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THE SOLUTION OPERATOR OF THE INHOMOGENEOUS DIRICHLET PROBLEM IN THE UNIT BALL

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Abstract. In this paper we estimate norms of integral operator induced by the Green function related to the Poisson equation in the unit ball with vanishing boundary data.

1. Introduction and notation

Throughout the paper we will assume that n is an integer greater than 2. We denote by B^n and S^{n-1} the unit ball and unit sphere in $\mathbf{R^n}$, respectively. For two normed spaces X and Y with the vector norm $|\cdot|$ defined as $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, by the norm of a linear operator $T : X \to Y$ we mean

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

Let P be the Poisson kernel, i.e., the function

$$P(x,\eta) = \frac{1-|x|^2}{|x-\eta|^n},$$

and let G be the Green function of the unit ball w.r.t. the Laplace operator, i.e., the function

$$G(x,y) = c_n\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{[x,y]^{n-2}}\right),$$

where

$$c_n = \frac{1}{(n-2)\omega_{n-1}}, \tag{1.1}$$

where ω_{n-1} is the Hausdorff measure of S^{n-1} and

$$[x,y] := |x|y| - y/|y|| = |y|x| - x/|x||.$$

It is known that both functions P and G are harmonic for $|x| < 1$ with $x \neq y$.

Let $f : S^{n-1} \to \mathbf{R}^n$ be an L^1 integrable function on the unit sphere S^{n-1}, and let $g : B^n \to \mathbf{R}^n$ be an L^1 integrable function in the unit ball. The solution of the Poisson equation $\triangle u = g$ (in the sense of distributions) in the unit ball, satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) - \mathcal{G}[g](x) := \int_{S^{n-1}} P(x,\eta)f(\eta)d\sigma(\eta) - \int_{B^n} G(x,y)g(y)dy, \tag{1.2}$$

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for $|x| < 1$. Here $d\sigma$ is the normalized Lebesgue $(n-1)$-dimensional measure on the Euclidean sphere.

We consider the Poisson equation with inhomogeneous Dirichlet boundary condition

$$\begin{cases} \triangle u(x) & = g, x \in B^n \\ & u|_{\partial B^n} = 0, \end{cases} \tag{1.3}$$

where $g \in L^p(B^n)$, $p \geqslant 1$. The weak solution is then given by

$$u(x) = -\mathcal{G}[g](x) = -\int_{B^n} G(x,y)g(y)dy, |x| < 1. \tag{1.4}$$

The main goal of our paper is to estimate various norms of the integral operator $\mathcal{G}$. We call it the *solution operator* of the Dirichlet problem. The compressive study of this problem for $n = 2$ has been done by the first author in [12]. In [13] its counterpart is considered for *the differential operator* of the Dirichlet problem. For some related results concerning the planar case, we refer to the papers [2, 4–7]. In [3], Anderson, Khavinson, and Lomonosov considered the L^2 norm of the operator

$$\mathcal{N}[f](x) =: \frac{1}{(n-2)\omega_{n-1}} \int_{B^n} \frac{1}{|x-y|^{n-2}} f(y)dy.$$

The following two results extend and generalize the corresponding results obtained in [3].

Theorem 1.1. *For $p > n/2$, $\mathcal{G}(L^p(B^n)) \subseteq L^\infty(B^n)$. Moreover, for $1 \leqslant q < \frac{n}{n-2}$, $1/p + 1/q = 1$, we have*

$$\|\mathcal{G}\| = \|\mathcal{G} : L^p(B^n) \to L^\infty(B^n)\| = c_n \left(\frac{\pi^{n/2}\Gamma(1+q)\Gamma\left(\frac{n-q(n-2)}{n-2}\right)}{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{n}{n-2}\right)} \right)^{\frac{1}{q}}.$$

In particular, for $p = \infty$

$$\|\mathcal{G}\|_\infty = \frac{1}{2n} \ \ (n \geq 3).$$

Remark 1.2. The particular case $p = \infty$ (i.e., $q = 1$) of Theorem 1.1 is simple and follows from the observation below. Since the function $u(x) = -\frac{1}{2n}(1-|x|^2)$ represents a unique solution of Poisson equation

$$\begin{cases} \triangle u(x) & = 1, x \in \Omega \\ & u|_{\partial\Omega} = 0, \end{cases}$$

it follows that for any integer $n \geq 3$, we have

$$\|\mathcal{G}\|_\infty = \sup_{x \in B^n} \left| \int_{B^n} G(x,y)dy \right| = \frac{1}{2n} \sup_{x \in B^n} (1-|x|^2) = \frac{1}{2n}. \tag{1.5}$$

The mentioned case when $p = \infty$ has been also considered in connection to the problem of best approximation of $|x|^2$ by harmonic functions (see [15]).

