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PREPERIODIC PORTRAITS FOR UNICRITICAL POLYNOMIALS

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ABSTRACT. Let K be an algebraically closed field of characteristic zero, and for $c \in K$ and an integer $d \geq 2$, define $f_{d,c}(z) := z^d + c \in K[z]$. We consider the following question: If we fix $x \in K$ and integers $M \geq 0$, $N \geq 1$, and $d \geq 2$, does there exist $c \in K$ such that, under iteration by $f_{d,c}$, the point x enters into an N-cycle after precisely M steps? We conclude that the answer is generally affirmative, and we explicitly give all counterexamples. When $d = 2$, this answers a question posed by Ghioca, Nguyen, and Tucker.

1. INTRODUCTION

Throughout this article, K will be an algebraically closed field of characteristic zero. Let $\varphi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. For $n \geq 0$, let φ^n denote the n-fold composition of φ; that is, φ^0 is the identity map, and $\varphi^n = \varphi \circ \varphi^{n-1}$ for each $n \geq 1$. A point $x \in K$ is **preperiodic** for φ if there exist integers $M \geq 0$ and $N \geq 1$ for which $\varphi^{M+N}(x) = \varphi^M(x)$. In this case, the minimal such M is called the **preperiod** of x, and the minimal such N is called the **eventual period** of x. If the preperiod M is zero, then we say that x is **periodic** of **period** N. If $M \geq 1$, then we call x **strictly preperiodic**. If M and N are the preperiod and period, respectively, then we call the pair (M, N) the **preperiodic portrait** (or simply **portrait**) of x under φ.

A natural question to ask is the following:

Question 1.1. Given a polynomial $\varphi \in K[z]$ of degree at least 2, and given integers $M \geq 0$ and $N \geq 1$, does there exist an element $x \in K$ with portrait (M, N) for φ?

This question was completely answered by Baker [1] in the case that $M = 0$. (See also [9, Thm. 1] for the corresponding statement for rational functions.) Before stating Baker's result, though, we give an example of a polynomial that fails to admit points with a certain portrait.

Consider the polynomial $\varphi(z) = z^2 - 3/4$. A quadratic polynomial $z^2 + c$ typically admits two points of period 2, forming a single two-cycle; however, the polynomial φ admits no such points. Indeed, such a point x would satisfy $\varphi^2(x) = x$, but one can see that

$$\varphi^2(z) - z = (z - 3/2)(z + 1/2)^3,$$

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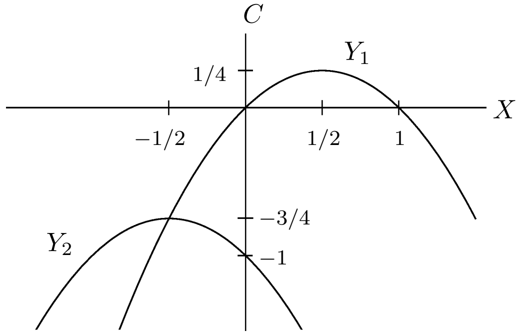

FIGURE 1. The affine curve $Y : (X^2 + C)^2 + C - X = 0$, with components Y_1 and Y_2.

and each of the points $3/2$ and $-1/2$ is actually a *fixed point* for φ. This example stems from the fact that $c = -3/4$ is the root of the period 2 hyperbolic component of the Mandelbrot set. In other words, $c = -3/4$ is a *bifurcation point* — it is the parameter at which the two points forming a two-cycle for $z^2 + c$ merge into one point, effectively collapsing the two-cycle to a single fixed point. To illustrate this, we let Y be the affine curve defined by $(X^2 + C)^2 + C - X = 0$. For a given $c \in K$, if x is a fixed point or a point of period 2 for $z^2 + c$, then $(x, c) \in Y(K)$. This suggests a natural decomposition of Y into two irreducible components — a "period 1 curve" Y_1, defined by $X^2+C-X = 0$, and a "period 2 curve" Y_2, defined by $\frac{(X^2+C)^2+C-X}{X^2+C-X} = X^2 + X + C + 1 = 0$, illustrated in Figure 1. The bifurcation at $c = -3/4$ may be seen by letting c tend to $-3/4$ and observing that the two points on Y_2 lying over c (corresponding to the two points of period 2 for $z^2 + c$) approach a single point on Y_1 (corresponding to a *fixed point* for $z^2 - 3/4$).

