

INVARIANT HOLOMORPHIC FOLIATIONS ON KOBAYASHI HYPERBOLIC HOMOGENEOUS MANIFOLDS

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ABSTRACT. Let M be a Kobayashi hyperbolic homogeneous manifold. Let \mathcal{F} be a holomorphic foliation on M invariant under a transitive group G of bi-holomorphisms. We prove that the leaves of \mathcal{F} are the fibers of a holomorphic G -equivariant submersion $\pi: M \rightarrow N$ onto a G -homogeneous complex manifold N . We also show that if \mathcal{Q} is an automorphism family of a hyperbolic convex (possibly unbounded) domain D in \mathbb{C}^n , then the fixed point set of \mathcal{Q} is either empty or a connected complex submanifold of D .

1. INTRODUCTION

The existence of a nontrivial holomorphic foliation on a complex manifold M gives rise to restrictions on the geometry of the manifold itself. For instance, by the Baum–Bott index theorem [BaBo], the existence of such a foliation implies the vanishing of certain characteristic classes of M . As a consequence, complex projective spaces do not admit nonsingular holomorphic foliations. Recently, M. Brunella, M. McQuillan and L. G. Mendes (see [Bru]) gave a birational classification of (singular) holomorphic foliations on projective surfaces in the spirit of the Enriques–Kodaira classification. On noncompact manifolds all characteristic classes tend to vanish and a similar result is not to be expected without further assumptions.

Let G be a Lie group acting by biholomorphisms on M preserving a holomorphic foliation \mathcal{F} , i.e. if F is a leaf of \mathcal{F} , then so is $g \cdot F$, for all $g \in G$. In this context, A. Behague and B. Scárdua [BeSc] gave a complex version of a classical result of D. Tischler [Tis]. Namely, they showed that a holomorphic foliation with closed leaves which is invariant under a holomorphic transverse action of a complex Lie group of dimension equal to the codimension of the foliation is given by a holomorphic fibration.

In the case of a transitive action of a Lie group G , the leaves of a G -invariant foliation are nonsingular but might not be closed. For instance a foliation defined by a generic translation vector field on a complex torus \mathbb{C}^n/Λ , or on $(\mathbb{C}^*)^n$, has no closed leaves.

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Here we assume the manifold to be homogeneous and Kobayashi hyperbolic which, by a result of K. Nakajima [Nak], implies that M is biholomorphic to a homogeneous Siegel domain of type II. The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let M be a hyperbolic homogeneous complex manifold and G a group of automorphisms acting transitively on M . Let \mathcal{F} be a G -invariant holomorphic foliation on M . Then there exists a G -homogeneous complex manifold N and a G -equivariant holomorphic submersion $\pi: M \rightarrow N$ such that the leaves of \mathcal{F} are the fibers of π .*

The base N might not be hyperbolic (cf. Example 4.2). The hyperbolicity of M implies that the automorphism group $\text{Aut}(M)$ of M is a Lie group acting properly on M . The foliation \mathcal{F} is invariant with respect to the closure of G in $\text{Aut}(M)$. Thus, without loss of generality we may assume G to be closed in $\text{Aut}(M)$, implying that G is a Lie group acting properly on M .

Examples of equivariant submersions whose base and total space are hyperbolic homogeneous Siegel domains are given, e.g., in [Mie], Prop. 5.6, where C. Miebach remarked that such submersions do not need to be holomorphic fiber bundles (cf. Example 4.3). In fact, if the submersion π in the above theorem is a holomorphic fiber bundle, then M is biholomorphic to a product $N \times F$ of hyperbolic, homogeneous Siegel domains and G is a subgroup of $\text{Aut}(N) \times \text{Aut}(F)$ (cf. Lemma 4.1).

If the foliation is not invariant, then hyperbolicity is not a sufficient condition for the leaves to be closed. In Section 4 we present an example (Example 3.2) suggested by John Erik Fornæss of a nonsingular holomorphic foliation on the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ having some nonclosed leaves.

The paper is organized as follows. In Section 2 we introduce “uniform Bochner–Frobenius local coordinates” and present some preliminary material. In Section 3 we show that the leaves of \mathcal{F} are closed. In Appendix A we give a different proof of this fact by exploiting the Lie group structure of the automorphism group of a Siegel domain. In Section 4 we prove Theorem 1.1 and discuss some examples.