Remark 1.3. According to Theorem 1.1, the operator $\mathcal{G} : L^p(B^n) \to L^\infty(B^n)$ is bounded when $p > n/2$. A natural question arises: what happens in the case when $p \leq n/2$? If we consider the function $f(x) = |x|^{-2}\log^{-1}(2/|x|)$, $x \neq 0$, then it can be easily shown that $f \in L^p(B^n)$ $(p \leq n/2)$, and the appropriate function $\mathcal{G}[f](x)$ is not in L^∞, i.e., $\mathcal{G}(L^p(B^n)) \not\subseteq L^\infty(B^n)$ for $p \leq n/2$. However, for $p \geqslant 1$ we have $\mathcal{G}(L^p(B^n)) \subseteq L^p(B^n)$, or more precisely we have the following theorem.

Theorem 1.4. *For $p \geqslant 1$, the operator $\mathcal{G}$ is a bounded operator on the space $L^p(B^n)$ onto itself with the norm $\|\mathcal{G}\|_p$ satisfying the inequalities*

$$\|\mathcal{G}\|_p \leq (2n)^{\frac{p-2}{p}} \lambda_1^{\frac{2(1-p)}{p}}, \quad 1 \leqslant p \leqslant 2,$$

and

$$\|\mathcal{G}\|_p \leq \lambda_1^{-\frac{2}{p}} (2n)^{\frac{p-2}{p}}, \quad 2 \leqslant p \leqslant \infty,$$

which reduce to equalities for $p = 1, 2, \infty$, where $\lambda_1 = \lambda_1(B^n)$ is the first eigenvalue of the Dirichlet Laplacian of the unit ball defined in subsection 2.3.

The proof of Theorem 1.1 is postponed until Section 4, and it is obtained via Möbius transformations of the unit ball. It depends on Lemma 3.1, which is somehow very involved, and presents itself a subtle integral inequality. The proof of Theorem 1.4 uses the eigenvalues of the Dirichlet Laplacian and follows from the Riesz-Thorin interpolation theorem.

2. Preliminaries

2.1. **Gauss hypergeometric function.** Through the paper we will often use the properties of the hypergeometric functions. First of all, the hypergeometric function $F(a, b, c, t) = {}_2F_1(a, b; c; t)$ is defined by the series expansion

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} t^n, \text{ for } |t| < 1,$$

and by the continuation elsewhere. Here $(a)_n$ denotes the shifted factorial, i.e., $(a)_n = a(a+1)\cdots(a+n-1)$ with any real number a.

The following identities will be used in the proof of the main results of this paper: *Euler's identity*

$$F(a, b; c; t) = (1-t^2)^{c-a-b} F(c-a, c-b; c; t), \ \operatorname{Re}(c) > \operatorname{Re}(b) > 0; \tag{2.1}$$

Pfaff's identity

$$F(a, b; c; t) = (1-t^2)^{-a} F(a, c-b; c; \frac{t}{t-1}), \ \operatorname{Re}(c) > \operatorname{Re}(b) > 0; \tag{2.2}$$

Differentiation identity

$$\frac{\partial}{\partial t} F(a, b; c; t) = \frac{ab}{c} F(a+1, b+1; c+1; t); \tag{2.3}$$

and *Kummer's Quadratic Transformation*

$$F\left(a, b; 2b; \frac{4t}{(1+t)^2}\right) = (1+t)^{2a} F(a, a+\frac{1}{2}-b; b+\frac{1}{2}; t^2), \tag{2.4}$$

where the above identity is true for every t for which both series converge.

By using the Chebychev's inequality, one can easily obtain the following inequalities for the Gamma function (see [8]).

Proposition 2.1. *Let m, p, and k be real numbers with $m, p > 0$ and $p > k > -m$. If*

$$k(p-m-k) \geqslant (\leqslant) 0, \tag{2.5}$$

then we have

$$\Gamma(p)\Gamma(m) \geqslant (\leqslant) \Gamma(p-k)\Gamma(m+k). \tag{2.6}$$

2.2. **Möbius transformations of the unit ball.** The set of isometries of the hyperbolic unit ball B^n is a Kleinian subgroup of the group of all Möbius transformations of the extended space $\overline{\mathbf{R}}^n$ onto itself denoted by $\mathbf{Conf}(B^n) = \mathbf{Isom}(B^n)$. We refer to the Ahlfors' book [1] for a detailed survey of this class of important mappings. In general, a Möbius transform $T_x : B^n \to B^n$ has the form

