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DOUBLE CENTRALISER PROPERTY AND MORPHISM CATEGORIES

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Abstract. Given a ring A and an idempotent $e \in A$, double centraliser property on the bimodule eA is characterised in terms of equivalences of additive categories, which are related to morphism categories. The results and methods then are applied to gendo-Gorenstein algebras.

1. Introduction

Given a ring A and an idempotent $e = e^2 \in A$, the (eAe, A)-bimodule eA is said to satisfy a *double centraliser property* (or equivalently, eA is a faithfully balanced bimodule) if and only if $\mathrm{End}_{eAe}(eA) = A$. Note that the inclusion $A \subset \mathrm{End}_{eAe}(eA)$ follows if eA is faithful. Another equality – $\mathrm{End}_A(eA) = eAe$ – is automatic. Double centraliser properties are fundamental in representation theory and its applications. Major examples include Schur-Weyl duality relating the algebraic group GL_n with finite symmetric groups [Gr], Soergel's double centraliser theorem for the BGG-category of a semisimple complex Lie algebra [S], and many variations. If the projective A-module eA is injective as well, then double centraliser property is equivalent to A being eA-copresented, which by definition is equivalent to A having dominant dimension at least two. In this case, the celebrated Morita-Tachikawa correspondence connects the algebra A exactly with the pair (eAe, eA). This kind of double centraliser property recently has led to defining new classes of algebras such as gendo-symmetric [FK1, FK2], Morita [KY] and gendo-d-Gorenstein [GK] algebras.

Instead of writing double centraliser property as equalities of endomorphism rings, one may also phrase it as an equivalence of additive categories $\mathrm{add}_{eAe}(eA) \simeq \mathrm{add}_A(A)$. If A is an artin algebra, eA is injective over A and A has dominant dimension at least two, then a classical result of Auslander [A] implies a further equivalence of (much larger) categories: $\mathrm{copres}(eA) \simeq eAe - \mathrm{mod}$, as additive categories, with A being an object in $\mathrm{copres}(eA)$. Here an object X in $\mathrm{copres}(eA) \subset \mathrm{mod} - A$ is given as the kernel of an A-module morphism f between two objects in $\mathrm{add}_A(eA)$. The category $\mathrm{copres}(eA)$ can be seen as a special case of a morphism category. These observations motivate the question to be discussed in this article, as well as the choice of categories to be considered.

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Question. Suppose A and e are given.

Does double centraliser property on eA imply equivalences of additive categories, which are morphism categories or closely related to such?

Can one even characterise double centraliser property in terms of equivalences of additive categories?

For practical purposes, we want the morphism categories to be large, and we also want to relate them to other categories arising in this context; in particular, to the module category of the quotient algebra $\overline{A} := A/AeA$.

The above question came up when studying gendo-Gorenstein algebras introduced in [GK] in terms of a double centraliser property, and we are going to apply our results to these algebras. Before specialising to these algebras, we will, however, answer the above question for general rings A.

In the following, we use the notation Mor for morphism categories, $\mathcal{S}$ for monomorphism categories and $\mathcal{F}$ for epimorphism categories (see Definitions 2.1, 2.3).

Theorem on Characterisations (Theorems 3.2 and 3.3)**.** *Let A be a ring and e be an idempotent of A. Let $\mathcal{X}$ consist of the objects* $(1 : eA \to eA)$ *and* $(eA \to 0)$ *in* $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))$.

(1) *The (eAe, A)-bimodule eA has double centraliser property if and only if the functor* $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$ *induces an equivalence*

$$\mathcal{S}(\mathrm{add}_{eAe}(eA)) \overset{\cong}{\to} \mathcal{S}(\mathrm{add}_A A)$$

of additive categories.

(2) *The (eAe, A)-bimodule eA has double centraliser property if and only if the composition functor* $\mathrm{Cok} \circ \mathrm{Hom}_{eAe}(eA, -) : \mathrm{Mor}(\mathrm{add}_{eAe}(eA)) \to A\text{-mod}$ *yields an equivalence*

$$\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to} A\text{-mod}$$

of additive categories.

Assuming double centraliser property, it is also possible to describe $\overline{A} - \mathrm{mod}$ as a category of morphisms:

Corollary 3.5. *Let A be a ring and e be an idempotent of A. Assume that the (eAe, A)-bimodule eA has double centraliser property. Then the functor G induces an equivalence* $G : \mathcal{F}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \xrightarrow{\cong} \overline{A}\text{-mod}$ *of additive categories.*

Another description of $\overline{A} - \mathrm{mod}$ as a category of morphisms is possible under additional assumptions.

Theorem 4.2. *Let A be an artin algebra and e be an idempotent of A such that the (eAe, A)-bimodule eA has double centraliser property. Assume that eA is a finitely generated Gorenstein projective left eAe-module such that* $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$. *Then there exists a fully faithful functor*

$$F : \overline{A}\text{-mod} \to \mathcal{S}(\mathrm{add}_A A)/\mu.$$

The organisation of this article is as follows: In Section 2, the various objects and categories to be studied will be defined. In Section 3, double centraliser property for A a general ring, is characterised in terms of equivalences of additive categories, proving in particular the Theorem on Characterisations. Moreover, we provide further equivalences of categories implied by double centraliser property, proving in particular Corollary 3.5.

