

A SHARP LOWER BOUND ON THE POLYGONAL ISOPERIMETRIC DEFICIT

EMANUEL INDREI

(Communicated by Jeremy Tyson)

ABSTRACT. It is shown that the isoperimetric deficit of a convex polygon P admits a lower bound in terms of the variance of the radii of P , the area of P , and the variance of the barycentric angles of P . The proof involves circulant matrix theory and a Taylor expansion of the deficit on a compact manifold.

1. INTRODUCTION

The polygonal isoperimetric inequality states that if $n \geq 3$ and P is an n -gon with area $|P|$ and perimeter $L(P)$, then the deficit is nonnegative,

$$\delta(P) := L^2(P) - 4n \tan \frac{\pi}{n} |P| \geq 0,$$

and uniquely minimized when P is convex and regular. A sharp stability result for this classical inequality has recently been obtained in [IN15] via a novel approach involving a functional minimization problem on a compact manifold and the spectral theory for circulant matrices. The heart of the matter is a quantitative polygonal isoperimetric inequality for convex polygons which states that

$$(1.1) \quad \sigma_s^2(P) + \sigma_r^2(P) \lesssim \delta(P),$$

where $\sigma_s^2(P)$ is the variance of the side lengths of P and $\sigma_r^2(P)$ is the variance of its radii (i.e. the distances between the vertices and their barycenter).

The starting point of the proof is the following inequality [FRS85, pg. 35] which holds for any n -gon:

$$(1.2) \quad 8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq nS(P) - 4n \tan \frac{\pi}{n} |P|,$$

where $S(P)$ is the sum of the squares of the side lengths of P . Since $n^2 \sigma_s^2(P) = nS(P) - L^2(P)$, it follows that (1.2) is equivalent to

$$(1.3) \quad 8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq \delta(P) + n^2 \sigma_s^2(P).$$

In order to establish (1.1), it is shown in [IN15] that

$$(1.4) \quad \sigma_s^2(P) \lesssim \delta(P)$$

whenever P is a convex n -gon; thereafter, a more general stability result (e.g. valid for simple n -gons) is deduced via a version of the Erdős-Nagy theorem which states that a polygon may be convexified in a finite number of “flips” while keeping the

Received by the editors February 20, 2015 and, in revised form, August 28, 2015.
 2010 *Mathematics Subject Classification*. Primary 52Bxx, 58Cxx; Secondary 51Kxx.
 The author is a PIRE Postdoctoral Fellow.

perimeter invariant. The method of proof of (1.2) given in [FRS85] is based on a polygonal Fourier decomposition, whereas the technique in [IN15] is based on a Taylor expansion of the deficit (in a suitable sense). It is natural to wonder whether one can directly deduce (1.1) via the method in [IN15] without relying on [FRS85]. A positive answer is given in this paper. In fact, a new inequality is established which combined with (1.4) improves (1.1).

Let $\sigma_a^2(P)$ denote the variance of the barycentric angles of P (i.e. the angles generated by the vertices and barycenter of the set of vertices of P ; see §2). Then the following is true.

Theorem 1.1. *Let $n \geq 3$ and let P be a convex n -gon. There exists $c_n > 0$ such that*

$$c_n \delta(P) \geq \sigma_r^2(P) + |P| \sigma_a^2(P),$$

and the exponent on the deficit is sharp.

This result directly combines with (1.4) and yields:

Corollary 1.2. *Let $n \geq 3$ and let P be a convex n -gon. There exists $c_n > 0$ such that*

$$c_n \delta(P) \geq \sigma_s^2(P) + \sigma_r^2(P) + |P| \sigma_a^2(P).$$

Remark 1.3. The theorem holds for a more general class of polygons. The only requirement in the proof is that the barycentric angles of P sum to 2π .

Remark 1.4. An inequality of the form

$$\sigma_a^2(P) \leq c_n \delta(P)$$

cannot hold in general. One can see this by a simple scaling consideration: let P be a convex polygon and let P_α be the convex polygon obtained by dilating the radii of P by $\alpha > 0$. Then $\delta(P_\alpha) = \alpha^2 \delta(P)$, but $\sigma_a^2(P_\alpha) = \sigma_a^2(P)$.