In Section 5 we consider the foliation \mathcal{F}^K induced on the fixed point set M^K of the isotropy subgroup K of G at one point. The set M^K is a connected homogeneous complex submanifold of M . We show that every leaf of \mathcal{F}^K is the intersection of a leaf of \mathcal{F} with M^K , i.e. such an intersection is connected. The result is based on the following proposition, which might be of interest on its own.

Proposition 1.2. *Let D be a hyperbolic convex (possibly unbounded) domain in \mathbb{C}^n and let \mathcal{Q} be a family of automorphisms of D . Then the fixed point set $D^{\mathcal{Q}}$ of \mathcal{Q} is either empty or a connected complex submanifold of D .*

Note that if $D^{\mathcal{Q}}$ is not empty, then \mathcal{Q} is relatively compact in $\text{Aut}(D)$. Using Bochner coordinates one easily sees that each connected component of $D^{\mathcal{Q}}$ is a complex submanifold of the domain. The main issue in the above proposition is the connectness of $D^{\mathcal{Q}}$.

2. PRELIMINARIES

Let M be a hyperbolic complex manifold. We refer to [Kob] for the definition of Kobayashi distance, hyperbolic manifolds and their properties. Let G be a

connected, closed subgroup of $\text{Aut}(M)$. Then the G -action is proper (cf. [Kob], Thm. 5.4.2). In particular every isotropy subgroup

$$G_p := \{ g \in G : g \cdot p = p \}$$

is compact. Denote by \mathbb{B}^m the unit radius Euclidean ball in \mathbb{C}^m centered at 0.

Lemma 2.1. *Let $M \cong G/K$ be a hyperbolic, homogeneous n -dimensional complex manifold and let \mathcal{F} be a k -dimensional, G -invariant holomorphic foliation on M . Then for any $p \in M$ there exist holomorphic local coordinates (U_p, ψ_p) , with U_p a G_p -invariant neighborhood of p , such that:*

- (i) $\psi_p(U_p) = \mathbb{B}^{n-k} \times \mathbb{B}^k$,
- (ii) the plaques of \mathcal{F} in U_p correspond to $\{ (z, w) \in \mathbb{B}^{n-k} \times \mathbb{B}^k : z = \text{const} \}$,
- (iii) there is a faithful representation $L : G_p \rightarrow U(n-k) \times U(k)$ so that for every $g \in G_p$ and $q \in U_p$ one has $\psi_p(g \cdot q) = L(g) \cdot \psi_p(q)$.

Proof. Since the foliation is nonsingular near p , there are local holomorphic coordinates (z, w) centered at p (called ‘‘Frobenius coordinates’’) in which the plaques of \mathcal{F} are $\{z = \text{const}\}$. Since G_p is compact, averaging any inner product on T_pM over Haar measure gives a G -invariant Riemannian metric on M inducing the standard topology. Since G_p fixes p , we can pick a G_p -invariant open set U (for example, a ball in such a metric) sufficiently small as to lie in the domain of the coordinates. So without loss of generality, we can assume that $U \subset \mathbb{C}^n$ is a bounded domain, $p = 0$ and the foliation \mathcal{F} on U is given by $\{z = \text{const}\}$.

Identify \mathbb{C}^n with its tangent space at the origin and for $g \in G_p$ consider the linear operator on \mathbb{C}^n defined by $L(g) = dg_p$. Then $L(G_p)$ is a compact group of linear transformations of \mathbb{C}^n permuting the affine subspaces $\{z = \text{const}\}$. By choosing an $L(G_p)$ -invariant Hermitian inner product on \mathbb{C}^n we realize $L(G_p)$ as a closed subgroup of the unitary group $U(n)$. Moreover, since the subspace $\{z = 0\}$ is invariant, one has $L(G_p) \subseteq U(n - k) \times U(k)$.

We conclude by showing that such an $L(G_p)$ -action on \mathbb{C}^n is a local linearization of the G_p -action on M . Following [Boc], p. 375, consider the Haar measure μ on G_p and define a holomorphic map on U by

$$\psi_p(z, w) := \int_{G_p} L(h)^{-1} \cdot (h \cdot (z, w)) d\mu(h).$$

By construction $(d\psi_p)_p = \text{Id}$ and, shrinking U if necessary, ψ_p is a biholomorphism onto its image. Also, $\psi_p \circ g = L(g) \circ \psi_p$ for all $g \in G_p$. Note that ψ_p preserves \mathcal{F} , since both g and $L(g)$ permute the affine subspaces $\{z = \text{const}\}$. Finally, arrange so that the image of ψ_p coincides with a copy of $\mathbb{B}^{n-k} \times \mathbb{B}^k$. □

The above coordinates, which we refer to as *Bochner–Frobenius local coordinates*, are *uniform* with respect to the Kobayashi distance of M in the following sense.

