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MORITA EQUIVALENCE CLASSES OF 2-BLOCKS OF DEFECT THREE

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Abstract. We give a complete description of the Morita equivalence classes of blocks with elementary abelian defect groups of order 8 and of the derived equivalences between them. A consequence is the verification of Broué's abelian defect group conjecture for these blocks. It also completes the classification of Morita and derived equivalence classes of 2-blocks of defect at most three defined over a suitable field.

1. Introduction

Throughout let k be an algebraically closed field of prime characteristic ℓ and let $\mathcal{O}$ be a discrete valuation ring with residue field k and field of fractions K of characteristic zero. We assume that K is large enough for the groups under consideration. We consider blocks B of $\mathcal{O}G$ with defect group D, for finite groups G.

We are concerned with the description of the Morita and derived equivalence classes of (module categories for) blocks of finite groups with a given defect group D. We briefly review progress on this problem to date. If D is an abelian ℓ-group whose automorphism group is an ℓ-group, then any block with defect group D must be nilpotent and so Morita equivalent to $\mathcal{O}D$ (see [7] and [24]). There are many other examples of ℓ-groups for which it has been proved that every fusion system is nilpotent, but we do not list these here. If D is cyclic, then the Morita equivalence classes can be characterised in terms of Brauer trees, in work going back to Brauer ([4], [5]) and Dade ([11]), culminating in the determination of the source algebras in [31]. In a series of papers Erdmann characterises the Morita equivalence classes of tame blocks defined over k except when D is generalised quaternion and B has two simple modules (see [17]), although the problem remains open for blocks defined over $\mathcal{O}$. The (three) Morita equivalence classes of blocks defined over $\mathcal{O}$ with defect group $C_2 \times C_2$ are determined in [30]. When $D = \langle x, y : x^{2^r} = y^{2^s} = [x,y]^2 = [x,[x,y]] = [y,[x,y]] = 1\rangle$, where $r \geq s \geq 1$ (non-metacylic minimal non-abelian 2-group), the Morita equivalence classes of blocks are determined in [40] and [14]. When D is a homocyclic 2-group, the Morita equivalence classes of blocks are determined in [13].

In this paper we use the classification given in [13] to completely determine the Morita and derived equivalence classes of blocks defined over $\mathcal{O}$ with defect group $D \cong C_2 \times C_2 \times C_2$. As a consequence it follows that Broué's abelian defect group

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conjecture holds for 2-blocks of defect three. This of course is the culmination of the work of many, both on classifications and on constructions of categorical equivalences between blocks. We also note that this completes the classification of Morita equivalence classes of 2-blocks of defect at most three, for blocks defined over k. Blocks with elementary abelian defect groups of order 8 have already been studied in [21], where it is shown that Alperin's weight conjecture and the isotypy version of Broué's abelian defect group conjecture hold for these blocks. The results of [21] are needed here, in particular to achieve Morita equivalences over $\mathcal{O}$ rather than k.

Before stating the main theorem, we recall the definition of the inertial quotient of B. Let b_D be a block of $\mathcal{O}DC_G(D)$ with Brauer correspondent B, and write $N_G(D, b_D)$ for the stabilizer in $N_G(D)$ of b_D under conjugation. Then the *inertial quotient* of B is $E = N_G(D, b_D)/DC_G(D)$, an ℓ'-group unique up to isomorphism.

Theorem 1.1. *Let B be a block of $\mathcal{O}G$, where G is a finite group. If B has defect group D isomorphic to $C_2 \times C_2 \times C_2$, then B is Morita equivalent to the principal block of precisely one of the following:*
(i) D;
(ii) $D \rtimes C_3$;
(iii) $C_2 \times A_5$, and the inertial quotient is C_3;
(iv) $D \rtimes C_7$;
(v) $SL_2(8)$, and the inertial quotient is C_7;
(vi) $D \rtimes (C_7 \rtimes C_3)$;
(vii) J_1, and the inertial quotient is $C_7 \rtimes C_3$;
(viii) ${}^2G_2(3) \cong \mathrm{Aut}(SL_2(8))$, and the inertial quotient is $C_7 \rtimes C_3$.