In Section 4, we specialise to A being an artin algebra, and derive further equivalences of categories from double centraliser property, proving in particular Theorem 4.2. Moreover, we spell out some consequences for gendo-d-Gorenstein algebras; in particular, Corollary 4.3 characterises these algebras using an equivalence of additive categories.

2. Preliminaries

Throughout, A is a ring and $e = e^2 \in A$ an idempotent. Here and in Section 3, we will put no further assumptions on A. Later, in Section 4, we will assume A to be an artin algebra (over an artinian ring not to be mentioned explicitly). When proving Theorems on Characterisation (Theorems 3.2 and 3.3) and related results, A is an arbitrary ring. Moreover, eA is not assumed to be finitely generated over eAe. The category $\mathrm{add}_{eAe}(eA)$ by definition consists of summands of finite direct sums of eA. Morphism categories will be defined to have finitely generated objects. The functor α to be used in proving Theorem on Characterisation does send finitely generated left or right modules to finitely generated left or right modules, denoted by $A - \mathrm{mod}$ or $\mathrm{mod} - A$, respectively. And the image categories in Theorems 3.2, 3.3 and 4.2 always are categories of finitely generated modules, even if eA is not finitely generated over eAe.

In the special case of artin algebras we use further notation as follows: Let A be an artin algebra and A-mod (resp. mod-A) be the category of finitely generated left (resp. right) A-modules. Denote by A-proj (resp. proj-A) the full subcategory of projective A-modules in A-mod (resp. mod-A) and by $P^{\leq 1}(A\text{-mod})$ the full subcategory of A-modules in A-mod having projective dimension at most 1. For any A-module M, we denote by $\mathrm{injdim}_A M$ its injective dimension.

An A-module M is said to be *Gorenstein projective* (see [EJ]) in A-mod, if there is an exact sequence $P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots$ in A-proj with $\mathrm{Hom}_A(P^\bullet, Q)$ exact for any A-module Q in A-proj, such that $M \cong \ker d^0$. Denote by A-$\mathcal{G}$proj the full subcategory of Gorenstein projective modules in A-mod. If G is a finitely generated Gorenstein projective A-module, then we denote by $(A\text{-}\mathcal{G}\mathrm{proj})_G^{\leq 1}$ the full subcategory of A-$\mathcal{G}$proj consisting of all modules M admitting an exact sequence $0 \to G_1 \to G_0 \to M \to 0$, where the G_i lie in add$G$. An artin algebra A is called *Iwanaga-Gorenstein* (for short, *Gorenstein*) if $\mathrm{injdim}_A A < \infty$ and $\mathrm{injdim} A_A < \infty$. A Gorenstein algebra A is *d-Gorenstein* if $\mathrm{injdim}_A A \leq d < \infty$; see [Iwa].

Let N_A be a right A-module in mod$A$ and $B = \mathrm{End}_A(N)$. Then $\mathrm{End}_B(N)$ is called the double centraliser of N_A. In [GK] an algebra A has been defined to be a *gendo-d-Gorenstein algebra* for some non-negative integer d if A is isomorphic to the endomorphism algebra of a finitely generated Gorenstein projective generator over a d-Gorenstein algebra B. Equivalently, there is an associated idempotent e of A such that eAe is a d-Gorensein algebra and eA is a finitely generated Gorenstein projective left eAe-module, and the (eAe, A)-bimodule eA has double centraliser property. In particular, if d is zero, then A is a *Morita algebra* (see [Mo, KY]). For unspecified parameter d, we just call A a gendo-Gorenstein algebra.

Definition 2.1. Let A be a ring and $\mathcal{C}$ be a full subcategory of A-mod. The *morphism category* $\mathrm{Mor}(\mathcal{C})$ of $\mathcal{C}$ is defined as follows: An object is a morphism $f : X \to Y$ in $\mathcal{C}$, and a morphism from $f : X \to Y$ to $f' : X' \to Y'$ is a pair (g_1, g_0), where $g_1 : X \to X'$ and $g_0 : Y \to Y'$, such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_1"'] & Y \arrow[d, "g_0"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commutes.

Remark 2.2. $\mathrm{Mor}(\mathcal{C})$ is an additive category, but not an abelian category in general (for example, if $\mathcal{C}$ is not an abelian subcategory of A-mod).

Definition 2.3. Define $\mathcal{S}(\mathcal{C})$ to be the full additive subcategory of $\mathrm{Mor}(\mathcal{C})$ in which the objects are $u : X \to Y$ in $\mathcal{C}$ such that u are monomorphisms in A-Mod. We call $\mathcal{S}(\mathcal{C})$ the *monomorphism category* of $\mathcal{C}$.

Define $\mathcal{F}(\mathcal{C})$ to be the full additive subcategory of $\mathrm{Mor}(\mathcal{C})$ in which the objects are $v : X \to Y$ in $\mathcal{C}$ such that v are epimorphisms in A-Mod. We call $\mathcal{F}(\mathcal{C})$ the *epimorphism category* of $\mathcal{C}$.

Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ be an ideal in $\mathcal{A}$. We denote by $\mathcal{A}/\mathcal{I}$ the corresponding factor category. If $\mathcal{K}$ is a class of objects of the category $\mathcal{A}$, we denote by $\langle\mathcal{K}\rangle$ the ideal of $\mathcal{A}$ given by all maps which factor through an object in $\mathrm{add}\mathcal{K}$. For simplicity, we just write $\mathcal{A}/\mathcal{K}$, instead of $\mathcal{A}/\langle\mathcal{K}\rangle$.

For subcategories $\mathcal{C}$ and $\mathcal{D}$ of an abelian category $\mathcal{A}$ we write $\mathcal{C} \perp_l \mathcal{D}$ when $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for any $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $0 < i \leq l$.

Let A be an artin algebra and $\mathcal{B}$ be a resolving subcategory of A-mod. Let $\mathcal{C}$ be a functorially finite subcategory of $\mathcal{B}$ and l be a non-negative integer. Recall from [Iy, Section 2.4] that $\mathcal{C}$ is a *maximal l-orthogonal subcategory* of $\mathcal{B}$, if $\mathcal{C} \perp_l \mathcal{C}$ and $\mathcal{C} = \mathcal{C}^{\perp l} \cap \mathcal{B} = {}^{\perp l}\mathcal{C} \cap \mathcal{B}$, where $\mathcal{C}^{\perp l} := \{X \in \mathcal{B} \,|\, \mathcal{C} \perp_l X\}$ and ${}^{\perp l}\mathcal{C} := \{X \in \mathcal{B} \,|\, X \perp_l \mathcal{C}\}$.

3. Characterising double centraliser property by equivalences of categories of morphisms, and consequences

Let A be a ring and e be an idempotent of A. In this section, we will show that the (eAe, A)-bimodule eA has double centraliser property if and only if there is an equivalence between $\mathcal{S}(\mathrm{add}_{eAe}(eA))$ and $\mathcal{S}(\mathrm{add}_A A)$ if and only if there is an equivalence between some factor category of $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))$ and A-mod. Moreover, we restrict the second equivalence to get the subcategory A/AeA-mod of A-mod.

The following well-known result is a reformulation of double centraliser property in terms of equivalences of categories.

Lemma 3.1. *Let A be a ring and e be an idempotent of A. Then the (eAe, A)-bimodule eA has double centraliser property if and only if the functor* $\mathrm{Hom}_{eAe}(eA, -)$ *restricts to an equivalence of additive categories from* $\mathrm{add}_{eAe}(eA)$ *to* $\mathrm{add}_A A$.

Proof. We first prove necessity. Since $A^{\mathrm{op}} \cong \mathrm{End}_{eAe}(eA)$ by assumption, it follows that the functor $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$ is indeed an equivalence of additive categories.

Now we prove sufficiency, using an elementary fact: If $\mathrm{add}(M)_A = \mathrm{add}(N)_A$ for two A-modules M and N in $\mathrm{mod}A$, then the double centralisers of M_A and N_A are isomorphic. Since the double centralisers of eA and A are $\mathrm{End}_{eAe}(eA)$ and A^{op}

respectively, it follows that $\mathrm{End}_{eAe}(eA)$ and A^{op} are isomorphic, which means that the (eAe, A)-bimodule eA has double centraliser property. □

The following result is our first characterisation of double centraliser property in terms of equivalences of categories of morphisms.

Theorem 3.2. *Let A be a ring and e be an idempotent of A. Then the (eAe, A)-bimodule eA has double centraliser property if and only if the functor*

$$\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$$

induces an equivalence $\mathcal{S}(\mathrm{add}_{eAe}(eA)) \overset{\cong}{\to} \mathcal{S}(\mathrm{add}_A A)$ of additive categories.

Proof. "only if": $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$ is an equivalence of additive categories by Lemma 3.1. Hence, $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{Mor}(\mathrm{add}_{eAe}(eA)) \to \mathrm{Mor}(\mathrm{add}_A A)$ is an equivalence of additive categories. Moreover, if $f : X \to Y$ is an object in $\mathcal{S}(\mathrm{add}_{eAe}(eA))$, then $\mathrm{Hom}_{eAe}(eA, f) : \mathrm{Hom}_{eAe}(eA, X) \to \mathrm{Hom}_{eAe}(eA, Y)$ is an object in $\mathcal{S}(\mathrm{add}_A A)$, because for any eAe-morphism $g : eA \to Y$, $fg = 0$ implies $g = 0$. Therefore, the restriction of

$$\mathrm{Hom}_{eAe}(eA, -) : \mathcal{S}(\mathrm{add}_{eAe}(eA)) \to \mathcal{S}(\mathrm{add}_A A)$$

is a fully faithful functor. Given any object $g : X \to Y$ in $\mathcal{S}(\mathrm{add}_A A)$, we have an object $eA \otimes_A g : eX \to eY$ in $\mathcal{S}(\mathrm{add}_{eAe}(eA))$ such that $\mathrm{Hom}_{eAe}(eA, eA \otimes_A g) \cong g$. This means that $\mathrm{Hom}_{eAe}(eA, -) : \mathcal{S}(\mathrm{add}_{eAe}(eA)) \to \mathcal{S}(\mathrm{add}_A A)$ is a dense functor. So, $\mathcal{S}(\mathrm{add}_{eAe}(eA))$ and $\mathcal{S}(\mathrm{add}_A A)$ are equivalent as additive categories.