Lemma 2.2. *Let $M \cong G/K$ be a hyperbolic, homogeneous complex manifold and let \mathcal{F} be a G -invariant holomorphic foliation on M . Then there exists $r > 0$ such that every point $p \in M$ lies in a system of Bochner–Frobenius local coordinates, centered at p , and defined on the Kobayashi ball of radius $r > 0$ centered at p .*

Proof. Since M is complete hyperbolic (see [Kob], Thm. 3.6.22), the topology of M coincides with that defined by the Kobayashi distance. Thus, given $p_0 \in M$ there

exists $r > 0$ such that the Kobayashi ball centered at p_0 of radius r lies inside the domain of Bochner–Frobenius coordinates (U_{p_0}, ψ_{p_0}) . Let $p \in M$ and $g \in G$ be such that $g \cdot p_0 = p$. It is easy to check that $(g \cdot U_{p_0}, \psi_{p_0} \circ g^{-1})$ are Bochner–Frobenius local coordinates centered at p . Then the result follows by recalling that g is an isometry for the Kobayashi distance. \square

The following fact will be used in Section 4 in order to show that the fibration onto the leaf space of the foliation is a holomorphic submersion.

Proposition 2.3. *Let M be a complex manifold and \mathcal{F} a holomorphic foliation on M . Assume there exists a real manifold N and a C^∞ submersion $\pi: M \rightarrow N$ such that the leaves of \mathcal{F} are the fibers of π . Then there exists a unique complex structure on N so that π is holomorphic.*

Proof. We say that a complex-valued function on an open subset of N is an *up-function* if its composition with π is holomorphic. A complex structure on N is *down* if the functions which are holomorphic with respect to that complex structure are precisely the up-functions, on every open subset of N .

Note that any two complex structures are equal if they have the same holomorphic functions on every open set. Therefore a down complex structure on N is unique, if it exists. The problem of existence is local: should we prove existence of a down complex structure on each open set in a covering of N , then by uniqueness these complex structures agree where any two are defined, so they glue together to a unique down complex structure on N .

Fix a point $p_0 \in M$. Choose local Frobenius coordinates $(U, (z, w))$ centered at p_0 , so that the plaques of \mathcal{F} are given by $\{z = \text{const}\}$. The complex submanifold $Z := \{(z, w) : w = 0\}$ of M is transverse to the foliation and, by shrinking U if necessary, the map π restricts to Z to be a local diffeomorphism $\pi|_Z: Z \rightarrow \pi(Z)$. This diffeomorphism defines a (integrable) complex structure on the open set $\pi(Z) \subset N$. In this complex structure, a holomorphic function on $\pi(Z)$ is precisely one whose pull-back is holomorphic on Z . In particular, up-functions on open subsets of $\pi(Z)$ are holomorphic.

In order to conclude the proof, we need to show that every function on an open subset W of $\pi(Z)$, which is holomorphic with respect to such a complex structure, is in fact an up-function. The pull-back $f: \pi^{-1}(W) \rightarrow \mathbb{C}$ of such a function is constant along the fibers of π over W and holomorphic on $\pi^{-1}(W) \cap Z$. With respect to the above Frobenius coordinates, f is given by $(z, w) \mapsto f(z, w) = f(z, 0)$. Thus it is holomorphic on $\pi^{-1}(W) \cap U$.

In order to check that f is holomorphic on $\pi^{-1}(W)$, let $q_1 \in \pi^{-1}(W)$. Then there exists a unique point $q_0 \in Z$ such that $q_1 \in \pi^{-1}(\pi(q_0))$. By assumption, the fiber $\pi^{-1}(\pi(q_0))$ is the leaf of \mathcal{F} through q_0 , hence it is connected. Therefore there exists a continuous curve $\rho: [0, 1] \rightarrow \pi^{-1}(\pi(q_0))$ such that $\rho(0) = q_0$ and $\rho(1) = q_1$. The set

$$A := \{t \in [0, 1] : f \text{ is holomorphic in a neighborhood of } \rho(t) \text{ in } M\}$$

is clearly open in $[0, 1]$. Moreover $0 \in A$, since f is holomorphic on $U \ni \rho(0)$. Suppose that $A \neq [0, 1]$; take $t_1 \in [0, 1]$ to be the smallest real number not in A . Take a new Frobenius coordinate system $(U_1, (z, w))$ centered at $\rho(t_1)$ such that $U_1 \subset \pi^{-1}(W)$. Then, for t_2 close enough to t_1 , with $t_2 < t_1$, the point $\rho(t_2)$ belongs to U_1 . Thus $\rho(t_2) = (0, w_2)$, for some w_2 in $\mathbb{C}^{\dim \mathcal{F}}$.