$$z = T_x y = \frac{(1-|x|^2)(y-x) - |y-x|^2 x}{[x,y]^2}. \tag{2.7}$$

Then we have

$$|T_x y| = \left| \frac{x-y}{[x,y]} \right|. \tag{2.8}$$

If dy denotes the volume measure in the ball, since $y = T_{-x} z$ is a conformal mapping, in view of (2.8) we have

$$dy = \left(\frac{1-|x|^2}{[z,-x]^2} \right)^n dz. \tag{2.9}$$

2.3. **Eigenvalues of the Dirichlet Laplacian.** First of all, it is known that there exists an orthonormal basis of $L^2(B^n)$ consisting of eigenfunctions $(\varphi_n)_n$ of the Dirichlet Laplacian

$$\begin{cases} -\Delta u = \lambda u, & z \in B^n \\ u|_{\partial B^n} = 0 \end{cases} \tag{2.10}$$

with corresponding eigenvalues $\lambda_1 < \lambda_2 \le \cdots \le \lambda_n \cdots$. The functions φ_n are real valued.

It is well known that $\lambda_1(B^n)$ is given by the square of the first positive zero of the Bessel function $J_{(n-1)/2}(t)$ of the first kind of order $\alpha = (n-1)/2$:

$$J_\alpha(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{t}{2} \right)^{2m+\alpha}. \tag{2.11}$$

3. The main lemma

Lemma 3.1. *Let*

$$I(t) = (1-t^2)^{n-q(n-2)} \int_0^1 \frac{(1-r^{n-2})^q r^{n-q(n-2)-1}}{(1-r^2t^2)^{n-q(n-2)+1}} dr, \quad 0 \le t < 1,$$

where $n \ge 3$ is a natural number and $1 < q < \frac{n}{n-2}$. Then the maximal value of function $I(t)$ is attained at $t = 0$, i.e.,

$$\begin{aligned} \max_{0 \le t < 1} I(t) = I(0) &= \int_0^1 (1-r^{n-2})^q r^{n-q(n-2)-1} dr \\ &= \frac{\Gamma(1+q)\Gamma\left(\frac{n-q(n-2)}{n-2}\right)}{(n-2)\Gamma\left(1+q+\frac{n-q(n-2)}{n-2}\right)}. \end{aligned} \tag{3.1}$$

Proof. At the beginning we will observe the case $n > 3$. For $a = n - q(n-2)$, we have $0 < a < 2$ and the next expansion

$$\begin{aligned} I(t) &= (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^{\frac{n-a}{n-2}} r^{a-1}}{(1-r^2t^2)^{a+1}} dr \\ &= (1-t^2)^a \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)}{\Gamma(a+1)k!} t^{2k} \int_0^1 (1-r^{n-2})^{\frac{n-a}{n-2}} r^{2k+a-1} dr \\ &= \frac{\Gamma(2+\frac{2-a}{n-2})(1-t^2)^a}{(n-2)\Gamma(a+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)\Gamma(\frac{a+2k}{n-2})}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right) k!} t^{2k}. \end{aligned} \tag{3.2}$$

Assume that $n \geqslant 3$ and $k \geqslant 0$. Let

$$K = \frac{2k}{n-2}, \quad M = 2 + \frac{2k+a}{n-2}, \quad P = 2 + \frac{2}{n-2}.$$

From (2.6) we have

$$\Gamma(M)\Gamma(P) \leqslant \Gamma(M-K)\Gamma(P+K). \tag{3.3}$$

By using the formula $\Gamma(x+1) = x\Gamma(x)$ and (3.3), we have

$$\begin{aligned} \frac{\Gamma\left(\frac{a+2k}{n-2}\right)}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right)} &= \frac{\Gamma(2+\frac{a+2k}{n-2})}{\Gamma(2+\frac{2+2k}{n-2})} \frac{1}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2}+1\right)} \\ &\leq \frac{\Gamma(2+\frac{a}{n-2})}{\Gamma(2+\frac{2}{n-2})} \frac{1}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2}+1\right)}. \end{aligned}$$