"if": Assume that $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$ induces an equivalence $\mathcal{S}(\mathrm{add}_{eAe}(eA)) \overset{\cong}{\to} \mathcal{S}(\mathrm{add}_A A)$ of additive categories. Let $(0 \to \mathrm{add}_{eAe}(eA))$ denote the full subcategory of $\mathcal{S}(\mathrm{add}_{eAe}(eA))$ consisting of objects of the forms $(0 \to X)$ with $X \in \mathrm{add}_{eAe}(eA)$ and by $(0 \to \mathrm{add}_A A)$ the full subcategory of $\mathcal{S}(\mathrm{add}_A A)$ consisting of objects of the forms $(0 \to Y)$ with $Y \in \mathrm{add}_A A$. Then $\mathrm{Hom}_{eAe}(eA, -)$ restricts to an equivalence $(0 \to \mathrm{add}_{eAe}(eA)) \overset{\cong}{\to} (0 \to \mathrm{add}_A A)$. This implies that $\mathrm{add}_{eAe}(eA) \overset{\cong}{\to} \mathrm{add}_A A$ as additive categories. So the (eAe, A)-bimodule eA has double centraliser property, again by Lemma 3.1. □

Let $\mathcal{X}'$ consist of the two objects $(1 : A \to A)$ and $(A \to 0)$ in the morphism category $\mathrm{Mor}(\mathrm{add}_A A)$ of A. Recall from [ARS] that the Cokernel functor $\mathrm{Cok} : \mathrm{Mor}(\mathrm{add}_A(A)) \to A\text{-mod}$ induces an equivalence $\mathrm{Mor}(\mathrm{add}_A A)/\mathcal{X}' \overset{\cong}{\to} A\text{-mod}$ of additive categories. Now we consider the functor

$$\alpha : \mathrm{Mor}(\mathrm{add}_{eAe}(eA)) \to A\text{-mod}$$

defined by $\alpha(f) = \mathrm{Cok} \circ \mathrm{Hom}_{eAe}(eA, f)$ for a morphism f in $\mathrm{Mor}(\mathrm{add}_{eAe}\ (eA))$. Let $\mathcal{X}$ consist of the two objects $(1 : eA \to eA)$ and $(eA \to 0)$ in $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))$.

The following result is another characterisation of double centraliser property, by a different equivalence of categories.

Theorem 3.3. *Let A be a ring and e be an idempotent of A. Then the (eAe, A)-bimodule eA has double centraliser property if and only if $\alpha : \mathrm{Mor}(\mathrm{add}_{eAe}(eA)) \to A\text{-mod}$ yields an equivalence $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to} A\text{-mod}$ of additive categories.*

Proof. By Lemma 3.1, the (eAe, A)-bimodule eA has double centraliser property if and only if the functor $\mathrm{Hom}_{eAe}(eA, -) : \mathrm{add}_{eAe}(eA) \to \mathrm{add}_A A$ induces an equivalence $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to} \mathrm{Mor}(\mathrm{add}A)/\mathcal{X}'$ of additive categories. Since $\mathrm{Cok} : \mathrm{Mor}(\mathrm{add}_A A)/\mathcal{X}' \to A$-mod is an equivalence of additive categories by [ARS], the (eAe, A)-bimodule eA has double centraliser property if and only if the functor $\alpha : \mathrm{Mor}(\mathrm{add}_{eAe}(eA)) \to A$-mod yields an equivalence $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to}$ A-mod of additive categories.

In the following, we give a direct proof of the necessity. We first claim that α is dense. Given an A-module M, take a projective resolution $P_1 \overset{g}{\to} P_0 \to M \to 0$. By Lemma 3.1 there are eAe-modules X and Y in $\mathrm{add}_{eAe}(eA)$ such that $P_1 = \mathrm{Hom}_{eAe}(eA, X)$ and $P_0 = \mathrm{Hom}_{eAe}\ (eA, Y)$, and there is an eAe-morphism $f : X \to Y$ such that $\mathrm{Hom}_{eAe}(eA, f) = g$. It follows that $M \cong \alpha(f)$. Next we prove that α is full. An A-morphism $M \to M'$ can be lifted to projective presentations of M and M'. Using Lemma 3.1, we see that $M \to M'$ is in the image of the functor α by a similar argument as above.