Baker showed that the polynomial $\varphi(z) = z^2 - 3/4$ is, in some sense, the *only* polynomial of degree at least 2 that fails to admit points of a given period. To make this more precise, we first recall the following terminology and notation: two polynomials $\varphi, \psi \in K[z]$ are **linearly conjugate** if there exists a linear polynomial $\ell(z) = az + b$ such that $\psi = \ell^{-1} \circ \varphi \circ \ell$, and in this case we write $\varphi \sim \psi$. Note that $\psi^n = \ell^{-1} \circ \varphi^n \circ \ell$, so this relation is the appropriate notion of equivalence in dynamics. In particular, $x \in K$ has portrait (M, N) for ψ if and only if $\ell(x)$ has portrait (M, N) for φ.

Theorem 1.2 (Baker [1, Thm. 2])**.** *Let $\varphi(z) \in K[z]$ with degree $d \geq 2$, and let $N \geq 1$ be an integer. If $\varphi(z) \not\sim z^2 - 3/4$, then φ admits a point of period N. If $\varphi(z) \sim z^2 - 3/4$, then φ admits a point of period N if and only if $N \neq 2$.*

Though Baker was only considering *periodic* points, and therefore only answered Question 1.1 for $M = 0$, it is not difficult to extend his result to the case $M > 0$.

Proposition 1.3. *Let $\varphi(z) \in K[z]$ with degree $d \geq 2$, and let $M \geq 0$ and $N \geq 1$ be integers. If $\varphi(z) \not\sim z^2 - 3/4$, then φ admits a point of portrait (M, N). If $\varphi(z) \sim z^2 - 3/4$, then φ admits a point of portrait (M, N) if and only if $N \neq 2$.*

Proof. The claim that if $\varphi(z) \sim z^2 - 3/4$, then φ does not admit points of portrait $(M, 2)$ follows immediately from Theorem 1.2. We now suppose either that $\varphi \not\sim z^2 - 3/4$, or that $\varphi \sim z^2 - 3/4$ and $N \neq 2$, and we show that there exists a point of portrait (M, N) for φ.

The $M = 0$ case is precisely Theorem 1.2, and the $M = 1$ case follows from the fact (see [7, Lem. 4.24]) that if a polynomial admits a point of period N, then it also admits a point of portrait $(1, N)$. Now suppose $M \geq 2$. By induction, there exists $y \in K$ with portrait $(M - 1, N)$ for φ. Since y is itself strictly preperiodic, it is easy to see that any preimage x of y has portrait (M, N). □

We now consider the dual question to Question 1.1: given an element $x \in K$, and given integers $M \geq 0$ and $N \geq 1$, does there exist a polynomial $\varphi(z) \in K[z]$ of degree at least 2 for which x has portrait (M, N)?

It is not difficult to see that the answer to this question is "yes." Let $\psi(z)$ be any polynomial not linearly conjugate to $z^2 - 3/4$, so that ψ is guaranteed to admit a point ζ of portrait (M, N) by Proposition 1.3. If we let $\ell(z) := z + (\zeta - x)$, so that $\ell(x) = \zeta$, then x has portrait (M, N) for $\varphi := \ell^{-1} \circ \psi \circ \ell$.

This suggests that an appropriate dual question should only allow us to consider one polynomial (or, at worst, finitely many) from each linear conjugacy class. Also, since we are imposing a single condition on the polynomial φ — namely, that the given point x have portrait (M, N) under φ — we ought to consider a one-parameter family of maps for each degree $d \geq 2$.

This naturally leads us to consider the class of *unicritical polynomials*; i.e., polynomials with a single (finite) critical point. Every unicritical polynomial is linearly conjugate to a polynomial of the form

$$f_{d,c}(z) := z^d + c,$$

so we consider only this one-parameter family of polynomials for each $d \geq 2$. Note that $f_{d,c} \sim f_{d,c'}$ if and only if c/c' is a $(d-1)$th root of unity, so this family contains only finitely many polynomials from a given conjugacy class.

We now ask the following more restrictive question:

Question 1.4. Given $(x, M, N, d) \in K \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$, does there exist $c \in K$ for which x has portrait (M, N) under $f_{d,c}$?

If there does exist such an element $c \in K$, we will say that x **realizes portrait** (M, N) **in degree** d. Before stating our main result, we give some examples of tuples (x, M, N, d) for which the answer to Question 1.4 is negative.

