > K$ such that T intersects O_k which is contractible in M_K . This is impossible. The closedness of $\text{Sing}(v)$ implies the first assertion. Since $M = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{LD}} \sqcup \text{P} \sqcup \text{E}$, we obtain that E is contained in an open subset $M - \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{LD}} \subseteq \text{P} \sqcup \text{E}$. Therefore $\text{E} \subseteq \text{int}(\text{P} \sqcup \text{E})$. Since $\overline{\text{E}}$ is a finite union of closures of exceptional orbits, we have that E is nowhere dense. This implies $\text{E} \subset \overline{\text{P}}$ and so $\text{E} \subset \text{int}(\text{P} \sqcup \text{E}) \subset \text{int}\overline{\text{P}}$. \square

From now on, we consider only non-wandering cases in this section.

Lemma 2.4. *Let v be a non-wandering flow on a compact surface M . Then $\text{Per}(v)$ is open, $M = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{LD}} \sqcup \text{P}$, and $\overline{\text{LD}} \sqcup \text{Per}(v) \supseteq M - \text{Sing}(v)$. In particular, each quasi-minimal set has non-empty interior.*

Proof. By taking a double covering of M if necessary, we may assume that v is transversally orientable. By Theorem III.2.12, III.2.15 [4], the set of recurrence points is dense in M . Write $U := M - \overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{LD}}}$. By Lemma 2.3, this $U \subseteq \text{P} \sqcup \text{E}$ is an open neighbourhood of E . Since each point of P is not weakly recurrent, we have $\overline{\text{E}} \supseteq U$. Since E is nowhere dense, we have U is empty and so is E (i.e. each quasi-minimal set has non-empty interior). Hence $M = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \overline{\text{LD}} \sqcup \text{P}$. By Lemma 2.1, there are no limit cycles. Since there are no limit cycles and since $\overline{\text{LD}}$ is a finite union of quasi-minimal sets, we obtain $\overline{\text{LD}} \cap \text{Per}(v) = \emptyset$. Fix a periodic orbit O . Then there is an annular neighbourhood V of O which is a finite union of flow boxes such that $V \cap (\text{Sing}(v) \sqcup \overline{\text{LD}}) = \emptyset$. Hence $V \subseteq \text{Per}(v) \sqcup \text{P}$. Since the set of recurrence points is dense in M , we have that $V \cap \text{Per}(v)$ is dense in V . Therefore the holonomy of O is identical and so $V \subseteq \text{Per}(v)$. This means that $\text{Per}(v)$ is open. For any $x \in \text{P}$, since the regular weakly recurrent points form $\text{Per}(v) \sqcup \overline{\text{LD}}$, by non-wandering property, each neighbourhood of x meets $\text{Per}(v) \sqcup \overline{\text{LD}}$ and so $\overline{\text{LD}} \sqcup \text{Per}(v) \supseteq \text{P}$. \square

Now we state the characterization of non-wandering flows.

Theorem 2.5. *Let v be a continuous flow on a compact surface M . Then v is non-wandering if and only if $\overline{\text{LD}} \sqcup \text{Per}(v) \cup \text{Sing}(v) = M$. In particular, if v is non-wandering, then $\text{Per}(v)$ is open and there are no exceptional orbits.*

Proof. Suppose that v is non-wandering. By Lemma 2.4, we have $\overline{\text{LD} \sqcup \text{Per}(v)} \cup \text{Sing}(v) = M$. Conversely, suppose that $\overline{\text{LD} \sqcup \text{Per}(v)} \cup \text{Sing}(v) = M$. For any regular point x of M , we have $x \in \overline{\text{LD} \sqcup \text{Per}(v)}$. This shows that v is non-wandering. \square

We state the following characterization of the union $\text{P} \sqcup \text{Sing}(v)$ of non-periodic proper orbits.