By the definition of t_1 , the function f is holomorphic near $(0, w_2)$. Since the function f is independent of w in the Frobenius coordinate system $(U_1, (z, w))$, it follows that $f(z, 0) = f(z, w_2)$. This implies that $f(z, w) = f(z, 0)$ is holomorphic in a neighborhood of $\rho(t_1) = (0, 0)$, giving a contradiction. Thus $A = [0, 1]$ and f is holomorphic in a neighborhood of q_1 for every $q_1 \in \pi(W)$, i.e. f is holomorphic on $\pi^{-1}(W)$. Hence the defined complex structure on $\pi(Z)$ is down, concluding the proof. \square

3. LEAVES ARE CLOSED

In this section we show that under the assumptions of Theorem 1.1 the leaves of the foliation are closed.

Proposition 3.1. *Let G be a Lie group acting transitively on a hyperbolic, complex manifold M and let \mathcal{F} be a G -invariant holomorphic foliation on M . Then the leaves of \mathcal{F} are closed.*

Proof. Let p be a point in M and assume by contradiction that the leaf $F_p \in \mathcal{F}$ through p is not closed.

Claim. There exist Bochner–Frobenius coordinates (U_p, ψ_p) at p and a sequence $\{p_n\} \subset U_p \cap F_p$ such that $p_n \rightarrow p$ and, for every $n \in \mathbb{N}$, the points p_n do not belong to the plaque of F_p in U_p containing p .

Proof of claim. The plaques of \mathcal{F} on U_p are given by $\{(z, w) \in \mathbb{B}^{n-k} \times \mathbb{B}^k : z = \text{const}\}$, and p belongs to the plaque $\{z = 0\}$. If a sequence as in the claim does not exist, then, by shrinking U_p if necessary, we may assume that $F_p \cap U_p$ consists of a unique plaque, namely $\{z = 0\}$. Since F_p is not closed, there exists a sequence of points $q_n \in F_p$ such that $q_n \rightarrow q$, with $q \notin F_p$. By homogeneity (see Lemma 2.2), there exist Bochner–Frobenius coordinates (U_{q_n}, ψ_{q_n}) at q_n such that $F_p \cap U_{q_n}$ consists of a unique plaque and, for n large enough, $q \in U_{q_n}$. In particular $F_p \cap U_{q_n}$ is closed in U_{q_n} and since $q_m \in F_p \cap U_{q_n}$, for m large enough, this gives a contradiction and proves the claim.

The leaf F_p can be regarded as an immersed complex submanifold $\varphi : N \rightarrow M$, with $\varphi(N) = F_p$. Let p_n be as in the claim, $\zeta, \zeta_n \in N$ such that $\varphi(\zeta) = p$ and $\varphi(\zeta_n) = p_n$. Since p_n does not belong to the same plaque of p and φ is an immersion, it follows that ζ is not in the closure of the set $\{\zeta_n\}$. Pick $g_n \in G$ such that $g_n \cdot p = p_n$ and, consequently, $g_n \cdot F_p = F_p$. Note that g_n maps plaques of \mathcal{F} on a neighborhood U onto plaques of \mathcal{F} on $g_n \cdot U$, and the plaques are open in $F_p \cong N$. Thus, $h_n := \varphi^{-1} \circ g_n \circ \varphi : N \rightarrow N$ is a biholomorphism of N such that $h_n \cdot \zeta = \zeta_n$.

Since M is complete hyperbolic and hence taut (see, e.g., [Kob]) and $p_n = g_n \cdot p \rightarrow p$, we can assume without loss of generality that $\{g_n\}$ converges uniformly on compacta to some $g \in G_p$. Hence $h_n \rightarrow h := \varphi^{-1} \circ g \circ \varphi : N \rightarrow N$, with h an automorphism of N such that $h \cdot \zeta = \zeta$. This implies $\zeta_n = h_n \cdot \zeta \rightarrow \zeta$ and gives a contradiction. \square

We conclude this section with an example of a nonsingular holomorphic foliation on the unit ball \mathbb{B}^2 of \mathbb{C}^2 having nonclosed generic leaves.