Blocks are derived equivalent if and only if they have the same inertial quotient.

For completeness we note that by [15] representatives for the Morita equivalence classes of blocks (with respect to k) with defect group D_8 are the principal blocks of kD_8, kA_7, $kPSL_2(7)$ and $kPSL_2(9)$. By [16] representatives for the Morita equivalence classes of blocks with defect group Q_8 are the principal blocks of kQ_8, $kSL_2(3)$ and $kSL_2(5)$.

A block with defect group $C_2 \times C_2 \times C_2$ cannot be Morita equivalent to a block with non-isomorphic defect group. This is since Morita equivalence preserves defect and (i) 2-blocks of defect three with abelian defect groups other than $C_2 \times C_2 \times C_2$ must be nilpotent (and so Morita equivalent to the group algebra of a defect group), and (ii) 2-blocks of defect three with non-abelian defect groups have five irreducible characters (whilst the number is eight for blocks with defect group $C_2 \times C_2 \times C_2$).

Corollary 1.2. *Broué's abelian defect group conjecture (as stated in* [6]*) holds for all 2-blocks with defect at most three. That is, let B be a block of $\mathcal{O}G$ for a finite group G with defect group D of order dividing 8, and let b be the unique block of $\mathcal{O}N_G(D)$ with Brauer correspondent B. Then B and b have derived equivalent module categories.*

Proof. If a defect group D is isomorphic to C_2, C_4, $C_4 \times C_2$ or C_8, then the block is nilpotent, in which case the conjecture holds automatically since $\mathrm{Aut}(D)$ is a 2-group. If $D \cong C_2 \times C_2$, then the result follows from [30]. Suppose that $D \cong C_2 \times C_2 \times C_2$. By Theorem 1.1 the derived equivalence class of B is uniquely determined by the inertial quotient and the result follows. $\square$

Note that we do not prove that there are splendid derived equivalences of blocks.

Corollary 1.3. *Let B be a 2-block of defect at most three. Then the Cartan invariants of B are at most the order of a defect group.*

The above does not hold in complete generality.

Since we now have a complete list of Cartan matrices (up to ordering of the simple modules), and indeed the decomposition matrices, for 2-blocks of defect at most three, it would be interesting to look for possible concrete restrictions on Cartan matrices.

2. Quoted results

The following proposition will be used when considering automorphism groups of simple groups. It gathers together two propositions from [21], which in turn gathers results from [12] and [26].

Proposition 2.1. *Let ℓ be any prime and let G be a finite group and $N \lhd G$ with $[G : N]$ a prime not equal to ℓ. Let b be a G-stable ℓ-block of $\mathcal{O}N$. Then either each block of $\mathcal{O}G$ covering b is Morita equivalent to b or there is a unique block of $\mathcal{O}G$ covering b. In the former case, B and b have isomorphic inertial quotient.*

Proof. Note that the group $G[b]$ of elements of G acting as inner automorphisms on b is a normal subgroup of G containing N. If $G[b] = G$, then each block of G covering b is source algebra equivalent to b by [21, 2.2] and has inertial quotient isomorphic to that of b by [21, 3.4]. If $G[b] = N$, then there is a unique block of G covering b by [21, 2.3]. □

The following is a distillation of those results in [27] which are relevant here.

Proposition 2.2 ([27])**.** *Let G be a finite group and $N \lhd G$. Let B be a block of $\mathcal{O}G$ with defect group D covering a G-stable nilpotent block b of $\mathcal{O}N$ with defect group $D \cap N$. Then there are finite groups L and $M \lhd L$ such that (i) $M \cong D \cap N$, (ii) $L/M \cong G/N$, (iii) there is a subgroup D_L of L with $D_L \cong D$ and $D_L \cap M \cong D \cap N$, and (iv) there are a central extension $\tilde{L}$ of L by an ℓ'-group and a block $\tilde{B}$ of $\mathcal{O}\tilde{L}$ which is Morita equivalent to B and has defect group $\tilde{D} \cong D$.*