Proposition 2.6. *Let v be a continuous non-wandering flow on a compact surface M . Then $\text{P} \sqcup \text{Sing}(v) = \{x \in M \mid \omega(x) \cup \alpha(x) \subseteq \text{Sing}(v)\}$.*

Proof. We may assume M is connected. By taking a double covering of M and the doubling of M if necessary, we may assume that v is transversally orientable and M is closed and orientable. Obviously $\text{P} \sqcup \text{Sing}(v) \supseteq \{x \in M \mid \omega(x) \cup \alpha(x) \subseteq \text{Sing}(v)\}$. Therefore it suffices to show this converse relation. Fix a point $y \in \text{P}$. By Lemma 2.1, the non-wandering property implies $\overline{O_v(y)} \cap \text{Per}(v) = \emptyset$. We show $\overline{O_v(y)} \cap \text{LD} = \emptyset$. Otherwise $\overline{O_v(y)}$ contains a locally dense orbit O . Since $\text{int} \overline{O}$ is an open neighbourhood of O , we have $O_v(y) \cap \text{int} \overline{O} \neq \emptyset$. The relation $y \in \overline{O} \subseteq \overline{O_v(y)}$ implies that y is not proper, which is a contradiction. Since $\text{E} = \emptyset$, we have $\overline{O_v(y)} \subset \text{P} \sqcup \text{Sing}(v)$. Suppose that $y \in \overline{\text{LD}}$. Thus there is a recurrent point in LD whose orbit closure contains $O_v(y)$. Removing the singular points, Theorem 3.1, [15] implies $\omega(y) \cup \alpha(y) \subseteq \text{Sing}(v)$. This means $y \in \{x \in M \mid \omega(x) \cup \alpha(x) \subseteq \text{Sing}(v)\}$. Otherwise $y \notin \overline{\text{LD}}$. Suppose that each periodic orbit is null homotopic (i.e. non-essential). Fix a saturated neighbourhood W of y . Then the intersection $W \cap \text{Per}(v)$ is open dense in W . To apply Theorem 3.1, [15], we need to replace a small flow box near $O_v(y)$. Take a flow box $B \subset W$ which can be identified with $[-1, 1] \times [-1, 1]$ such that $O_v(y) \cap B = [-1, 1] \times \{0\}$ and each orbit in B is $[-1, 1] \times \{a\}$ for some $a \in [-1, 1]$. Fix a small transverse arc γ through y in B . Let γ_+ be a connected component of $\gamma \setminus O$. Since each periodic orbit is null homotopic, each periodic orbit intersects γ_+ at most one point. This means that each point $p := (1, \varepsilon)$ in $\text{Per}(v) \cap (\{1\} \times [-1, 1])$ intersecting γ_+ goes back to a point $(-1, \varepsilon)$ in B (i.e. $O_v(p) \cap B = [-1, 1] \times \{\varepsilon\}$). Replacing B with γ , we may assume that the flow $v|_B$ induces a homeomorphism from $\{-1\} \times [-1, 1]$ to $\{1\} \times [-1, 1]$ which can be identified with the identity mapping $1_{[-1, 1]}$ on $[-1, 1]$. Replacing $1_{[-1, 1]}$ with a homeomorphism f which is contracting near 0 (e.g. $f(x) = x^3$), we obtain the resulting continuous flow w such that $O_v(y) = O_w(y)$ and $\text{Sing}(w) = \text{Sing}(v)$, modifying v in B . We identify $O_v(y) \cap B = [-1, 1] \times \{0\}$ with $\text{dom}(f) = [-1, 1]$. By the Baire category theorem, the countable intersection $\bigcap_{n \in \mathbb{Z}} f^n([-1, 1] \cap \text{Per}(v))$ is dense in $[-1, 1]$. Thus there is a point $z \in \text{Per}(v)$ such that $O_w(z)$ is proper and $y \in \overline{O_w(z)} - O_w(z)$. Removing the singular points, Theorem 3.1, [15] for w implies $\overline{O_w(y)} - O_w(y) = \omega_w(y) \cup \alpha_w(y) \subseteq \text{Sing}(w)$. Since $O_v(y) = O_w(y)$ and $\text{Sing}(w) = \text{Sing}(v)$, we have $\omega(y) \cup \alpha(y) \subseteq \text{Sing}(v)$. Suppose that there are essential periodic orbits. Let O be an essential periodic orbit. Cutting this periodic orbit O , the remainder $M - O$ has two new boundaries. Adding two center disks to the two boundaries of $M - O$, we obtain a new closed surface M' whose genus is less than one of M and the resulting non-wandering flow v' on M' with $\text{Sing}(v') = \text{Sing}(v)$ such that $\alpha(y) = \alpha_{v'}(y)$ and $\omega(y) = \omega_{v'}(y)$, where $\alpha_{v'}(y)$ (resp. $\omega_{v'}(y)$) is the alpha (resp. omega) limit set of y with respect to v' . By finite iterations, we obtain a new closed surface M^\dagger and the resulting non-wandering flow v^\dagger on M^\dagger with $\text{Sing}(v^\dagger) = \text{Sing}(v)$ such that $\alpha(y) = \alpha_{v^\dagger}(y)$ and $\omega(y) = \omega_{v^\dagger}(y)$ such that each periodic orbit of v^\dagger is null homotopic. This implies $\omega(y) \cup \alpha(y) \subseteq \text{Sing}(v)$. \square