Example 3.2. Let X be the real vector field in \mathbb{R}^2 given by

$$X(x, y) := -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \left(\frac{1}{2} - x^2 - y^2 \right) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right).$$

Then the circle $C := \{x^2 + y^2 = \frac{1}{2}\}$ is a closed integral curve of X . All nearby leaves accumulate to that curve. Indeed for each (x, y) close enough to $x^2 + y^2 = \frac{1}{2}$ the standard scalar product

$$\left\langle X(x, y), x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\rangle$$

is positive if (x, y) belongs to the disc of radius $1/2$, and negative otherwise. Hence X pushes towards C .

Let Z be the complexification of X in the (z, w) -plane, i.e. $Z(z, w)$ is given by replacing x with z and y with w . Let \mathcal{F} be the holomorphic foliation defined by Z . A direct computation shows that the only singularity of \mathcal{F} is $(0, 0)$. Moreover, the complex conic $C^{\mathbb{C}} := \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1/2\}$ is a closed leaf of \mathcal{F} and nearby leaves accumulate to it. Consider the compact, polynomially convex set

$$K := \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1/2, |z|^2 + |w|^2 \leq 2\}$$

not containing the origin. By [For], Prop. 2.1, given a small positive $\varepsilon > 0$, there exists an automorphism $\Phi \in \text{Aut}(\mathbb{C}^2)$ such that $\Phi(0, 0) = (2, 0)$ and $|\Phi(z) - z| < \varepsilon$ for all $z \in K$.

Consider the foliation $\tilde{\mathcal{F}}$ given by the restriction to \mathbb{B}^2 of the holomorphic foliation $(\Phi^{-1})^*\mathcal{F}$. By construction the foliation $\tilde{\mathcal{F}}$ has no singularities in \mathbb{B}^2 and, for ε small enough, $\Phi(C)$ is contained in \mathbb{B}^2 . Then the connected component of $\Phi(C^{\mathbb{C}}) \cap \mathbb{B}^2$ containing $\Phi(C)$ is a closed leaf of $\tilde{\mathcal{F}}$ and nearby leaves accumulate on it, thus they are not closed.

4. THE EQUIVARIANT SUBMERSION

Proof of Theorem 1.1. Without loss of generality we may assume that G is a connected, closed subgroup of $\text{Aut}(M)$. The leaf F through the base point is closed by Proposition 3.1, therefore so is its stabilizer L in G . Let $N := G/L$, a homogeneous manifold with G -equivariant fiber bundle map $\pi: M = G/K \rightarrow N = G/L$. By Proposition 2.3, there is a unique complex structure on the leaf space N such that π is holomorphic. Since the real analytic G -action on M is by biholomorphisms, so is the G -action on N . □

Let M be a hyperbolic G -homogeneous manifold, let N be a G -homogeneous manifold and let $\pi: M \rightarrow N$ be a G -equivariant holomorphic submersion. The G -equivariance implies that such a submersion is a smooth fiber bundle. However, in general π does not admit any local holomorphic trivialization, as noticed in the case of certain equivariant submersions by C. Miebach in [Mie], Prop. 5.6 and Rem., p. 347. The fibers of Miebach’s submersions are biholomorphic to balls and the bases of his submersions are hyperbolic. In this section we construct an example where the base of the submersion is not hyperbolic. We first collect some useful facts.

Lemma 4.1. *Assume that the G -equivariant submersion $\pi: M \rightarrow N$ in Theorem 1.1 is a holomorphic fiber bundle. Then π is a trivial holomorphic fiber bundle, M*

is biholomorphic to a product $N \times F$ of hyperbolic, homogeneous Siegel domains and G is a subgroup of $\text{Aut}(N) \times \text{Aut}(F)$.

Proof. Since the total space of the holomorphic fiber bundle $\pi: M \rightarrow N$ is hyperbolic, the base N is also hyperbolic by a result of S. Nag [Nag]. As a consequence, N is biholomorphic to a homogeneous Siegel domain of type II [Nak]. In particular it is simply connected. Then, Royden [Roy], Cor. 1, implies that π is a trivial holomorphic fiber bundle and consequently M is biholomorphic to $N \times F$.