It remains to calculate the kernel of α. Obviously, under this functor α the two objects in $\mathcal{X}$ are sent to zero. Thus the ideal $\langle\mathcal{X}\rangle$ is contained in the Kernel of α. We claim that $\mathrm{Ker}\alpha = \langle\mathcal{X}\rangle$. Assume that a map $(g_1, g_0) : (f : X \to Y) \to (f' : X' \to Y')$ is given such that $\alpha(g_1, g_0) = 0$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{eAe}(eA, X) \arrow[r, "\mathrm{Hom}_{eAe}(eA,f)"] \arrow[d, "\mathrm{Hom}_{eAe}(eA,g_1)"] & \mathrm{Hom}_{eAe}(eA, Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{Hom}_{eAe}(eA,g_0)"] & \alpha(f) \arrow[r] \arrow[d, "\alpha(g_1,g_0)=0"] & 0 \\
\mathrm{Hom}_{eAe}(eA, X') \arrow[r, "\mathrm{Hom}_{eAe}(eA,f')"] & \mathrm{Hom}_{eAe}(eA, Y') \arrow[r, "\pi'"] & \alpha(f') \arrow[r] & 0
\end{tikzcd}$$

Since $\pi'\mathrm{Hom}_{eAe}(eA, g_0) = 0$, there is a map $\widetilde{h} : \mathrm{Hom}_{eAe}(eA, Y) \to \mathrm{Hom}_{eAe}\ (eA, X')$ such that $\mathrm{Hom}_{eAe}(eA, f')\widetilde{h} = \mathrm{Hom}_{eAe}(eA, g_0)$. Again using Corollary 3.5, we get a map $h : Y \to X'$ with $\widetilde{h} = \mathrm{Hom}_{eAe}(eA, h)$ and $g_0 = f'h$. So $f'hf = g_0 f = f'g_1$, and also $f'(h, g_1 - hf)\begin{pmatrix} f \\ 1 \end{pmatrix} = g_0 f$ and $f'(h, g_1 - hf) = g_0(1, 0)$. Hence, in the following diagram all squares commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\begin{pmatrix} f \\ 1 \end{pmatrix}"] & Y \arrow[d, "1"] \\
Y \oplus X \arrow[r, "{(1,\ 0)}"] \arrow[d, "{(h,\ g_1-hf)}"] & Y \arrow[d, "g_0"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

By $(h, g_1 - hf)\begin{pmatrix} f \\ 1 \end{pmatrix} = g_1$, we get that (g_1, g_0) factors through the map $(1, 0) : Y \oplus X \to Y$, which belongs to $\mathrm{add}\mathcal{X}$. Therefore $\alpha : \mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \to A$-mod is an equivalence. $\square$

Given A and e as before, there is a quotient ring $\overline{A} := A/AeA$. Assuming double centraliser property, we are going to describe $\overline{A}$-mod of A-mod in terms of categories of morphisms. We first state a crucial lemma.

Lemma 3.4. *Let A be a ring and e be an idempotent of A. Then a morphism $f : X \to Y$ in $\mathrm{add}_{eAe}(eA)$ is an epimorphism if and only if $e\alpha(f) = 0$ if and only if $\alpha(f)$ is in $\overline{A}$-mod.*

Proof. Let $f : X \to Y$ be in $\mathrm{add}_{eAe}(eA)$. Then, by the definition of α there is an exact sequence

$$\mathrm{Hom}_{eAe}(eA, X) \xrightarrow{\mathrm{Hom}_{eAe}(eA,f)} \mathrm{Hom}_{eAe}(eA, Y) \to \alpha(f) \to 0.$$

Applying $eA \otimes_A -$ yields the following exact sequence:

$$\mathrm{Hom}_{eAe}(eAe, X) \xrightarrow{\mathrm{Hom}_{eAe}(eAe,f)} \mathrm{Hom}_{eAe}(eAe, Y) \to e\alpha(f) \to 0.$$

Thus f is an epimorphism if and only if $\mathrm{Hom}_{eAe}(eAe, f)$ is an epimorphism if and only if $e\alpha(f) = 0$ if and only if $\alpha(f)$ is in $\overline{A}$-mod. □

Now we consider the functor

$$G : \mathcal{F}(\mathrm{add}_{eAe}(eA)) \to A\text{-mod}$$

defined by $G(\nu) := \alpha(v)$ for a morphism v in $\mathcal{F}(\mathrm{add}_{eAe}(eA))$.

Corollary 3.5. *Let A be a ring and e be an idempotent of A. Assume that the (eAe, A)-bimodule eA has double centraliser property. Then the functor G induces an equivalence $G : \mathcal{F}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \xrightarrow{\cong} \overline{A}$-mod of additive categories.*

Proof. By Theorem 3.3 and Lemma 3.4, G is a full functor from $\mathcal{F}(\mathrm{add}_{eAe}(eA))$ onto the category $\overline{A}$-mod and its kernel is the set of morphisms which factor through an object of the form $((1, 0) : X \oplus X' \to X)$, which is the ideal $\langle \mathcal{X} \rangle$. Thus $G : \mathcal{F}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \to \overline{A}$-mod is an equivalence of additive categories. □

Remark 3.6. (1) There is an equivalence $\mathcal{S}(eAe\text{-mod}) \to \mathcal{F}(eAe\text{-mod})$, which sends an object $(u : X \to Y)$ to the canonical map $(Y \to Y/X)$. The restriction of this equivalence to $\mathcal{S}(\mathrm{add}_{eAe}(eA))$ does not provide an equivalence with $\mathcal{F}(\mathrm{add}_{eAe}(eA))$ in general. For example, if eA is a left Gorenstein projective eAe-module such that $\mathrm{add}_{eAe}(eA) = eAe\text{-}\mathcal{G}\mathrm{proj}$, then $\mathrm{add}_{eAe}(eA)$ is not closed under Cokernels of monomorphisms, and therefore the restriction of the equivalence does not have its image inside $\mathcal{F}(\mathrm{add}_{eAe}(eA))$.