Taking a suspension of a non-wandering circle homeomorphism, we have the following statement.

Corollary 2.7. *Each non-wandering continuous homeomorphism on \mathbb{S}^1 is topologically conjugate to a rotation.*

Proof. Let v be the suspension of a non-wandering circle homeomorphism f . Then v is a non-wandering continuous flow on \mathbb{T}^2 . Since v is a suspension, there are no singular points. By Proposition 2.6, we have $P = \emptyset$. Theorem 2.5 implies $\text{Per}(v) \sqcup \text{LD} = \mathbb{T}^2$. Since $\text{Per}(v)$ is open, the complement $\text{LD} = \mathbb{T}^2 - \text{Per}(v)$ is closed. By Lemma 2.3, we obtain $\overline{\text{Per}(v)} \cap \overline{\text{LD}} = \overline{\text{Per}(v)} \cap \text{LD} = \emptyset$. This implies that both of $\text{Per}(v)$ and LD are open and closed. Since \mathbb{T}^2 is connected, we have that v is pointwise periodic or minimal. Then f is topologically conjugate to periodic or minimal. This means that f is topologically conjugate to a rotation. \square

By the smoothing result [9], we have the following result.

Corollary 2.8. *Each non-wandering continuous flow on a compact surface M is topologically equivalent to a C^∞ -flow.*

Proof. We may assume that M is connected. By Theorem 2.5, there are no exceptional orbits. By Proposition 2.6, the closure of each proper orbit contains singular points. We show that if the closure of a locally dense orbit is minimal, then it is the whole surface which is \mathbb{T}^2 . Indeed, let O be a locally dense orbit whose closure is minimal. Then \overline{O} contains neither singular points nor periodic points. Since the closure of a proper orbit contains singular points, the minimal set \overline{O} consists of locally dense orbits. Since each point of \overline{O} is contained in the interior of \overline{O} , this minimal set \overline{O} is closed and open, and so is the whole surface M which is homeomorphic to \mathbb{T}^2 . Therefore each minimal set is either a closed orbit or \mathbb{T}^2 . By the smoothing theorem [9], this continuous flow is topologically equivalent to a C^∞ -flow. \square

We state the uniformity of $\text{Per}(v)$ of a non-wandering continuous flow v .