Quantitative polygonal isoperimetric inequalities turn out to be useful tools in geometric problems. For instance (1.1) was recently utilized in [CM14] to improve a result of Hales which showed up in his proof of the honeycomb conjecture [Hal01]. This was achieved by showing that the notion of asymmetry in (1.1) directly controls the Hausdorff distance between P and a specific regular polygon. Moreover, [IN15] has also been employed in [CN15] to prove a quantitative version of a Faber-Krahn inequality for the Cheeger constant of n -gons obtained in [BF15]. Related stability results for the isotropic, anisotropic, and relative isoperimetric inequalities have been obtained in [FMP08, FMP10, FI13], respectively.

2. PRELIMINARIES

Let $n \geq 3$ and let $P \subset \mathbb{R}^2$ be an n -gon generated by the set of vertices $\{A_1, A_2, \dots, A_n\} \subset \mathbb{R}^2$ whose center of mass O is taken to be the origin. For $i \in \{1, 2, \dots, n\}$, the i -th side length of P , denoted by $l_i := |A_i A_{i+1}|$, is the length of the vector $\overrightarrow{A_i A_{i+1}}$ which connects A_i to A_{i+1} , where $A_i = A_j$ if and only if $i = j \pmod{n}$; with this notation in mind, $\{r_i := |\overrightarrow{OA_i}|\}_{i=1}^n$ is the set of radii. Furthermore, x_i is the angle between the vectors $\overrightarrow{OA_i}$ and $\overrightarrow{OA_{i+1}}$ and the set $\{x_i\}_{i=1}^n$ comprises the *barycentric angles* of P .

The circulant matrix method introduced in [IN15] is based on the idea that a large class of polygons can be viewed as points in \mathbb{R}^{2n} satisfying some constraints. More precisely, consider

$$\mathcal{M} := \left\{ (x; r) \in \mathbb{R}^{2n} : x_i, r_i \geq 0, (2.1), (2.2), (2.3) \text{ hold} \right\},$$

where

$$(2.1) \quad \sum_{i=1}^n x_i = 2\pi,$$

$$(2.2) \quad \sum_{i=1}^n r_i = n,$$

$$(2.3) \quad \begin{cases} \sum_{i=1}^n r_i \cos \left(\sum_{k=1}^{i-1} x_k \right) = 0, \\ \sum_{i=1}^n r_i \sin \left(\sum_{k=1}^{i-1} x_k \right) = 0. \end{cases}$$

Note that \mathcal{M} is a compact $2n - 4$ dimensional manifold and each point $(x; r) \in \mathcal{M}$ represents a polygon centered at the origin with barycentric angles x and radii r ; therefore, it is appropriate to name such objects *polygonal manifolds*. Indeed, a point O is the barycenter of the set of vertices of P if and only if

$$\sum_{i=1}^n \overrightarrow{OA_i} = 0,$$

which is equivalent to saying that the projections of $\sum_{i=1}^n \overrightarrow{OA_i}$ onto $\overrightarrow{OA_1}$ and $\overrightarrow{OA_1}^\perp$ vanish; in other words, $(x; r)$ satisfies (2.3). Furthermore, (2.1) is satisfied by all convex polygons (also many nonconvex ones) and (2.2) is a convenient technical assumption which derives from scaling considerations. Note that the convex regular n -gon corresponds to the point $(x_*; r_*) = (\frac{2\pi}{n}, \dots, \frac{2\pi}{n}; 1, \dots, 1)$. With this in mind, the variance of the interior angles and radii of P are represented, respectively, by the quantities

$$\begin{aligned} \sigma_a^2(P) &= \sigma_a^2(x; r) := \frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{1}{n^2} \left(\sum_{i=1}^n x_i \right)^2, \\ \sigma_r^2(P) &= \sigma_r^2(x; r) := \frac{1}{n} \sum_{i=1}^n r_i^2 - \frac{1}{n^2} \left(\sum_{i=1}^n r_i \right)^2. \end{aligned}$$