First, observe that 0 cannot realize portrait $(1, N)$ in degree d for any $N \geq 1$ and $d \geq 2$. Indeed, suppose $f_{d,c}(0) = c$ is periodic of period N, which is equivalent to saying that 0 either has portrait $(1, N)$ or is periodic of period N itself. Since a periodic point must have precisely one periodic preimage, and since the only preimage of c under $f_{d,c}$ is 0, we must have that 0 is periodic of period N. This particular counterexample is special to unicritical polynomials, since the failure of 0 to realize portrait $(1, N)$ is due to the fact that $f_{d,c}$ is totally ramified at 0 for all $c \in K$, as illustrated in Figure 2.

Next, consider $x = -1/2$. We show that x cannot realize portrait $(0, 2)$ in degree 2; that is, there is no $c \in K$ such that $-1/2$ has period 2 for $f_{2,c}$. If there were such a parameter c, then we would have

$$0 = f_{2,c}^2(-1/2) - (-1/2) = (c + 3/4)^2.$$

However, if we take $c = -3/4$, then $-1/2$ is a fixed point for $f_{2,c}$. There is therefore no $c \in K$ such that $-1/2$ has period 2 for $f_{2,c}$. This is illustrated in Figure 1, which shows that the only point on the "period 2 curve" Y_2 lying over $x = -1/2$ also lies on the "period 1 curve" Y_1.

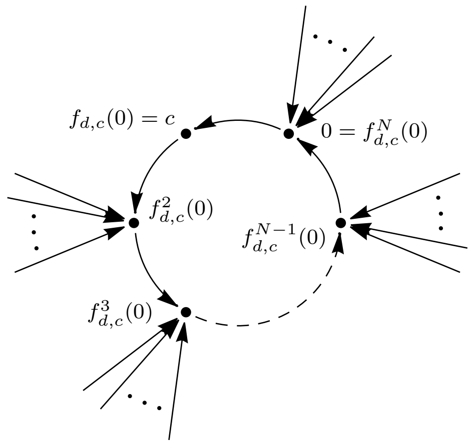

FIGURE 2. If c is periodic for $f_{d,c}$, then c has no strictly preperiodic preimage

An argument similar to the one in the preceding paragraph shows that $x = 1/2$ cannot realize portrait $(1, 2)$ in degree 2 and that $x = \pm 1$ cannot realize portrait $(2, 2)$ in degree 2. Our main result states that these are the only instances where the answer to Question 1.4 is negative.

Theorem 1.5. *Let K be an algebraically closed field of characteristic zero, and let $(x, M, N, d) \in K \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. Then there exists $c \in K$ for which x has portrait (M, N) under $f_{d,c}$ if and only if*

$$(x, M) \neq (0, 1) \quad \textit{and} \quad (x, M, N, d) \notin \left\{ \left(-\frac{1}{2}, 0, 2, 2\right), \left(\frac{1}{2}, 1, 2, 2\right), (\pm 1, 2, 2, 2) \right\}.$$

Ghioca, Nguyen, and Tucker [7] consider the more general problem of "simultaneous multi-portraits" for polynomial maps: Given a $(d-1)$-tuple of points $(x_1, \ldots, x_{d-1}) \in K^{d-1}$, and given $(d-1)$ portraits $(M_1, N_1), \ldots, (M_{d-1}, N_{d-1})$, does there exist a degree d polynomial in standard form

$$\varphi(z) := z^d + c_{d-2} z^{d-2} + \cdots + c_1 z + c_0$$

such that, for each $i \in \{1, \ldots, d-1\}$, x_i has portrait (M_i, N_i) for φ? The $d = 2$ case of this question is precisely the $d = 2$ case of Question 1.4 in the present article. In an earlier version of their article, the authors of [7] provided $(x, M) = (0, 1)$ and $(x, M, N) = (-1/2, 0, 2)$ as examples of the failure of a given point to realize a given portrait in degree 2, and they asked whether there were any other such failures. Theorem 1.5 completely answers this question.