Proof. We give some guidance on the extraction of the result from [27]. The construction of L, D_L and M is given in [27, 1.8]. The construction of $\tilde{L}$ and $\tilde{B}$ is addressed in [27, 1.18], using 5.5 and 5.15 of [35] and the twisted group algebra constructed in [27, 1.12]. Following the discussion in [27, 1.18] the Morita equivalence is a consequence of [27, 1.12], again making use of [35, 5.15]. □

Proposition 2.3 ([43])**.** *Let B be an ℓ-block of $\mathcal{O}G$ for a finite group G and let $Z \leq O_\ell(Z(G))$. Let $\bar{B}$ be the unique block of $\mathcal{O}(G/Z)$ corresponding to B. Then B is nilpotent if and only if $\bar{B}$ is nilpotent.*

Proposition 2.4 (Extracted from [13])**.** *Let B be a block of $\mathcal{O}G$ for a quasi-simple group G with elementary abelian defect group D of order 8. Then one of the following occurs:*

(i) $G \cong SL_2(8)$ and B is the principal block;

(ii) $G \cong {}^2G_2(q)$, where $q = 3^{2m+1}$ for some $m \in \mathbb{N}$, and B is the principal block;

(iii) $G \cong J_1$ and B is the principal block;

(iv) $G \cong Co_3$ and B is the unique non-principal 2-block of defect three;

(v) G is of type $D_n(q)$, $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$ for some q of odd prime power order, $O_2(G) = 1$ and B is Morita equivalent to the principal block of $C_2 \times A_5$ or of $C_2 \times A_4$;
(vi) $|O_2(G)| = 2$ and $D/O_2(G)$ is a Klein four group;
(vii) B is nilpotent.

Proof. Note that by [21, 12.1, 13.1] there are no blocks of quasisimple groups G with defect group D when $Z(G)$ has odd order and $G/Z(G) \cong PSL_n(q)$ or $PSU_n(q)$ for q a power of an odd prime (there may be such blocks when $Z(G)$ has even order, in which case we are in situation (vi)).

Note also that blocks of $E_6(q)$ and ${}^2E_6(q)$ with defect group D as in [13, 5.4] are nilpotent covered in the sense of [13, 2.3] and have inertial index C_3. Hence by [13, 2.4] such blocks are Morita equivalent to the principal block of $C_2 \times A_4$.

The result then follows from Propositions 3.4, 5.3, 5.4, Lemma 4.2 and Theorem 6.1 of [13]. □

Lemma 2.5. *Let B be a block of $\mathcal{O}G$ for a finite group G with normal defect group $D \cong C_2 \times C_2 \times C_2$. Then B is Morita equivalent to $\mathcal{O}(D \rtimes E)$, where E has odd order and acts faithfully on D.*

Proof. This follows from [25, A. Theorem], noting that the inertial quotient is one of 1, C_3, C_7 and $C_7 \rtimes C_3$, each having trivial Schur multiplier (by, for example, [20, 11.21]. □

3. Preliminary results

Although not necessary in their application here, the following two results are stated for source algebras in the sense of [34, 3.2]. Source algebra equivalence (that is, an isomorphism of source algebras) implies Morita equivalence.

Proposition 3.1. *Let $N = {}^2G_2(q)$, where $q = 3^{2m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, and $N \leq G \leq \mathrm{Aut}(N)$. Let b be the principal 2-block of $\mathcal{O}N$. Then every block of $\mathcal{O}G$ covering b is source algebra equivalent to b. Further, each of these blocks shares a defect group with b and has isomorphic inertial quotient.*

Proof. G/N is cyclic of odd order (see, for example, [8, Table 5]). Let $N = G_0 \leq G_1 \leq \cdots \leq G_n = G$, with each $|G_{i+1}/G_i|$ prime. By [37, Theorem 8.5] b has defect groups of the form $C_2 \times C_2 \times C_2$, and by [42, p. 74] it has irreducible character degrees occurring with multiplicity either one or two, so that each irreducible character is G-stable. Since $[G : N]$ is odd each block of $\mathcal{O}G_i$ covering b shares a defect group with b. By [21], every block with defect group $C_2 \times C_2 \times C_2$ (in particular b and every block of $\mathcal{O}G_i$ covering it) has precisely eight irreducible characters, and it follows that for each i there are $[G_{i+1} : N]$ 2-blocks of $\mathcal{O}G_{i+1}$ covering b, and amongst these there are $[G_{i+1} : G_i]$ blocks of $\mathcal{O}G_{i+1}$ covering each such block of $\mathcal{O}G_i$. It follows from Proposition 2.1 that each block of $\mathcal{O}G_i$ covering b is source algebra equivalent to b. That the blocks have isomorphic inertial quotient follows from [21, 3.4]. □