Moreover, in $(x; r)$ coordinates, the deficit is given by the formula

$$\begin{aligned} \delta(P) &= \delta(x; r) := \left(\sum_{i=1}^n (r_{i+1}^2 + r_i^2 - 2r_{i+1}r_i \cos x_i)^{1/2} \right)^2 \\ &\quad - 2n \tan \frac{\pi}{n} \sum_{i=1}^n r_i r_{i+1} \sin x_i. \end{aligned}$$

3. PROOF OF THEOREM 1.1

By a simple reduction argument, it suffices to prove the inequality on \mathcal{M} : let P be a convex n -gon and note that it is represented by $(x; r) \in \mathbb{R}^{2n}$, where $x \in \mathbb{R}^n$ denotes its interior angles and $r \in \mathbb{R}^n$ its radii. Convexity implies (2.1), and (2.3) follows from the definition of barycenter. If

$$\sum_{i=1}^n r_i = s \neq n,$$

consider (by a slight abuse of notation) the polygon $P_s = (x; \frac{n}{s}r)$ obtained by scaling the radii of P . Evidently $\sigma_a^2(P_s) = \sigma_a^2(P)$, $|P_s| = (n/s)^2|P|$, $\sigma_r^2(P_s) = (n/s)^2\sigma_r^2(P)$, and $\delta(P_s) = (n/s)^2\delta(P)$. Hence if the inequality stated in the theorem holds for $P_s \in \mathcal{M}$, then it also holds for P . Now let

$$\begin{aligned} \phi(x; r) &:= n^2(|P|\sigma_a^2 + \sigma_r^2) \\ &= \frac{1}{2} \left(\sum_{i=1}^n r_i r_{i+1} \sin x_i \right) \left(n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2 \right) + n \sum_{i=1}^n r_i^2 - \left(\sum_{i=1}^n r_i \right)^2, \end{aligned}$$

and note that it suffices to show

$$(3.1) \quad \phi(x; r) \leq c \delta(x; r)$$

for all $(x; r) \in \mathcal{M}$. The polygonal isoperimetric inequality implies $\delta(x; r) \geq 0$ for every $(x; r) \in \mathcal{M}$ with $\delta(x; r) = 0$ if and only if $(x; r) = z_* := (x_*, r_*)$. Since \mathcal{M} is compact and δ is continuous it follows that for $\mu > 0$,

$$\inf_{\mathcal{M} \setminus B_\mu(z_*)} \delta > 0,$$

and so (3.1) follows easily on $\mathcal{M} \setminus B_\mu(z_*)$. Thus it suffices to prove (3.1) for some neighborhood B_μ of the point z_* . Direct calculations imply (recall that the notation is periodic mod n)

$$(3.2) \quad D\phi(z_*) := (D_x\phi(z_*), D_r\phi(z_*)) = 0,$$

$$D_{x_k x_l} \phi(z_*) = \begin{cases} n(n-1) \sin \frac{2\pi}{n}, & k = l, \\ -n \sin \frac{2\pi}{n}, & k \neq l, \end{cases}$$

$$D_{r_k r_l} \phi(z_*) = \begin{cases} 2(n-1), & k = l, \\ -2, & k \neq l, \end{cases}$$

and $D_{r_k x_l} \phi(z_*) = 0$. Thus by letting $\Phi := D^2\phi(z_*)$ it follows that

$$\Phi = \begin{pmatrix} n \sin \frac{2\pi}{n} \mathcal{C} & 0_{n \times n} \\ 0_{n \times n} & 2\mathcal{C} \end{pmatrix},$$

where $0_{n \times n}$ is the $n \times n$ zero matrix and

$$\mathcal{C} = \begin{pmatrix} n-1 & -1 & & \cdots & & -1 \\ -1 & n-1 & -1 & & \cdots & \\ & -1 & n-1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & \vdots & \ddots & -1 & n-1 & -1 \\ -1 & & \cdots & & -1 & n-1 \end{pmatrix}_{n \times n}.$$

Moreover, $D\delta(z_*)$ is given by

$$\begin{cases} D_{x_k} \delta(z_*) = 2n \tan \frac{\pi}{n}, \\ D_{r_k} \delta(z_*) = 0; \end{cases}$$

hence, (2.1) implies

$$\begin{aligned} \langle D\delta(z_*), (x - x_*; r - r_*) \rangle &= \langle D_x \delta(z_*), x - x_* \rangle + \langle D_r \delta(z_*), r - r_* \rangle \\ (3.3) \qquad \qquad \qquad &= 2n \tan \frac{\pi}{n} \sum_{i=1}^n (x_i - (x_*)_i) = 0. \end{aligned}$$