For $a \in (0,2)$ we obtain

$$\begin{aligned} \frac{I(t)}{\frac{\Gamma\left(2+\frac{2-a}{n-2}\right)}{n-2}} : \frac{\Gamma(2+\frac{a}{n-2})}{\Gamma(2+\frac{2}{n-2})} &\leqslant \frac{\left(1-t^2\right)^a}{\Gamma(a+1)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k+1)}{\Gamma(1+k)} \frac{t^{2k}}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2}+1\right)} \\ &= \frac{(n-2)(1-t^2)^a}{a} F\left(\frac{a}{2}, 1+a, \frac{2+a}{2}, t^2\right) \\ &\quad - \frac{(n-2)(1-t^2)^a}{(n+a-2)} F\left(1+a, \frac{1}{2}(n+a-2), \frac{a+n}{2}, t^2\right) \\ &= \frac{(n-2)}{a} F\left(1, -\frac{a}{2}, 1+\frac{a}{2}, t^2\right) \\ &\quad - \frac{(n-2)a}{a(n+a-2)} F\left(1, \frac{1}{2}(n-a-2), \frac{a+n}{2}, t^2\right) \\ &:= J(t). \end{aligned}$$

The last expression for the function $J(t)$ was obtained by using the identity (2.4). Further, we have

$$\begin{aligned} \frac{\partial J(t)}{\partial t} &= \frac{-2t(n-2)}{a+2} F\left(2, \frac{2-a}{2}, 2+\frac{a}{2}, t^2\right) \\ &\quad - \frac{2t(n-2)(n-a-2)}{(n+a-2)(a+n)} F\left(2, 1+\frac{1}{2}(n-a-2), 1+\frac{a+n}{2}, t^2\right) \\ &< 0. \end{aligned} \tag{3.4}$$

We conclude that the maximal value of the function $I(t)$ is attained at $t = 0$.

In order to prove Lemma 3.1 for the special case $n = 3$ with $1 < q < 3$, we should notice that

$$\begin{aligned}\max_{0\le t<1} I(t) &= \max_{0\le t<1}(1-t^2)^{3-q}\int_0^1 \frac{(1-r)^q r^{2-q}}{(1-r^2t^2)^{4-q}}dr \\ &\le \max_{0\le t<1}(1-t^2)^{3-q}\int_0^1 \frac{(1-r)^q r^{2-q}}{(1-rt^2)^{4-q}}dr.\end{aligned} \tag{3.5}$$

Put

$$J(t) := (1-t^2)^{3-q}\int_0^1 \frac{(1-r)^q r^{2-q}}{(1-rt^2)^{4-q}}dr, 0 \le t < 1.$$

By using the Taylor expansion, we obtain

$$J(t) = \frac{\Gamma(1+q)\Gamma(3-q)}{6}(1-t^2)^{3-q}F\left(4-q, 3-q, 4; t^2\right), 0 \le t < 1. \tag{3.6}$$

By using (2.1) and (2.2), respectively, on the expression for $J(t)$, we find that

$$J(t) = \frac{\Gamma(1+q)\Gamma(3-q)}{6}F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right), 0 \le t < 1. \tag{3.7}$$

So,

$$\begin{aligned}\max_{0\le t<1} J(t) &= \frac{\Gamma(1+q)\Gamma(3-q)}{6}\max_{0\le t<1} F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right) \\ &= \frac{\Gamma(1+q)\Gamma(3-q)}{6}\max_{0\le t<1} F\left(q, 3-q, 4; 0\right).\end{aligned} \tag{3.8}$$

The last equality is a consequence of the facts that $t^2/(t^2-1) < 0$ and that the coefficients

$$\frac{(q)_k(3-q)_k}{(1)_k(4)_k}$$

of the hypergeometric function

$$F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right)$$

are decreasing with respect to $k \ge 1$.
Thus,

$$\max_{0\le t<1} I(t) = I(0) = \int_0^1 (1-r)^q r^{2-q}dr = \frac{\pi q(1-q)(2-q)}{6\sin \pi q}. \tag{3.9}$$

□

4. Proof of Theorem 1.1

We start this section with an easy lemma.

Lemma 4.1. *Let $p > n/2$ and $\|\mathcal{G}\| := \|\mathcal{G} : L^p(B^n) \to L^\infty(B^n)\|$. Then*

$$\|\mathcal{G}\| = \sup_{x\in B^n}\left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

Proof. Let $u(x) = \mathcal{G}[g](x)$, $g \in L^p(B)$. The Hölder inequality implies

$$\|u\|_\infty \le \sup_{x\in B} \left(\int_B |G(x,y)|^q\, dy\right)^{\frac{1}{q}} \left(\int_B |g(y)|^p dy\right)^{\frac{1}{p}},$$

i.e.,

$$\|\mathcal{G}\| \le \sup_{x\in B^n} \left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}}.$$

On the other hand, there exists $x_0 \in B^n$ so that

$$\left(\int_{B^n} |G(x_0,y)|^q\, dy\right)^{\frac{1}{q}} > \sup_{x\in B^n} \left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}} - \epsilon.$$