(2) The categories $\mathcal{S}(eAe\text{-mod})$ and $\mathcal{F}(eAe\text{-mod})$ are not equivalent in general. For example, take any artin algebra A, $e = 1$. Then there is a double centraliser property on the left A-module A. The objects in $\mathcal{F}(\mathrm{add_A A})$ are just surjections onto direct summands, since the modules are projective. To each such surjection, there is an injection of the complement, which is in $\mathcal{S}(\mathrm{add_A A})$. But $\mathcal{S}(\mathrm{add_A A})$ will usually contain many more, non-split, objects. Thus, in general there cannot be a bijection between isomorphism classes of indecomposable objects in $\mathcal{S}(\mathrm{add_A A})$ and in $\mathcal{F}(\mathrm{add_A A})$.

4. Gorenstein projectives and gendo-Gorenstein algebras

Now we are going to move towards applications to gendo-Gorenstein algebras. Therefore, *from now on, we assume A to be an artin algebra.*

Our first aim is to compare the subcategory $\overline{A}$-mod with the monomorphism category $\mathcal{S}(\mathrm{add}_A A)$ of $\mathrm{add}_A A$. Let μ consist of the objects $(1 : A \to A)$ and $(0 \to A)$ in $\mathcal{S}(\mathrm{add}_A A)$.

Lemma 4.1. *Let A be an artin algebra. The functor $F_1 : \mathcal{S}(\mathrm{add}_A A) \to A$-mod, defined by $F_1(u) = \mathrm{Cok}(u)$ for a morphism u in $\mathcal{S}(\mathrm{add}_A A)$, yields an equivalence - also denoted by F_1 - of additive categories between the factor category $\mathcal{S}(\mathrm{add}_A A)/\mu$ and the stable category $\underline{P^{\leq 1}(A\text{-mod})}$ of $P^{\leq 1}(A\text{-mod})$.*

Proof. By definition of $\mathrm{Cok} : \mathrm{Mor}(\mathrm{add}_A(A)) \to A$-mod, it induces a full and dense functor $\mathcal{S}(\mathrm{add}_A A) \to P^{\leq 1}(A\text{-mod})$. So there is an induced full and dense functor $F_1 : \mathcal{S}(\mathrm{add}_A A) \to \underline{P^{\leq 1}(A\text{-mod})}$.

Now we compute the kernel of F_1. Obviously, under this functor F_1 the two objects in μ are sent to zero. Thus the ideal $\langle \mu \rangle$ is contained in the Kernel of F_1. We claim that $\mathrm{Ker} F_1 = \langle \mu \rangle$. Assume that there is a map $(g_1, g_0) : (f : P \to Q) \to (f' : P' \to Q')$ in $\mathcal{S}(\mathrm{add}_A A)$ such that $F_1(g_1, g_0) = 0$ in $\underline{P^{\leq 1}(A\text{-mod})}$. Then the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "f"] \arrow[d, "g_1"] & Q \arrow[r, "\pi"] \arrow[d, "g_0"] & F_1(f) \arrow[r] \arrow[d, "F_1(g_1,g_0)"] & 0 \\
0 \arrow[r] & P' \arrow[r, "f'"] & Q' \arrow[r, "\pi'"] & F_1(f') \arrow[r] & 0
\end{tikzcd}$$

Since $F_1(g_1, g_0)$ factors through Q', there is a map $h : Q \to P'$ such that $f'h = g_0$ and also $f'hf = g_0 f = f' g_1$. So $f'h = (g_0, 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence the following all squares commute:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "f"] & Q \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\
Q \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, "h"] & Q \oplus P \arrow[d, "{(g_0,0)}"] \\
P' \arrow[r, "f'"] & Q'
\end{tikzcd}$$

Because f' is a monomorphism, we have $hf = g_1$. This means that (g_1, g_0) factors through the map $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : Q \to Q \oplus P$, which belongs to $\mathrm{add}\mu$. Therefore, F_1 induces an equivalence $\mathcal{S}(\mathrm{add}_A A)/\mu \xrightarrow{\cong} \underline{P^{\leq 1}(A\text{-mod})}$. $\square$

Now we fix a notation. Let A be an artin algebra and e be an idempotent. We put $\mathrm{End}_{\underline{eAe}}(eA)^{\mathrm{op}} := \mathrm{End}_{eAe}(eA)^{\mathrm{op}}/\langle eAe \rangle$, where $\langle eAe \rangle$ is the ideal of eAe-mod given by all maps which factor through an object in $\mathrm{add}_{eAe}(eAe)$.