Recall that a block of a finite group G is *quasiprimitive* if each block of every normal subgroup that it covers (in the sense of [1, §15]) is G-stable.

Proposition 3.2. *Let G be a finite group and $N \lhd G$ with G/N abelian of odd order. Let b be a G-stable block of $\mathcal{O}N$ with defect group $C_2 \times C_2 \times C_2$ and inertial quotient C_3. Suppose that $l(b) = 3$. Let B be a block of $\mathcal{O}G$ covering b.*

(i) If $[G : N]$ is prime, then B is either source algebra equivalent to b or nilpotent. In the former case B has inertial quotient C_3 and $[G : N] = 3$.

(ii) Returning to the general case that G/N is abelian of odd order, suppose that B is quasiprimitive. Then B either covers a nilpotent block of some normal subgroup containing N or is source algebra equivalent to b and has inertial quotient C_3.

Proof. (i) By [21] we have $l(B) \leq 7$. Suppose first that $[G : N] \geq 5$. Since we are assuming that $l(b) = 3$, there cannot be a unique block of $\mathcal{O}G$ covering b (since each irreducible Brauer character of b is G-stable and so the total number of irreducible Brauer characters in blocks covering B is at least 15), so by Proposition 2.1 B is source algebra equivalent to b and has the same inertial quotient.

Suppose now that $[G : N] = 3$. If every irreducible Brauer character of b is G-stable, in which case again by Proposition 2.1 B is source algebra equivalent to b and has the same inertial quotient. If the three irreducible Brauer characters are permuted transitively, then $l(B) = 1$, so that by [21] B is nilpotent.

(ii) follows by applying (i) to intermediate (normal) subgroups between N and G in a composition series including N. □

The following is a strengthening of a special case of the main result of [22], which is only known to hold for blocks defined over k.

Proposition 3.3. *Let G be a finite group and $N \lhd G$. Let B be a block of $\mathcal{O}G$ with elementary abelian defect group D of order 8 and let C be a non-nilpotent G-stable block of $\mathcal{O}N$ covered by B. Write $P = N \cap D$ and suppose that $D = P \times Q$ for some Q of order 2 such that $G = N \rtimes Q$. Then $B \cong C \otimes_{\mathcal{O}} \mathcal{O}Q$. In particular B and the block $C \otimes_{\mathcal{O}} \mathcal{O}Q$ of $\mathcal{O}(N \times Q)$ are Morita equivalent.*

Proof. Write $Q = \langle x \rangle$. As noted in [22] B and C share a block idempotent e, so that B is a crossed product of C with Q and it suffices to find a graded unit of $Z(B)$ of degree x and order two. We do this by exploiting the existence of a perfect isometry as shown in [21, 5.1], although we must show that this perfect isometry satisfies additional properties. Part of the proof follows that of [21, 5.1], and we take facts from there without explicit further reference. Note however that for convenience we use a different labeling of the irreducible characters.

Denote by E the inertial quotient of B, so that $|E| = 3$ and E acts faithfully on D. Write $H = D \rtimes E$. Then $Q \leq Z(H)$ and so $H = (P \rtimes E) \times Q$.

By [28] we have $k(B) = 8$. Label the irreducible characters θ_i of H so that $\theta_1, \ldots, \theta_4$ have Q in their kernel, $\theta_1(1) = \theta_2(1) = \theta_3(1) = 1$, $\theta_4(1) = 3$ and $\theta_i(g) = \theta_{i-4}(g)$ for all $i = 5, \ldots, 8$ and all $g \in P \rtimes E$. We have $\theta_i(x) = -\theta_i(1)$ for $i = 5, \ldots, 8$. Similarly label the irreducible characters $\chi_1, \ldots, \chi_8$ of B so that $\mathrm{Res}^G_N(\chi_i) = \mathrm{Res}^G_N(\chi_{i-4})$ for all $i = 5, \ldots, 8$. Note that $\chi_i(x) = -\chi_{i-4}(x)$ for all $i = 5, \ldots, 8$.