We are left to show that G is contained in the product $\text{Aut}(N) \times \text{Aut}(F)$ (cf. [Kob], Cor. 5.4.12, for G connected).

Write $M = N \times F$ and for g in G let $g \cdot (z, w) = (\alpha_g(z, w), \beta_g(z, w))$. Since g preserves the leaves $\{*\} \times F$ of \mathcal{F} , it follows that α_g does not depend on w . Then one checks that the map $G \rightarrow \text{Aut}(N)$, given by $g \rightarrow \alpha_g$, is a group homomorphism.

Fix $z \in N$ and consider the map $\beta_g(z, *): F \rightarrow F$, defined by $w \rightarrow \beta_g(z, w)$. Let $g^{-1} \cdot (z, w) = (\alpha_g^{-1}(z), \beta_{g^{-1}}(z, w))$. Since $g^{-1}g \cdot (z, w) = (z, w)$, it follows that

$$\beta_{g^{-1}}(\alpha_g(z), \beta_g(z, *)) = \text{Id}_F.$$

Hence, $\beta_{g^{-1}}(\alpha_g(z), *)$ is a left inverse of $\beta_g(z, *)$. Similarly, one checks that it is a right inverse as well. Thus $\beta_g(z, *)$ is an automorphism of F . Moreover the map

$$N \times F \rightarrow F, \quad (z, w) \rightarrow \beta_g(z, w),$$

is holomorphic. Then the Proposition in [Roy] applies to show that β_g does not depend on z and one checks that the map

$$G \rightarrow \text{Aut}(N) \times \text{Aut}(F), \quad g \rightarrow (\alpha_g, \beta_g),$$

is a group homomorphism with trivial kernel. □

Example 4.2. Let $M = \mathbb{B}^2$ be the unit ball of \mathbb{C}^2 and G the five-dimensional isotropy subgroup of $\text{Aut}(\mathbb{B}^2) \cong SU(2, 1)$ at a boundary point $q \in \partial\mathbb{B}^2$. The group G is of the form TS , where T is a one-dimensional torus of $SU(1, 2)$ and $S = AN$ is the solvable factor of an Iwasawa decomposition KAN of $SU(1, 2)$. In particular G acts transitively on \mathbb{B}^2 and leaves invariant the foliation whose leaves are the complex geodesics whose closure contains q (cf. [Aba], Cor. 2.6.9, p. 308). Moreover, the space of leaves is biholomorphic to the space of complex lines through q which are not tangent to \mathbb{B}^2 , i.e. to \mathbb{C} .

In order to give a simple realization of this construction, it is convenient to consider the hyperbolic model $\mathbb{H}^2 := \{(z, w) \in \mathbb{C}^2 : \text{Im } w > |z|^2\}$ of \mathbb{B}^2 embedded in \mathbb{P}^2 via the map

$$\mathbb{H}^2 \rightarrow \mathbb{P}^2, \quad (z, w) \rightarrow [z : w : 1].$$

Any complex geodesic of \mathbb{H}^2 whose closure contains the boundary point $[0 : 1 : 0]$ is given by $\{z = \text{const}\}$. These complex geodesics define a holomorphic foliation on \mathbb{H}^2 which is invariant with respect to the five-dimensional isotropy group G of $\text{Aut}(\mathbb{H}^2)$ at $[0 : 1 : 0]$. The isotropy of G at $(0, i)$ is given by $\{(z, w) \rightarrow (e^{i\theta}z, w) : \theta \in \mathbb{R}\}$ and G contains the solvable subgroup generated by the elements of the form

$$\begin{aligned} (z, w) &\rightarrow (z, w + t), & (z, w) &\rightarrow (e^t z, e^{2t} w), \\ (z, w) &\rightarrow (z + t, w + 2itz + it^2), & (z, w) &\rightarrow (z + it, w + 2tz + it^2), \end{aligned}$$

for $t \in \mathbb{R}$ (cf. [Mie], Sect. 4). In particular, G acts transitively on \mathbb{H}^2 .

Every leaf of the foliation is biholomorphic to the unit disc in \mathbb{C} . The G -equivariant submersion of Theorem 1.1 is given by the projection

$$\pi: \mathbb{H}^2 \rightarrow \mathbb{C}, \quad (z, w) \rightarrow z,$$

and, as a consequence of the above lemma, it is not a holomorphic fiber bundle.