Fix $x_0 \in B^n$, and consider the function

$$g(y) = \frac{(G(x_0,y))^{q-1}}{\|(G(x_0,y))^{q-1}\|_p}.$$

Then

$$\begin{aligned} \|\mathcal{G}\| &\ge |\mathcal{G}[g](x_0)| \\ &= \left(\int_{B^n} |G(x_0,y)|^q\, dy\right)^{-\frac{1}{p}} \int_{B^n} |G(x_0,y)|^q dy \\ &= \left(\int_{B^n} |G(x_0,y)|^q\, dy\right)^{\frac{1}{q}} \\ &> \sup_{x\in B^n} \left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}} - \epsilon, \end{aligned} \tag{4.1}$$

i.e.,

$$\|\mathcal{G}\| = \sup_{x\in B^n} \left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}}. \qquad \square$$

Proof of Theorem 1.1. We divide the proof into two cases.

(i) This case includes the following range for (n,q): $n > 3$, with $1 < q < n/(n-2)$ and $n = 3$ with $q \in (2,3)$. According to Lemma 4.1,

$$\|\mathcal{G}\| = \sup_{x\in B^n} \left(\int_{B^n} |G(x,y)|^q\, dy\right)^{\frac{1}{q}}, \ for\ q > 1.$$

Further, we have

$$\|\mathcal{G}\|^q = c_n^q \sup_{x\in B^n} \int_{B^n} \frac{1}{|x-y|^{q(n-2)}} \left|1 - \left|\frac{x-y}{[x,y]}\right|^{n-2}\right|^q dy, \tag{4.2}$$

where c_n is defined by (1.1). We use the change of variable $z = T_x y$, i.e., $T_{-x} z = y$ in the previous integral, where $T_x y$ is the Möbius transform defined in (2.7). Denoting

$t=|x|$, by (2.9) we obtain

$$\begin{aligned}
&\sup_{x\in B^n}\int_{B^n}|G(x,y)|^q\,dy\\
&=\sup_{x\in B^n}c_n^q\int_{B^n}\frac{1}{|x-T_{-x}z|^{q(n-2)}}|1-|z|^{n-2}|^q\frac{(1-t^2)^n}{[z,-x]^{2n}}dz\\
&=c_n^q\sup_{x\in B^n}(1-t^2)^n\int_{B^n}\frac{(1-|z|^{n-2})^q}{\left|\frac{x[z,-x]^2-(1-t^2)(x+z)-|x+z|^2x}{[z,-x]^2}\right|^{q(n-2)}}\frac{dz}{[z,-x]^{2n}}\\
&=c_n^q\sup_{x\in B^n}(1-t^2)^n\int_{B^n}\frac{(1-|z|^{n-2})^q}{|z|^{q(n-2)}\left|\frac{1-t^2}{[z,-x]}\right|^{q(n-2)}}\frac{dz}{[z,-x]^{2n}}\\
&=c_n^q\sup_{x\in B^n}(1-t^2)^{n-q(n-2)}\int_{B^n}\left(\frac{1-|z|^{n-2}}{|z|^{n-2}}\right)^q[z,-x]^{q(n-2)-2n}dz\\
&=c_n^q\sup_{x\in B^n}(1-t^2)^{n-q(n-2)}\int_0^1\frac{(1-r^{n-2})^q}{r^{q(n-2)+1-n}}dr\int_S\frac{d\xi}{|rx+\xi|^{2n-q(n-2)}}\\
&=c_n^q\sup_{x\in B^n}(1-t^2)^{a}\int_0^1\frac{(1-r^{n-2})^q}{r^{1-a}}dr\int_S\frac{d\xi}{(r^2t^2+2rt\xi_1+1)^{\frac{n+a}{2}}}\\
&=c_n^qC_n\sup_{x\in B^n}(1-t^2)^{a}\int_0^1\frac{(1-r^{n-2})^q}{r^{1-a}}dr\int_{-1}^1\frac{(1-s^2)^{\frac{n-3}{2}}}{(r^2t^2+2rts+1)^{\frac{n+a}{2}}}ds,
\end{aligned}$$

where

$$a=n-q(n-2),\quad C_n=\frac{\omega_{n-1}\Gamma(n-1)}{2^{n-2}\Gamma^2(\frac{n-2}{2})}$$

and, without loss of generality, in last two equalities it was assumed that $x=te_1,\xi=(\xi_1,\dots,\xi_n)$. If we take the change of variable