Since $\phi(z_*) = \delta(z_*) = 0$, by utilizing (3.2), (3.3), and performing a Taylor expansion, it follows that for z close enough to z_* ,

$$\begin{aligned} \phi(z) &= \frac{1}{2} \langle D^2 \phi(z_*) (z - z_*), (z - z_*) \rangle \\ &\quad + \frac{1}{6} \sum_{i,j,k=1}^{2n} D_{ijk} \phi((1 - \theta_z)z_* + \theta_z z) (z - z_*)_i (z - z_*)_j (z - z_*)_k \end{aligned}$$

and

$$\begin{aligned} \delta(z) &= \frac{1}{2} \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \\ &\quad + \frac{1}{6} \sum_{i,j,k=1}^{2n} D_{ijk} \delta((1 - \tau_z)z_* + \tau_z z) (z - z_*)_i (z - z_*)_j (z - z_*)_k \end{aligned}$$

for some $\theta_z, \tau_z \in (0, 1)$. Furthermore, since in a neighborhood of z_* , ϕ and δ are C^3 and \mathcal{M} is compact, there exists $C > 0$ such that

$$\frac{1}{6} \left| \sum_{i,j,k=1}^{2n} D_{ijk} \phi((1 - \theta_z)z_* + \theta_z z) (z - z_*)_i (z - z_*)_j (z - z_*)_k \right| \leq C |z - z_*|^3,$$

$$\frac{1}{6} \left| \sum_{i,j,k=1}^{2n} D_{ijk} \delta((1 - \theta_z)z_* + \theta_z z) (z - z_*)_i (z - z_*)_j (z - z_*)_k \right| \leq C |z - z_*|^3$$

for $z \in \mathcal{M}$ sufficiently close to z_* . Thus there exists $C > 0$ for which

$$(3.4) \qquad \left| \phi(z) - \frac{1}{2} \langle D^2 \phi(z_*) (z - z_*), (z - z_*) \rangle \right| \leq C |z - z_*|^3,$$

$$(3.5) \qquad \left| \delta(z) - \frac{1}{2} \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \right| \leq C |z - z_*|^3$$

in a neighborhood of z_* . In particular, there exists $\eta = \eta(n) > 0$ such that

$$(3.6) \quad \phi(z) \leq \frac{1}{2} \|\Phi\|_2 |z - z_*|^2 + C |z - z_*|^3$$

for all $z \in B_\eta(z_*)$. By the results of [IN15, §3.6], it follows that

$$\inf_{w \in S_{\mathcal{H}}} \langle D^2 \delta(z_*) w, w \rangle =: \sigma > 0,^1$$

where \mathcal{H} is the tangent space of \mathcal{M} at z_* and $S_{\mathcal{H}}$ is the unit sphere in \mathcal{H} with center z_* . Moreover by continuity there exists a neighborhood $U \subset \mathbb{R}^{2n}$ of $S_{\mathcal{H}}$ such that

$$\langle D^2 \delta(z_*) w, w \rangle \geq \frac{\sigma}{2}$$

for all $w \in U$. Note that $\frac{z - z_*}{|z - z_*|} \in U$ for $z \in \mathcal{M}$ sufficiently close to z_* . Hence there exists $\mu = \mu(\eta, \sigma) \in (0, \eta]$ such that

$$\langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \geq \frac{\sigma}{2} |z - z_*|^2$$

for $z \in B_\mu(z_*)$. In particular, for $\tilde{\mu} := \min\{\mu, \frac{\sigma}{8C}\}$ and $z \in B_{\tilde{\mu}}(z_*)$,

$$\delta(z) \geq \frac{1}{4} \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle;$$

thus, recalling (3.6),

$$\phi(z) \leq \left(\frac{1}{\sigma} \|\Phi\|_2 + \frac{2C}{\sigma} |z - z_*| \right) \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \leq c_n \delta(z),$$

where $c_n := \frac{4}{\sigma} \|\Phi\|_2 + \frac{8C}{\sigma} \tilde{\mu}$. To achieve the second part of the theorem, it suffices to prove the existence of $c > 0$ such that

$$(3.7) \quad \langle \Phi(x; r), (x; r) \rangle \geq c |(x; r)|^2$$

for

$$(x; r) \in \mathcal{Z} := \left\{ (x; r) : \sum_{i=1}^n x_i = 0, \sum_{i=1}^n r_i = 0 \right\}.$$