There is a stable equivalence of Morita type between $\mathcal{O}H$ and B, leading to an isometry $L^0(H, \mathcal{O}H) \cong L^0(G, B)$ between the groups of generalised characters vanishing on 2-regular elements. $L^0(H, \mathcal{O}H)$ is generated by

$$\{\theta_1 - \theta_5, \theta_2 - \theta_6, \theta_3 - \theta_7, \theta_4 - \theta_8, \theta_1 + \theta_2 + \theta_3 - \theta_4\}.$$

We claim that if $\chi_i - \chi_j \in L^0(G,B)$, then $|i-j| = 4$, for suppose that $\chi_i(g) = \chi_j(g)$ for all $g \in G$ of odd order. Then $\mathrm{Res}^G_N(\chi_i)$ and $\mathrm{Res}^G_N(\chi_2)$ are irreducible characters of C agreeing on 2-regular elements. Noting that C is not nilpotent and that C has decomposition matrix that of the principal 2-block of A_4 or A_5, it follows that $\mathrm{Res}^G_N(\chi_i) = \mathrm{Res}^G_N(\chi_2)$, and the claim follows.

Hence the isometry takes elements of the form $\theta_i - \theta_{i-4}$ to elements of the form $\delta_j(\chi_j - \chi_{j-4})$. Now the isometry extends to a perfect isometry $I : \mathbb{Z}\operatorname{Irr}(H) \to \mathbb{Z}\operatorname{Irr}(B)$, and we have seen that $I(\theta_i)(g) = I(\theta_{i-4})(g)$ for every $i = 5, \ldots, 8$ and every $g \in N$.

Following [6] I induces an $\mathcal{O}$-algebra isomorphism $I^0 : Z(\mathcal{O}H) \to Z(B)$ with $I^0(x) = \frac{1}{|H|}\sum_{g \in G} \mu(g^{-1}, x)g$, where $\mu(g,h) = \sum_{i=1}^8 \theta_i(h)I(\theta_i)(g)$ for $g \in G$ and $h \in H$. We will show that $I^0(x) = ax$ for some $a \in \mathcal{O}N$, i.e., that $\mu(g,x) = 0$ whenever $g \in N$. Then $I^0(x)$ will be the required graded unit of $Z(B)$ of degree x and order two.

Let $g \in N$. Then

$$\mu(g,x) = \sum_{i=1}^{8} \theta_i(x)I(\theta_i)(g) = \sum_{i=5}^{8} \theta_{i-4}(1)\left(I(\theta_{i-4})(g) - I(\theta_i)(g)\right) = 0,$$

and we are done. □

4. Proof of the main theorem

We prove Theorem 1.1.

Proof. Let B be a block of $\mathcal{O}G$ for a finite group G with defect group $D \cong C_2 \times C_2 \times C_2$ with $[G : Z(G)]$ minimised such that B is not Morita equivalent to any of (i)-(viii). By minimality and the first Fong reduction, B is quasiprimitive; that is, for every $N \lhd G$ each block of $\mathcal{O}N$ covered by B is G-stable. By Proposition 2.2 if $N \lhd G$ and B covers a nilpotent block of $\mathcal{O}N$, then $N \leq Z(G)O_2(G)$. In particular $O_{2'}(G) \leq Z(G)$.

Following [3] write $E(G)$ for the *layer* of G, that is, the central product of the subnormal quasisimple subgroups of G (the *components*). Write $F(G)$ for the Fitting subgroup, which in our case is $F(G) = Z(G)O_2(G)$. Write $F^*(G) = F(G)E(G) \lhd G$, the generalised Fitting subgroup, and note that $C_G(F^*(G)) \leq F^*(G)$. Let b be the (unique) block of $\mathcal{O}F^*(G)$ covered by B.