Example 4.3. Let M be a hyperbolic homogeneous manifold. By a result of H. Ishi [Ish] the isotropy subgroups of $\text{Aut}(M)$ are at least one-dimensional. In fact, such a result is sharp. As an example, consider the tube domain M_V over the Vinberg cone

$$V := \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5 : y_3 > 0, y_1y_3 - y_4^2 > 0, y_2y_3 - y_5^2 > 0\}.$$

Then, by [Gea], Prop. 2.2, the isotropy subgroup of $\text{Aut}(M_V)$ at any point of M_V is one-dimensional. Let G be a transitive subgroup of $\text{Aut}(M_V)$ and \mathcal{F} a G -invariant foliation (e.g. the fibration constructed in [Mie]). Then the induced equivariant submersion is not a holomorphic fiber bundle. If it were, by the above lemma the one-dimensional isotropy subgroup K of $\text{Aut}(M_V)$ at a point $(z_0, w_0) \in N \times F \cong M$ would contain the product of the isotropy subgroups of $\text{Aut}(N)$ at z_0 and $\text{Aut}(F)$ at w_0 . Thus, by Ishi’s result, K would be at least two-dimensional, giving a contradiction.

5. FIXED POINT SETS OF ISOTROPY SUBGROUPS

In this section we first prove Proposition 1.2.

Lemma 5.1. *Let D be a hyperbolic convex (possibly unbounded) domain in \mathbb{C}^n . The Kobayashi balls of D are bounded convex domains.*

Proof. Let \mathbb{B}_N be the ball in \mathbb{C}^n with center 0 and radius $N \in \mathbb{N}$. Let $D_N := D \cap \mathbb{B}_N$. Note that D_N is a bounded convex domain for all $N \in \mathbb{N}$. Let k_{D_N} denote the Kobayashi distance of D_N . Since D is the increasing union of the D_N , it follows that $k_D = \lim_{N \rightarrow \infty} k_{D_N}$. For every N , the Kobayashi distance k_{D_N} is a convex function (see [Aba], Prop. 2.3.46). Passing to the limit, k_D is a convex function as well. By [BrSa], a hyperbolic convex domain is complete hyperbolic. Hence, the Kobayashi balls of D are bounded convex domains in \mathbb{C}^n . \square

Proof of Proposition 1.2. Assume that

$$D^{\mathcal{Q}} := \{z \in D : g \cdot z = z \text{ for all } g \in \mathcal{Q}\}$$

is nonempty and let $z \in D^{\mathcal{Q}}$. Then \mathcal{Q} is contained inside the isotropy subgroup of z in $\text{Aut}(D)$ which, by the hyperbolicity of D , is compact. Using Bochner’s local coordinates one sees that $D^{\mathcal{Q}}$ is a closed complex submanifold of D (cf. [Aba], Cor. 2.5.10). Moreover, there exists $s(z) > 0$ such that for all $0 < r < s(z)$ the intersection of the Kobayashi ball $B(z, r)$ with $D^{\mathcal{Q}}$ lies in the connected component $(D^{\mathcal{Q}})^z$ of z in $D^{\mathcal{Q}}$.

Assume by contradiction that $D^{\mathcal{Q}}$ is not connected. Define

$$R_z := \max\{r > 0 : (B(z, r) \cap D^{\mathcal{Q}}) \subset (D^{\mathcal{Q}})^z\}.$$

Then $R_z > 0$. Moreover, since $B(z, r) \subset B(z, s)$ for $0 < r < s$ and $\bigcup_{r>0} B(z, r) = D$, it follows that $R_z < +\infty$. For every $r > R_z$ the Kobayashi ball $B(z, r)$ intersects $D^{\mathcal{Q}}$ in another connected component different from $(D^{\mathcal{Q}})^z$. Therefore, there exists $w \in \partial B(z, R_z)$ such that $w \in D^{\mathcal{Q}}$ and $w \notin (D^{\mathcal{Q}})^z$. Let $(D^{\mathcal{Q}})^w$ be the connected

component of w in $D^{\mathcal{Q}}$. As before, there exists a maximal $R_w > 0$ such that $B(w, r) \cap D^{\mathcal{Q}} \subset (D^{\mathcal{Q}})^w$. Let $A := B(z, R_z) \cap B(w, R_w)$. By construction, $A \cap D^{\mathcal{Q}} = \emptyset$. Moreover, since both $B(z, R_z)$ and $B(w, R_w)$ are \mathcal{Q} -invariant, it follows that A is \mathcal{Q} -invariant.