$$\tau=\frac{1-s}{2}$$

in the previous integral, we have

(4.3)

$$\begin{aligned}
\|\mathcal{G}\|^q:c_n^q&=C_n\sup_{x\in B^n}(1-t^2)^{a}\int_0^1\frac{(1-r^{n-2})^q}{r^{1-a}}dr\int_{-1}^1\frac{(1-s^2)^{\frac{n-3}{2}}}{(r^2t^2+2rts+1)^{\frac{n+a}{2}}}ds\\
&=2^{n-2}C_n\sup_{x\in B^n}(1-t^2)^{a}\int_0^1\frac{(1-r^{n-2})^qr^{a-1}}{(1+rt)^{n+a}}dr\int_0^1\frac{\tau^{\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}}{(1-\frac{4rt\tau}{(1+rt)^2})^{\frac{n+a}{2}}}d\tau.
\end{aligned}$$

On the other hand, for a fixed r we have $\frac{4rt}{(1+rt)^2} < 1$ and

$$\begin{aligned}
&\int_0^1 \frac{\tau^{\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}}{(1-\frac{4rt\tau}{(1+rt)^2})^{\frac{a+n}{2}}}d\tau \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k)}{k!\Gamma(\lambda)}\left(\frac{4rt}{(1+rt)^2}\right)^k \int_0^1 \tau^{k+\frac{n-3}{2}}(1-\tau)^{\frac{n-3}{2}}d\tau \\
&= \Gamma\left(\frac{n-1}{2}\right)\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k)\Gamma(k+\frac{n-3}{2}+1)}{k!\Gamma(\lambda)\Gamma(n-1+k)}\left(\frac{4rt}{(1+rt)^2}\right)^k \\
&= \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(n-1)}F\left(\lambda, \frac{n-1}{2}; n-1; \frac{4rt}{(1+rt)^2}\right),
\end{aligned} \tag{4.4}$$

where $\lambda = \frac{n+a}{2}$.

By using the Kummer quadratic transformation and Euler's transformation for hypergeometric functions, for $t = |x|$ we obtain

$$\begin{aligned}
&\sup_{x\in B^n}(1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q r^{a-1}}{(1+rt)^{n+a}} F\left(\lambda, \frac{n-1}{2}; n-1; \frac{4rt}{(1+rt)^2}\right)dr \\
&= \sup_{x\in B^n}(1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1} F\left(\frac{n+a}{2}, \frac{a+2}{2}; \frac{n}{2}; r^2t^2\right)dr \\
&= \sup_{x\in B^n}(1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1}(1-r^2t^2)^{-a-1}\mathcal{F}(rt)dr \\
&\le \sup_{x\in B^n}(1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1}(1-r^2t^2)^{-a-1}\max_{t\le 1}\mathcal{F}(rt)dr,
\end{aligned} \tag{4.5}$$

where

$$\mathcal{F}(s) = F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; s^2\right).$$

By using the identity for the derivative of a hypergeometric function, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; r^2t^2\right) \\
&= -2r^2t\frac{2}{n}\frac{a}{2}\frac{q(n-2)-2}{2}F\left(\frac{q(n-2)-n+2}{2}, \frac{q(n-2)}{2}; \frac{n+2}{2}; r^2t^2\right) < 0,
\end{aligned} \tag{4.6}$$

for any $t \in [0,1]$, which implies

$$\max_{|x|\le 1} F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; r^2|x|^2\right) = F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; 0\right). \tag{4.7}$$

Finally, according to Lemma 3.1, for $n > 3$ the maximal value of the function

$$\begin{aligned}
\mathcal{I}(x) &= \int_{B^n} |G(x,y)|^q\, dy \\
&= c_n^q(1-|x|^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1}dr \int_S \frac{d\xi}{|rx+\xi|^{2n-q(n-2)}}
\end{aligned}$$

is attained at $x=0$. So,

(4.8)

$$\begin{aligned}\|G\|^q : c_n^q &= \sup_{x\in B^n}(1-|x|^2)^a\int_0^1(1-r^{n-2})^q r^{a-1}dr\int_S\frac{d\xi}{|rx+\xi|^{n+a}}\\ &= \omega_{n-1}\sup(1-|x|^2)^a\int_0^1\frac{(1-r^{n-2})^q r^{a-1}}{(1-r^2|x|^2)^{a+1}}\mathcal{F}(r|x|)dr\\ &= \omega_{n-1}\int_0^1(1-r^{n-2})^q r^{n-q(n-2)-1}drF\left(\frac{n+a}{2},\frac{q(n-2)-2}{2};\frac{n}{2};0\right)\\ &= \omega_{n-1}\int_0^1(1-r^{n-2})^q r^{a-1}dr=\frac{\omega_{n-1}\Gamma(1+q)\Gamma(\frac{n-q(n-2)}{n-2})}{(n-2)\Gamma(1+q+\frac{n-q(n-2)}{n-2})}.\end{aligned}$$