Corollary 2.9. *Let v be a non-wandering continuous flow on a compact surface M . Then each connected component of $\text{Per}(v)$ is either a connected component of M , an open annulus, or an open Möbius band. If $\text{Sing}(v)$ consists of finitely many contractible connected components, then P consists of finitely many separatrices, $\text{Sing}(v) \sqcup P$ is closed, and LD is open.*

Proof. Since $\text{Per}(v)$ is open, the flow box theorem implies that each periodic orbit has a saturated neighbourhood which is either an annulus or a Möbius band. Notice that a union of one saturated Möbius band and one saturated annulus with one intersection is a Möbius band and that a union of two saturated annuli (resp. Möbius bands) with one intersection is an annulus (resp. a Klein bottle). Fix a connected component U of $\text{Per}(v)$. If $\partial U = \emptyset$, then U is a connected component of M . If $\partial U \neq \emptyset$, then U is either an annulus or a Möbius band. Suppose that $\text{Sing}(v)$ consists of finitely many contractible connected components. Collapsing each connected component of $\text{Sing}(v)$ into a point, by Theorem 1, [20], the resulting space is homeomorphic to the original space M and the resulting flow is non-wandering. Thus we may assume that $\text{Sing}(v)$ is finite. By Theorem 3, [6], each singular point is either a center or a multi-saddle. By Proposition 2.6, each orbit in

P is a separatrix. Therefore P consists of finitely many separatrices and so $\text{Sing}(v) \sqcup P$ is closed. Since $\overline{\text{Per}(v)} \cap \text{LD} = \emptyset$, the complement $\text{LD} = M - (\text{Sing}(v) \sqcup P \sqcup \overline{\text{Per}(v)})$ is open. \square

We show that LD is not open in general. In fact, we construct a non-wandering flow on \mathbb{T}^2 such that P and LD are dense as follows.

Example 2.10. Consider an irrational rotation v on \mathbb{T}^2 . Fix any points $p \in \mathbb{T}^2$. Let $(t_i)_{i \in \mathbb{Z}}$ of \mathbb{R} be a sequence such that $\lim_{i \rightarrow \infty} t_i = \infty$, $\lim_{i \rightarrow -\infty} t_i = -\infty$, and $\lim_{i \rightarrow \infty} p_i = \lim_{i \rightarrow -\infty} p_i = p_0$, where $p_i := v(t_i, p_0) \in O_v(p)$. Using bump functions, replace $O_v(p)$ with a union of countably many singular points p_i ($i \in \mathbb{Z}$) and countably many proper orbits. Let v' be the resulting vector field. For any point $x \in \mathbb{T}^2 - O(p)$, we have $O_v(x) = O_{v'}(x)$. Moreover, $O_v(p) - \text{Sing}(v') = \text{LD}(v')$. By construction, LD is not open but $\overline{P} = \overline{\text{LD}} = \mathbb{T}^2$. This example also shows that the finiteness condition in Corollary 2.9 is necessary.

In [11], a foliation \mathcal{F} on a manifold M is said to be “rare species” if either all leaves are exceptional or \mathcal{F} has at least two of three types (i.e. proper, locally dense, exceptional) and the union of leaves of each type is dense. The author has constructed some kind of codimension one “rare species” foliations on compact 3-manifolds. Analogously we say that a flow v on M is “rare species” if either $E = M$ or v has at least two of three types (i.e. proper, locally dense, exceptional) and the union of orbits of each type is dense. Note that “rare species” surface flows are non-wandering and so have no exceptional orbits. In contrast to foliations, this implies there is only one possible kind of “rare species” flow on compact surfaces. Notice that the above example is a “rare species” flow on \mathbb{T}^2 . Therefore, we obtain the following statement.

Proposition 2.11. *There are smooth “rare species” flows on \mathbb{T}^2 . On the other hand, for each “rare species” flow v on a compact surface M , we have $M = \text{Sing}(v) \sqcup \text{LD} \sqcup P$ and $\overline{\text{LD}} = \overline{P} = M$.*

Note that all known codimension one foliations on compact manifolds are not C^1 but continuous. In contrast to codimension one foliations on compact manifolds, each “rare species” flow v on a compact surface is topologically equivalent to a C^∞ -flow. Moreover, we obtain the following examples.