Theorem 4.2. *Let A be an artin algebra and e be an idempotent of A such that the (eAe, A)-bimodule eA has double centraliser property. Assume that eA is a finitely generated Gorenstein projective left eAe-module such that $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$. Then there exists a fully faithful functor*

$$F : \overline{A}\text{-mod} \to \mathcal{S}(\mathrm{add}_A A)/\mu$$

Proof. Since $A^{\mathrm{op}} \cong \mathrm{End}_{eAe}(eA)$, we get from the dual of [Bu, Proposition 2.9] that $\mathrm{Hom_A(A/AeA, Ae)} = 0$ and $\mathrm{Ext^1_A(A/AeA, Ae)} = 0$. The proof of the dual of [Bu, Lemma 2.11] yields $\mathrm{Hom_A(AeA, Ae)} \cong \mathrm{Hom_A(Ae \otimes_{eAe} eA, Ae)}$ as left A-modules. So we obtain $\mathrm{Hom_A(AeA, Ae)} \cong \mathrm{Hom_A(A, Ae)}$ as left A-modules, from the short exact sequence $0 \to AeA \to A \to A/AeA \to 0$ of left A-modules. Moreover, there is an isomorphism $Ae \cong \mathrm{Hom_{eAe}(eA, eAe)}$ as left A-modules. Therefore, $\underline{\mathrm{End}}_{eAe}(eA)^{\mathrm{op}} = \mathrm{End}_{eAe}(eA)^{\mathrm{op}}/\langle eAe\rangle \cong A/\langle Ae\rangle = A/AeA = \overline{A}$. This implies that abelian categories $\underline{\mathrm{End}}_{eAe}(eA)^{\mathrm{op}}$-mod and $\overline{A}$-mod are equivalent.

Since ${}_{eAe}eA$ is a generator of eAe-mod and eA is a left Gorenstein projective eAe-module, it follows that $eAe\text{-proj} \subseteq \mathrm{add}_{eAe}(eA) \subseteq (eAe\text{-}\mathcal{G}\mathrm{proj})^{\leq 1}_{eA}$. Denote by $\underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}$ the stable category of $(eAe\text{-}\mathcal{G}\mathrm{proj})^{\leq 1}_{eA}$ modulo projectives. Since $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$, we get from [Be, Theorem 7.1] that there is an equivalence $\underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}/\mathrm{add}_{eAe}(\underline{eA}) \cong \underline{\mathrm{End}}_{eAe}(eA)^{\mathrm{op}}$-mod, where

$$\underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}/\mathrm{add}_{eAe}(\underline{eA})$$

is the quotient category of $\underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}$ modulo the subcategory $\mathrm{add}_{eAe}(\underline{eA})$. This implies an equivalence of additive categories

$$F_2 : \underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}/\mathrm{add}_{eAe}(\underline{eA}) \xrightarrow{\cong} \overline{A}\text{-mod}.$$

Now we claim that the functor $\mathrm{Hom}_{eAe}(eA, -) : eAe\text{-mod} \to A\text{-mod}$ induces a functor

$$F_3 : (eAe\text{-}\mathcal{G}\mathrm{proj})^{\leq 1}_{eA} \to P^{\leq 1}(A\text{-mod}).$$

Indeed, for any $X \in (eAe\text{-}\mathcal{G}\mathrm{proj})^{\leq 1}_{eA}$, there is an exact sequence $0 \to G_1 \to G_0 \to X \to 0$, where each $G_i \in \mathrm{add}_{eAe}(eA)$. Since $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$, applying the functor $\mathrm{Hom}_{eAe}(eA, -)$ shows that the projective dimension of $\mathrm{Hom}_{eAe}(eA, X)$ is at most 1. Since eA is a generator of eAe-mod, the induced functor F_3 is fully faithful. Moreover, F_3 induces a fully faithful functor, also denoted by F_3,

$$F_3 : \underline{(eAe\text{-}\mathcal{G}\mathrm{proj})}^{\leq 1}_{eA}/\mathrm{add}_{eAe}(\underline{eA}) \to \underline{P^{\leq 1}(A\text{-mod})}$$

By Lemma 3.7 and the above arguments, setting $F = F_1^{-1} \circ F_3 \circ F_2^{-1}$ yields that $F : \overline{A}\text{-mod} \to \mathcal{S}(\mathrm{add}_A A)/\mu$ is a fully faithful functor. $\square$

The following result characterises gendo-d-Gorenstein algebras in terms of equivalences of categories.

Corollary 4.3. *Let A be an artin algebra and d be a non-negative integer. Then the following statements are equivalent:*

(1) A is a gendo-d-Gorenstein algebra.

(2) There is an idempotent e of A with eA being a finitely generated Gorenstein projective left eAe-module, such that:

There is an exact sequence

$$0 \to (0 : 0 \to eAe) \to (0 : 0 \to I^0) \to \cdots \to (0 : 0 \to I^d) \to 0$$

in $\mathrm{Mor}(eAe\text{-mod})$ *with each* $I^j \in \mathrm{add}(D(eAe))$.