(ii) For the case $n=3$ with $1<q\le 2$, it is clear that

$$\mathcal{I}(x)=\int_{B^3}|G(x,y)|^q\,dy=\frac{1}{(2\pi)^q}\int_{B^3}\left(\frac{1}{|x-y|}-\frac{1}{[x,y]}\right)^q dy,$$

and that the same transforms for $I(x)$ as in the previous general case give

$$\mathcal{I}(x)=c_3(1-x^2)^{3-q}\int_0^1(1-r)^q r^{2-q}F[(6-q)/2,(5-q)/2,3/2,r^2x^2]dr,$$

where c_3 is an appropriate constant as in the general case. Put $t=|x|$. We can represent $\mathcal{I}(x)$ as

$$\mathcal{I}(x)=c_3\int_0^1\frac{(1-r)^q r^{1-q}\left(1-t^2\right)^{3-q}\left((1-rt)^{-4+q}-(1+rt)^{-4+q}\right)}{2(4-q)t}dr.$$

So,

$$\mathcal{I}(x)=c_3\frac{\left(1-t^2\right)^{3-q}}{2(4-q)t}\sum_{n=0}^{\infty}t^n\int_0^1(1-r)^q r^{1-q}(r^n-(-r)^n)\binom{-4+q}{n}dr,$$

and this implies

$$\mathcal{I}(x)=c_3\frac{\left(1-t^2\right)^{3-q}}{2(4-q)t}\sum_{n=0}^{\infty}\frac{\left(-1+e^{in\pi}\right)\binom{-4+q}{n}\Gamma(2+n-q)\Gamma(1+q)}{\Gamma(3+n)}t^n.$$

Thus,

$$\mathcal{I}(x)=c_3\frac{\pi(-1+q)q\left(1-t^2\right)^{3-q}\left(F(2-q,4-q;3;t)-F(2-q,4-q;3;-t)\right)}{4\sin(\pi q)(4-q)t}.$$

Let

$$c'(q):=c_3\frac{2^{-q}\pi^{2-q}(-1+q)q}{(4-q)\sin(\pi q)}.$$

Then

$$\mathcal{I}(x)/|c'|\leqslant I_1(x)=\frac{(1-t^2)}{t}\left(F(2-q,4-q;3;t)-F(2-q,4-q;3;-t)\right)$$

for $1<q<2$ and

$$I_1(x)=a_0+\sum_{n=1}^{\infty}a_nt^n,$$

where $a_0 > 0$ and

$$a_n = \frac{2\,(1+(-1)^n)\,\Gamma(3+n-q)(-(n-q)!\Gamma(4+n)+\Gamma(n)\Gamma(5+n-q))}{\Gamma(n)\Gamma(2+n)\Gamma(4+n)\Gamma(2-q)\Gamma(4-q)}.$$

Further, we have $a_n \leqslant 0$ because

$$\frac{(1+n-q)(2+n-q)(3+n-q)(4+n-q)}{n(1+n)(2+n)(3+n)} \leqslant 1,$$

which again implies that maximal value of the function $\mathcal{I}(x)$ is attained at $x = 0$. This finishes the proof of Theorem 1.1. □

5. Proof of Theorem 1.4

Let Ω be a domain of $\mathbf{R}^n$, and let $|\Omega|$ be its volume. For $\mu \in (0,1]$ define the operator V_μ on the space $L^1(\Omega)$ by the Riesz potential

$$(V_\mu f)(x) = \int_\Omega |x-y|^{n(\mu-1)} f(y)dy.$$

The operator V_μ is defined for any $f \in L^1(\Omega)$, and V_μ is bounded on $L^1(\Omega)$, or more generally we have the next lemma.