Corollary 2.12. *There are codimension one “rare species” foliations on open surfaces contained in \mathbb{R}^2 . In particular, the union of locally dense leaves is not open.*

Proof. Fix a minimal codimension one foliation \mathcal{F} on open surfaces contained in \mathbb{R}^2 (e.g. a foliation in Theorem, [8]). Replace a locally dense leaf L into countably many singular points and proper leaves connecting two singular points, and remove the singular points as the above construction for flows. Then we can obtain a desired foliation. \square

3. APPLICATIONS OF THIS CHARACTERIZATION

Recall that v is topologically transitive if it has a dense orbit. Closed orbits are singular points or periodic orbits. A subset is said to be co-connected if the complement of it is connected. We have the following characterization of transitivity for surface flows.

Theorem 3.1. *Let v be a continuous flow on a compact surface M . Then the following are equivalent:*

1. v is topologically transitive.
 2. v is non-wandering such that $P \sqcup \text{Sing}(v)$ is co-connected and $\text{int}(\text{Per}(v) \sqcup \text{Sing}(v)) = \emptyset$.
 3. v is non-wandering such that the set of regular weakly recurrent points is connected and the interior of the union of closed orbits is empty.
 4. $P \sqcup \text{Sing}(v)$ is co-connected and $\text{int} \text{Sing}(v) = \text{int} \text{Per}(v) = \text{int} P = \emptyset$.
- In each case, $M = \text{Sing}(v) \sqcup P \sqcup \text{LD} = \overline{\text{LD}}$ and each locally dense orbit is dense.

Proof. Obviously, the conditions 2 and 3 are equivalent. Since P is the complement of the set of the weakly recurrent points, v is non-wandering if and only if $\text{int} P = \emptyset$. Lemma 2.4 implies that the conditions 2 and 4 are equivalent. Suppose that v is topologically transitive. Then v is non-wandering. The openness of $\text{Per}(v)$ implies $\text{Per}(v) = \emptyset$. By transitivity, there is a dense orbit O and so $\overline{\text{LD}} = M$. Since O is connected and $O \subseteq \text{LD} \subseteq \overline{O} = M$, we have that $M - (P \sqcup \text{Sing}(v)) = \text{LD}$ is connected. Conversely, suppose v is non-wandering such that $\text{int}(\text{Per}(v) \sqcup \text{Sing}(v)) = \emptyset$ and $M - (P \sqcup \text{Sing}(v))$ is connected. Put $U = M - (P \sqcup \text{Sing}(v))$. Since $\text{Per}(v)$ is open, we have $\text{Per}(v) = \emptyset$ and so $\text{int} \text{Sing}(v) = \emptyset$. Since v is non-wandering, we have $\overline{\text{LD}} = M = \text{Sing}(v) \sqcup \text{LD} \sqcup P$. Thus $\text{LD} = U$ is connected. By Proposition 2.2, LD consists of finitely many orbit classes, each of which is closed in LD . This means that LD consists of one orbit class and so each locally dense orbit is dense. Therefore v is topologically transitive. \square

Since an area-preserving flow is non-wandering, Proposition 2.6 implies the following statement which is a generalization of Theorem A, [16].

Corollary 3.2. *Let v be a continuous flow on a compact surface M . Then v is topologically transitive if and only if v is non-wandering such that $\text{int}(\text{Per}(v) \sqcup \text{Sing}(v)) = \emptyset$ and $\{x \in M \mid \omega(x) \cup \alpha(x) \subseteq \text{Sing}(v)\}$ is co-connected.*

Notice that the non-wandering condition in Theorem 3.1 and Corollary 3.2 cannot be replaced by the chain recurrence. Indeed, consider an irrational rotation v_0 on \mathbb{T}^2 and a closed transverse arc T . Using a bump function and perturbing near T , we can replace v_0 with a new flow v_1 with $\text{Sing}(v_1) = T$ such that $\mathbb{T}^2 - T$ consists of proper orbits of v_1 and that each orbit of v_1 is contained in some orbit of v_0 . Then the resulting flow v_1 is not topologically transitive but chain recurrent, and satisfies all conditions except the non-wandering property in Theorem 3.1 and Corollary 3.2.

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