There is an exact sequence

$$0 \to (0 : 0 \to eAe) \to (0 : 0 \to E^0) \to \cdots \to (0 : 0 \to E^d) \to 0$$

in $\mathrm{Mor}(\text{mod-}eAe)$ *with each* $E^j \in \mathrm{add}(D(eAe))$.

The above functor α induces an equivalence $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \xrightarrow{\cong} A\text{-mod}$ *of additive categories.*

Proof. By Theorem 3.3, α induces an equivalence $\mathrm{Mor}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to} A\text{-mod}$ if and only if the (eAe, A)-bimodule eA has double centraliser property. Note that there is an exact sequence $0 \to (0:0 \to eAe) \to (0:0 \to I^0) \to \cdots \to (0:0 \to I^d) \to 0$ in $\mathrm{Mor}(eAe\text{-mod})$ with each $I^j \in \mathrm{add}(D(eAe))$ if and only if there is an exact sequence $0 \to eAe \to I^0 \to \cdots \to I^d \to 0$ in eAe-mod with each $I^j \in \mathrm{add}(D(eAe))$. Moreover, there is an exact sequence $0 \to (0:0 \to eAe) \to (0:0 \to E^0) \to \cdots \to (0:0 \to E^d) \to 0$ in $\mathrm{Mor}(\text{mod-}eAe)$ with each $E^j \in \mathrm{add}(D(eAe))$ if and only if there is an exact sequence $0 \to eAe \to E^0 \to \cdots \to E^d \to 0$ in mod-eAe with each $E^j \in \mathrm{add}(D(eAe))$. It follows that there is an exact sequence $0 \to (0:0 \to eAe) \to (0:0 \to I^0) \to \cdots \to (0:0 \to I^d) \to 0$ in $\mathrm{Mor}(eAe\text{-mod})$ with each $I^j \in \mathrm{add}(D(eAe))$ and an exact sequence $0 \to (0:0 \to eAe) \to (0:0 \to E^0) \to \cdots \to (0:0 \to E^d) \to 0$ in $\mathrm{Mor}(\text{mod-}eAe)$ with each $E^j \in \mathrm{add}(D(eAe))$ if and only if eAe is a d-Gorenstein algebra. Applying [GK, Theorem 4.2] completes the proof. □

Corollary 4.4. *Let A be a gendo-d-Gorenstein algebra for some non-negative integer d, and e be its associated idempotent.*

(a) Then there is an equivalence

$$\mathcal{F}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \overset{\cong}{\to} \overline{A}\text{-mod}$$

of additive categories.

(b) Assume that $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$. *Then there is a fully faithful functor from* $\overline{A}$-mod *to* $\mathcal{S}(\mathrm{add}_A A)/\mu$.

Proof. (a) Since A is a gendo-d-Gorenstein algebra and e is its associated idempotent, it follows from [GK, Theorem 4.2] that $A \cong \mathrm{End}_{eAe}(eA)$. Thus by Corollary 3.5 we get that $\mathcal{F}(\mathrm{add}_{eAe}(eA))/\mathcal{X} \cong \overline{A}$-mod as additive categories.

(b) Since A is a gendo-d-Gorenstein algebra for some non-negative integer d and e being its associated idempotent, it follows from [GK, Theorem 4.2] that $A \cong \mathrm{End}_{eAe}(eA)$ with eA being a finitely generated Gorenstein projective left eAe-module. Since $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$, it follows from Theorem 3.8 that there is a fully faithful functor from $\overline{A}$-mod to $\mathcal{S}(\mathrm{add}_A A)/\mu$. □

Corollary 4.5. *Let A be a gendo-d-Gorenstein algebra for some non-negative integer d, and e be an associated idempotent. Assume that* $\mathrm{add}_{eAe}(eA)$ *is a maximal 1-orthogonal subcategory of* eAe-$\mathcal{G}$proj. *Then there is an equivalence of additive categories*

$$\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}/\mathrm{add}_{eAe}(\underline{eA}) \cong \overline{A}\text{-mod}.$$

Proof. Since A is gendo-d-Gorenstein and e is its associated idempotent, it follows from [GK, Theorem 4.2] that eA is a finitely generated Gorenstein projective left eAe-module. Note from [Iy, Proposition 2.4.1] that $\mathrm{add}_{eAe}(eA)$ is a maximal 1-orthogonal subcategory of eAe-$\mathcal{G}$proj if and only if $\mathrm{Ext}^1_{eAe}(eA, eA) = 0$ and $(eAe\text{-}\mathcal{G}\mathrm{proj})^{\leq 1}_{eA} = eAe\text{-}\mathcal{G}\mathrm{proj}$. Therefore, by the proof of Theorem 3.8,

$$\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}/\mathrm{add}_{eAe}(\underline{eA}) = (\underline{eAe\text{-}\mathcal{G}\mathrm{proj}})^{\leq 1}_{eA}/\mathrm{add}_{eAe}(\underline{eA}) \cong \overline{A}\text{-mod}$$

as additive categories. □

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