Lemma 5.1 ([11, pp. 156–159])**.** *Let V_μ be defined on the $L^p(\Omega)$ with $p > 0$. Then V_μ is continuous as a mapping $V_\mu : L^p(\Omega) \to L^q(\Omega)$, where $1 \le q \le \infty$, and*

$$0 \le \delta = \delta(p,q) = \frac{1}{p} - \frac{1}{q} < \mu.$$

Moreover, for any $f \in L^p(\Omega)$

$$\|V_\mu f\|_q \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} (\omega_{n-1}/n)^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_p.$$

Theorem 5.2. *Let $\|\mathcal{G}\|_1 := \|\mathcal{G} : L^1(B) \to L^1(B)\|$. Then*

$$\|\mathcal{G}\|_1 = \frac{1}{2n}.$$

Proof. According to Theorem 1.1 we have

$$\|\mathcal{G}\|_{L^\infty \to L^\infty} = \frac{1}{2n}.$$

On the other hand, Lemma 5.1 states that the operator $\mathcal{G} : L^1 \to L^1$ is bounded. Then

$$\|\mathcal{G}\|_{L^1 \to L^1} = \|\mathcal{G}^*\|_{L^\infty \to L^\infty},$$

where $\mathcal{G}^*$ is the appropriate adjoint operator. Since

$$\mathcal{G}^*[f](x) = \int_{B^n} \overline{G(y,x)} f(y)dy = \int_{B^n} G(x,y)f(y)dy, f \in L^\infty(B),$$

we have

$$\|\mathcal{G}\|_{L^1 \to L^1} = \|\mathcal{G}\|_{L^\infty \to L^\infty}.$$ □

In the sequel we are going to observe the Hilbert case, i.e., the case when $p = 2$ concerning the operator $\mathcal{G} : L^2(B) \to L^2(B)$. It is well known that $\mathcal{G}^{-1} = -\triangle$ on the Sobolev space $H_0^1(\Omega)$, so the Hilbert norm $\mathcal{G}$ is precisely equal to the reciprocal value of the norm of $-\triangle$ (cf. [3]). So, we have the following theorem, whose proof is included here for the sake of completeness.

Theorem 5.3. *Let* $\|\mathcal{G}\|_2 := \|\mathcal{G} : L^2(B^n) \to L^2(B^n)\|$. *Then*

$$\|\mathcal{G}\|_2 = \frac{1}{\lambda_1}.$$

Thus,

$$\|\mathcal{G}[g]\|_2 \leq \frac{1}{\lambda_1}\|g\|_2, \;\; g \in L^2(B^n). \tag{5.1}$$

Equality is attained in (5.1) *for* $g(x) = c\varphi_1(x), a.e.\ x \in B^n$ *where* c *is a real constant.*

Proof. If $f \in L^2(B^n)$, then under the previous notation

$$f(x) = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \, \varphi_k(x).$$

Since $\mathcal{G}$ is bounded, we have

$$\mathcal{G}[f] = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \, \mathcal{G}[\varphi_k].$$

Also

$$\mathcal{G}[\varphi_k] = \frac{1}{\lambda_k}\mathcal{G}[\triangle\varphi_k] = -\frac{1}{\lambda_k}\varphi_k.$$

The fact that (φ_k) is orthonormal implies

$$\|\mathcal{G}[f]\|_2^2 = \sum_{k=1}^{\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\lambda_k^2}.$$

Since λ_1 is a simple eigenvalue and $0 < \lambda_1 < \lambda_2 \leq \cdots$, we have

$$\|\mathcal{G}[f]\|_2 \leq \frac{1}{\lambda_1}\|f\|_2.$$

Finally, we have

$$\|\mathcal{G}\|_2 = \frac{1}{\lambda_1}. \qquad \square$$

By using the Riesz-Thorin interpolation theorem [16], we obtain the following estimates for the norm of the operator $\mathcal{G} : L^p \to L^p$.

Denote by $\|\mathcal{G}\|_{L^1 \to L^1} = \|\mathcal{G}\|_{L^\infty \to L^\infty} = \|\mathcal{G}\|_1$ and $\|\mathcal{G}\|_{L^2 \to L^2} = \|\mathcal{G}\|_2$. Then

$$\|\mathcal{G}\|_p \leq \|\mathcal{G}\|_1^{\frac{2-p}{p}} \|\mathcal{G}\|_2^{\frac{2(p-1)}{p}} = (2n)^{\frac{p-2}{p}} \lambda_1^{\frac{2(1-p)}{p}},$$

where $\|\mathcal{G}\|_p$ represents the norm of the operator $\mathcal{G} : L^p(B^n) \to L^p(B^n)$ for $1 < p < 2$. Similarly,

$$\|\mathcal{G}\|_p \leq \|\mathcal{G}\|_2^{\frac{2}{p}} \|\mathcal{G}\|_1^{\frac{p-2}{p}} = \lambda_1^{-\frac{2}{p}} (2n)^{\frac{p-2}{p}},$$

where $\mathcal{G} : L^p(B^n) \to L^p(B^n)$ for $2 < p < \infty$. This yields the proof of Theorem 1.4.

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