Let $\overline{B}$ be the unique block of $\mathcal{O}(G/O_2(Z(G)))$ corresponding to B. First observe that $|O_2(Z(G))| \leq 2$, for otherwise $\overline{B}$ would have defect at most one and so would be nilpotent, which in turn would mean that B would be nilpotent by Proposition 2.3, a contradiction.

If $|O_2(G)| > 4$, then $O_2(G) = D$, a contradiction by Lemma 2.5. Hence $|O_2(G)| \leq 4$.

Claim. $O_2(G) \leq Z(G)$ and $|O_2(G)| \leq 2$.

Suppose that $O_2(G) \not\leq Z(G)$ (so $|O_2(G)| = 4$). If $O_2(Z(G)) \neq 1$, then $O_2(G/O_2(Z(G)))$ has order 2 and so is central in $G/O_2(Z(G))$, from which it follows using Proposition 2.3 that $\overline{B}$, and so B, is nilpotent, again a contradiction. If $O_2(Z(G)) = 1$, then $F^*(G) = O_2(G) \times (Z(G)E(G))$. Since $|O_2(G)| = 4$, B covers a nilpotent block of $F^*(G)$ and so $F^*(G) = O_2(G)Z(G)$. But $C_G(F^*(G)) \leq F^*(G)$

and so $D \leq C_G(O_2(G)) \leq O_2(G)Z(G)$, a contradiction. Hence $O_2(G) \leq Z(G)$ (and $|O_2(G)| \leq 2$) as claimed.

Write $E(G) = L_1 * \cdots * L_t$, where each L_i is a component of G (arguing as above we have that $t \geq 1$). Now B covers a block b_E of $\mathcal{O}E(G)$ with defect group contained in D, and b_E covers a block b_i of $\mathcal{O}L_i$. Since b_E is G-stable, for each i either $L_i \lhd G$ or L_i is in a G-orbit in which each corresponding b_i is isomorphic (with equal defect). Since B has defect three, it follows that if $t > 1$, then B covers a nilpotent block of a normal subgroup generated by components of G, a contradiction. Hence $t = 1$. So G has a unique component L_1, and $G/Z(G) \leq \mathrm{Aut}(L_1Z(G)/Z(G))$.

Suppose that $O_2(G) \not\leq [L_1, L_1]$. Then $F^*(G) = O_2(G) \times Z(G)L_1$. In this case $D \leq F^*(G)$, since otherwise b would be nilpotent. Since b is G-stable, this means $[G : F^*(G)]$ is odd, and so $O_2(G)$ is in fact a direct factor of G. By [30] it follows that B is Morita equivalent to one of (ii) or (iii), a contradiction. Hence $O_2(G) \leq [L_1, L_1]$.

We next show that $D \leq F^*(G)$. Suppose otherwise. Then since D is elementary abelian we may write $D = (D \cap F^*(G)) \times Q$ for some Q of order 2 (if Q were larger, then B would cover a nilpotent block of $\mathcal{O}F^*(G)$). By the Schreier conjecture $G/F^*(G)$ is solvable. Since b is G-stable, $DF^*(G)/F^*(G)$ is a Sylow 2-subgroup of $G/F^*(G)$. Hence $G = H \rtimes Q$ for some $H \lhd G$. By Proposition 3.3 $B \cong b \otimes_{\mathcal{O}} \mathcal{O}Q$ as $\mathcal{O}$-algebras. Now $b \otimes_{\mathcal{O}} \mathcal{O}Q$ is a block of $\mathcal{O}(H \times Q)$ with defect group $D = (D \cap H) \times Q$. Since b is Morita equivalent to the principal block of $\mathcal{O}A_4$ or $\mathcal{O}A_5$, it follows that B is Morita equivalent to one of (ii) or (iii). Hence $D \leq F^*(G)$. Since $[F^*(G) : L_1]$ is odd, this means D is also a defect group for b_1.

We now refer to Proposition 2.4. Suppose that $L_1 \cong SL_2(8)$ and b_1 is the principal block. Then G is $SL_2(8)$ or $\mathrm{Aut}(SL_2(8)) \cong SL_2(8) \rtimes C_3 \cong {}^2G_2(3)$, leading to (v) or (viii) of the theorem.