Indeed, if (3.7) holds, let $\omega : [0, \infty) \rightarrow [0, \infty)$ be any modulus of continuity (i.e. $\omega(0+) = 0$) such that

$$\phi(z) \leq c_n \omega(\delta(z)).$$

Then for $z \in \mathcal{M}$ close to z_* , (3.5) implies

$$\delta(z) \leq c_0 |z - z_*|^2$$

for some $c_0 > 0$. Moreover, $z - z_* \in \mathcal{Z}$ since $z \in \mathcal{M}$, and by combining (3.4) with (3.7) it follows that

$$(3.8) \quad \delta(z) \leq c_0 |z - z_*|^2 \leq c_1 \langle \Phi(z - z_*), (z - z_*) \rangle \leq c_2 \phi(z) \leq \tilde{c} \omega(\delta(z))$$

for some $\tilde{c} > 0$ provided z is close to z_* ; however, since $\delta(z) \rightarrow 0$ as $z \rightarrow z_*$ and $\delta(z) > 0$ for $z \neq z_*$, (3.8) leads to a contradiction if

$$\liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} = 0.$$

¹In fact, something stronger is proved: namely that $\inf_{w \in S_{\mathcal{H}}} \langle D^2 f(z_*) w, w \rangle =: \sigma > 0$, where f is an explicit function for which $D^2 f \leq D^2 \delta$. This is achieved via the spectral theory for circulant matrices and an analysis involving the tangent space of \mathcal{M} at z_* and the identification of a suitable coordinate system in which calculations can be performed efficiently. The barycentric condition (2.3) built into the definition of \mathcal{M} comes up in this analysis.

Thus the \liminf is strictly greater than zero and this implies ω is at most linear at zero.

To verify (3.7), note first that \mathcal{C} is a real, symmetric, circulant matrix generated by the vector $(n-1, -1, \dots, -1)$. A calculation shows that the eigenvalues of \mathcal{C} , say λ_k , are given by

$$(3.9) \quad \lambda_0 = 0 \quad \text{and} \quad \lambda_k = n \quad \text{for } k = 1, \dots, n-1.$$

Moreover, let $v_0 := (1, \dots, 1)$, and for $l \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ define

$$\begin{aligned} v_{2l-1} &:= \left(1, \cos \frac{2\pi l}{n}, \cos \frac{4\pi l}{n}, \dots, \cos \frac{2\pi l(n-1)}{n} \right), \\ v_{2l} &:= \left(0, \sin \frac{2\pi l}{n}, \sin \frac{4\pi l}{n}, \dots, \sin \frac{2\pi l(n-1)}{n} \right). \end{aligned}$$

One can readily check that v_k is an eigenvector of \mathcal{C} corresponding to the eigenvalue $\lambda_{\lceil \frac{k}{2} \rceil}$, and that the set $\{v_0, v_1, \dots, v_{n-1}\}$ forms a real orthogonal basis of \mathbb{R}^n (see e.g. Proposition 2.1 in [IN15]). For $k = 1, 2, \dots, n$, define $b_k := (v_{k-1}; 0, \dots, 0) \in \mathbb{R}^{2n}$ and $b_k := (0, \dots, 0; v_{k-n-1}) \in \mathbb{R}^{2n}$ for $k = n+1, \dots, 2n$. Since the set $\{b_k\}_{k=1}^{2n}$ forms a real orthogonal basis of \mathbb{R}^{2n} , given $(x; r) \in \mathbb{R}^{2n}$ there exist unique coefficients $\alpha_k \in \mathbb{R}$ such that