Since A is a nonempty bounded convex domain by Lemma 5.1 and \mathcal{Q} generates a relatively compact subgroup of $\text{Aut}(A)$, there exists a point of A fixed by \mathcal{Q} (see [Aba], Thm. 2.5.7). Hence $A \cap D^{\mathcal{Q}} \neq \emptyset$, giving a contradiction. \square

We conclude with a result on the fixed point sets of the isotropy subgroups of G .

Proposition 5.2. *Let $M \cong G/K$ be a hyperbolic homogeneous complex manifold and \mathcal{F} a G -invariant holomorphic foliation on M . Consider the fixed point set*

$$M^K = \{q \in M : k \cdot q = q \text{ for all } k \in K\}$$

and the foliation \mathcal{F}^K on M^K induced by \mathcal{F} . Then the hyperbolic complex submanifold M^K is connected and homogeneous with respect to a free action of a Lie group N . Moreover, \mathcal{F}^K is N -invariant and every leaf of \mathcal{F}^K is given by $F \cap M^K$, with F a leaf of \mathcal{F} .

Proof. By [Nak], M is biholomorphic to a convex domain in \mathbb{C}^n . Hence, by Proposition 1.2, the fixed point set M^K is a connected hyperbolic complex manifold. Moreover,

$$M^K = N_G(K) \cdot p = N \cdot p,$$

where $N := N_G(K)^e/K$ and $N_G(K)^e$ is the identity component in $N_G(K)$, the normalizer of K in G . The leaves of \mathcal{F}^K are the connected components of $F \cap M^K$, with F varying among the leaves of \mathcal{F} . By construction, \mathcal{F}^K is N -invariant. The fixed point set F^K of K in F is connected by Proposition 1.2. Since $F \cap M^K = F^K$, it follows that the intersection $F \cap M^K$ is a leaf of \mathcal{F}^K . \square

APPENDIX A

Here we give a different proof of Proposition 3.1 which exploits the Lie group structure of the automorphism group of a Siegel domain. The proof follows at once from the following lemmas.

Lemma A.1. *Let M be a hyperbolic homogeneous manifold and let G be a closed subgroup of $\text{Aut}(M)$ which acts transitively on M . Then G contains a closed, simply connected, solvable subgroup S which acts freely and transitively on M .*

Proof. Without loss of generality we may assume that G is connected. Thus it is contained in the identity component of the automorphism group of M . By [Nak], the manifold M is biholomorphic to a homogeneous Siegel domain of type II. By [Kan], p. 38, Thm. B, the group G admits a faithful finite-dimensional representation. Then from the Levi decomposition it follows that G decomposes as a semidirect product

$$G = (TL) \ltimes P,$$

where P is a simply connected solvable Lie group, T is a compact torus and L is a closed real semisimple subgroup of G centralizing T (see, e.g., [Var], Ex. 44e, p. 256). Let KAN be an Iwasawa decomposition of L (cf. [Hel]). We claim that the closed, simply connected, solvable subgroup $S := (AN) \ltimes P$ of G acts freely and transitively on M .

Indeed P is contractible ([Var], Thm. 3.18.11), therefore so is S . As a consequence, the group S admits no nontrivial compact subgroups and, since the S -action is proper, it acts freely on M .

Finally, TK is a compact subgroup of G , thus it is contained in (in fact, it coincides with) an isotropy subgroup of the transitive Lie group G (cf. [Kan], Prop. 5.7). As a consequence, S acts transitively on M . \square

Remark A.2. The above solvable subgroup S is not necessarily split. One can construct examples of simply connected, real nonsplit, solvable Lie groups acting freely and transitively on certain Hermitian symmetric spaces.

Lemma A.3. *Let M be a hyperbolic homogeneous manifold endowed with a holomorphic foliation which is invariant with respect to a closed, connected subgroup S of $\text{Aut}(M)$ acting freely and transitively on M . Then*

- (i) *S is solvable and simply connected. In particular, every connected Lie subgroup of S is closed.*
- (ii) *The leaves of the foliation are closed.*

Proof. (i) From Lemma A.1 it follows that S is simply connected and solvable. Thus every connected Lie subgroup of S is closed (cf. [Var], Thm. 3.18.12).

(ii) Let L_p be the stabilizer in S of a leaf F_p . Then F_p is the orbit of L_p . Since M is diffeomorphic to S via the S -action and F_p is connected, it follows that L_p is connected. Thus L_p is closed in S by (i) and F_p is closed in M . \square

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