If $L_1 \cong {}^2G_2(3^{2m+1})$ for some $m \in \mathbb{N}$, then $L_1 \leq G \leq \mathrm{Aut}({}^2G_2(3^{2m+1}))$, and by Proposition 3.1 B is Morita equivalent to b_1. By [33, Example 3.3], which in turn uses [29], b_1 is Morita equivalent to the principal block of $\mathcal{O}({}^2G_2(3))$.

If $L_1 \cong J_1$ or Co_3, then $G = L_1$. By [23, 1.5] the 2-block of $\mathcal{O}Co_3$ of defect three is Morita equivalent to the principal block of $\mathcal{O}({}^2G_2(3))$; hence we are done in this case. The principal block of $\mathcal{O}J_1$ is not Morita equivalent to that of $\mathcal{O}({}^2G_2(3))$, since their decomposition matrices are not similar (see [18] and [29]).

Suppose that L_1 is of type $D_n(q)$, $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$ and b_1 is Morita equivalent to the principal block of $\mathcal{O}(C_2 \times A_4)$ or $\mathcal{O}(C_2 \times A_5)$. Then G/L_1 is abelian and of odd order. By Proposition 3.2 B either covers a nilpotent block of a non-central normal subgroup (a contradiction) or is Morita equivalent to b_1, and we are done in this case.

This leaves the case that $|O_2(L_1)| = 2$ and $D/O_2(L_1)$ is a Klein four group. We have shown that $O_2(L_1) = O_2(G) \leq Z(G)$. Recall that $\overline{B}$ is the unique block of $\mathcal{O}(G/O_2(G))$ corresponding to B, and note that $\overline{B}$ has defect group $D/O_2(G)$. By [9] $\overline{B}$ is source algebra equivalent to the principal block of $\mathcal{O}A_4$ or of $\mathcal{O}A_5$. We now apply [36, Corollary 1.14], which uses the language of pointed groups, to show that B is Morita equivalent to the principal block of a central extension of A_4 or A_5 by a group of order 2, i.e., of $C_2 \times A_4$ or $C_2 \times A_5$. An excellent account of pointed groups may be found in [41], and we refer the reader there for notation. Since D is abelian, it follows that there are no essential pointed groups of B or of the principal block B' of $G' = C_2 \times A_4$ or $C_2 \times A_5$. Now the inertial quotient of B

(and of B') is C_3, and so the condition $E_G(P_\gamma) \cong E_{G'}(P'_{\gamma'})$ of [36, Corollary 1.14] is automatically satisfied (here $P_\gamma = D_\gamma$ is a defect pointed group of B and $P'_{\gamma'}$ is a defect pointed group of B', where $P' \in \mathrm{Syl}_2(G')$). Hence [36, Corollary 1.14] gives the required Morita equivalence.

To see that the blocks in cases (i)-(viii) represent distinct Morita equivalence classes it suffices to note that they have distinct decomposition matrices.

It remains to establish that representatives of the Morita equivalence classes are derived equivalent if and only if they have the same inertial quotient. First note that the number $l(B)$ of irreducible Brauer characters is: 1 in case (i); 3 in cases (ii)-(iii); 7 in cases (iv)-(v); 5 in cases (vi)-(viii). Hence blocks with different inertial quotient cannot be derived equivalent. The converse is already well known in the literature. Indeed, Broué's abelian defect group conjecture is known to hold in general for principal 2-blocks, as noted in [32], where one may find an account of progress on the conjecture. It is shown in [38, §3] that the principal blocks of A_4 and A_5 are derived equivalent, from which it follows that the blocks in cases (ii) and (iii) are derived equivalent. That the blocks in cases (iv) and (v) are derived equivalent has been shown in [39, §2.3, Example 2]. That the blocks in cases (vi) and (vii) are derived equivalent was first shown in [19], and a published proof may be found in [10, §6.2.3]. Finally, the derived equivalence between the blocks in cases (vi) and (viii) is given in [33, Remark 3.4]. $\square$

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