$$(x; r) = \sum_{k=1}^{2n} \alpha_k b_k.$$

Thus, by utilizing (3.9) it follows that

$$\begin{aligned} \langle \Phi(x; r), (x; r) \rangle &= \sum_{k, k'=1}^{2n} \alpha_k \alpha_{k'} \langle \Phi b_k, b_{k'} \rangle \\ &= n \sin \frac{2\pi}{n} \sum_{k=1}^n \alpha_k^2 \lambda_{\lceil \frac{k-1}{2} \rceil} |b_k|^2 + 2 \sum_{k=n+1}^{2n} \alpha_k^2 \lambda_{\lceil \frac{k-n-1}{2} \rceil} |b_k|^2 \\ &= n^2 \sin \frac{2\pi}{n} \sum_{k=2}^n \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+2}^{2n} \alpha_k^2 |b_k|^2. \end{aligned}$$

Furthermore, if $(x; r) \in \mathcal{Z}$,

$$\begin{aligned} \alpha_1 &= \frac{\langle (x; r), b_1 \rangle}{|b_1|^2} = \sum_{i=1}^n x_i = 0, \\ \alpha_{n+1} &= \frac{\langle (x; r), b_{n+1} \rangle}{|b_1|^2} = \sum_{i=1}^n r_i = 0; \end{aligned}$$

hence,

$$\begin{aligned} \langle \Phi(x; r), (x; r) \rangle &= n^2 \sin \frac{2\pi}{n} \sum_{k=1}^n \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+1}^{2n} \alpha_k^2 |b_k|^2 \\ &\geq 2n \sum_{k=1}^{2n} \alpha_k^2 |b_k|^2, \end{aligned}$$

and this concludes the proof.

ACKNOWLEDGEMENTS

The author is pleased to acknowledge support from NSF Grants OISE-0967140 (PIRE), DMS-0405343, and DMS-0635983 administered by the Center for Nonlinear Analysis at Carnegie Mellon University. Moreover, the excellent research environment provided by the Hausdorff Research Institute for Mathematics and the Rheinische Friedrich-Wilhelms-Universität Bonn is kindly acknowledged. Lastly, the author wishes to thank an anonymous referee for providing useful feedback on a preliminary version of this paper.

REFERENCES

- [BF15] M. Bucur and I. Fragala, *A Faber-Krahn inequality for the Cheeger constant of N -gons*, J. Geom. Anal., DOI 10.1007/s12220-014-9539-5 (2015).
- [CM14] M. Caroccia and F. Maggi, *A sharp quantitative version of Hales' isoperimetric honeycomb theorem*, arXiv:1410.6128 (2014).
- [CN15] M. Caroccia and R. Neumayer, *A note on the stability of the Cheeger constant of N -gons*, J. Convex Anal., to appear (2015).
- [FI13] A. Figalli and E. Indrei, *A sharp stability result for the relative isoperimetric inequality inside convex cones*, J. Geom. Anal. **23** (2013), no. 2, 938–969, DOI 10.1007/s12220-011-9270-4. MR3023863
- [FMP08] N. Fusco, F. Maggi, and A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. of Math. (2) **168** (2008), no. 3, 941–980, DOI 10.4007/annals.2008.168.941. MR2456887 (2009k:52021)
- [FMP10] A. Figalli, F. Maggi, and A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*, Invent. Math. **182** (2010), no. 1, 167–211, DOI 10.1007/s00222-010-0261-z. MR2672283 (2011f:49070)
- [FRS85] J. Chris Fisher, D. Ruoff, and J. Shilleto, *Perpendicular polygons*, Amer. Math. Monthly **92** (1985), no. 1, 23–37, DOI 10.2307/2322190. MR777566 (86e:51031)
- [Hal01] T. C. Hales, *The honeycomb conjecture*, Discrete Comput. Geom. **25** (2001), no. 1, 1–22, DOI 10.1007/s004540010071. MR1797293 (2002a:52020)
- [IN15] E. Indrei and L. Nurbekyan, *On the stability of the polygonal isoperimetric inequality*, Adv. Math. **276** (2015), 62–86, DOI 10.1016/j.aim.2015.02.013. MR3327086

CENTER FOR NONLINEAR ANALYSIS, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213

E-mail address: `egi@cmu.edu`