The main tool used in [7] to approach the multi-portrait problem is a result for *single* portraits, which they are then able to extend to multi-portraits by an iterative process. Their main result ([7, Thm. 1.3]), when applied to the case of unicritical polynomials, says the following: For a fixed $d \geq 2$, if $(x, M) \neq (0, 1)$ and if (M, N) avoids an effectively computable finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{N}$, then every $x \in K$ realizes portrait (M, N) in degree d. One might therefore be able to use the techniques of [7], involving Diophantine approximation, to prove Theorem 1.5 for fixed values of d. In this article, however, we take an entirely different approach by using properties of certain algebraic curves, which we call *dynamical modular curves*, that are defined in terms of the dynamics of the maps $f_{d,c}$.

We now briefly outline the rest of this article. In §2, we record a number of known properties of dynatomic polynomials and the corresponding dynamical modular curves. Section 3 contains the proof of Theorem 1.5, which is generally based on the following principle: For each portrait (M, N), there is a curve $Y_1(M, N)$ whose points parametrize maps $f_{d,c}$ together with points of portrait (M, N). If x does not achieve portrait (M, N) in degree d, then each point on $Y_1(M, N)$ lying above x must also lie on $Y_1(m, n)$ for some integers $m \le M$ and $n \le N$ with $m < M$ or $n < N$. Once the degree of $Y_1(M, N)$ becomes large enough, however, this proves to be impossible (excluding the special case $(x, M) = (0, 1)$, discussed above). In the final section, we discuss some related open problems.

2. Dynatomic polynomials and dynamical modular curves

2.1. **Dynatomic polynomials.** If c is an element of K and $x \in K$ is a point of period N for $f_{d,c}$, then (x, c) is a solution to the equation $f_{d,C}^N(X) - X = 0$. However, (x, c) is also a solution to this equation whenever x is a point of period dividing N for $f_{d,c}$. We therefore define the N**th dynatomic polynomial** to be the polynomial

$$\Phi_N(X, C) := \prod_{n|N} \left(f_{d,C}^n(X) - X\right)^{\mu(N/n)} \in \mathbb{Z}[X, C]$$

(where μ is the Möbius function), which has the property that

$$f_{d,C}^N(X) - X = \prod_{n|N} \Phi_n(X, C) \tag{2.1}$$

for all $N \in \mathbb{N}$ — see [13, p. 571]. For simplicity of notation, we omit the dependence on d. The fact that $\Phi_N(X, C)$ is a polynomial is shown in [17, Thm. 4.5], and it is not difficult to see that Φ_N is monic in both X and C.

If $(x, c) \in K^2$ is such that $\Phi_N(x, c) = 0$, we say that x has **formal period** N for $f_{d,c}$. Every point of exact period N has formal period N, but in some cases a point of formal period N may have exact period n a proper divisor of N. If x is such a point, then x appears in the cycle $\{x, f_{d,c}(x), \ldots, f_{d,c}^{N-1}(x)\}$ with multiplicity N/n, and this multiplicity is captured by Φ_N. In particular, x is a multiple root of the polynomial $\Phi_N(X, c) \in K[X]$, so we have the following:

Lemma 2.1. *Let $c \in K$. Suppose that $x \in K$ has formal period N and exact period $n < N$ for $f_{d,c}$. Then*

$$\left.\frac{\partial \Phi_N(X, C)}{\partial X}\right|_{(x,c)} = 0.$$

Moreover, $x, c \in \overline{\mathbb{Q}} \setminus \overline{\mathbb{Z}}$.

For an illustration of this phenomenon, see Figure 1, which shows the curves $Y_1 : \Phi_1(X, C) = 0$ and $Y_2 : \Phi_2(X, C) = 0$ in the degree $d = 2$ case. One can see in the figure that the X-partial of $\Phi_2(X, C)$ vanishes at the point $(x, c) = (-1/2, -3/4)$ on Y_2, where x actually has period 1 for $f_{2,c}$.

We also briefly explain the statement that $x, c \in \overline{\mathbb{Q}} \setminus \overline{\mathbb{Z}}$. (See also [13, p. 582].) Since x has formal period N and exact period $n < N$ for $f_{d,c}$, c is a root of the resultant

$$\mathrm{Res}_X\,(\Phi_N(X,C),\Phi_n(X,C))\in\mathbb{Z}[C].$$

Thus $c\in\overline{\mathbb{Q}}$, hence also $x\in\overline{\mathbb{Q}}$ since $\Phi_N(x,c)=0$. On the other hand, a multiple root x of the polynomial $\Phi_N(X,c)\in K[X]$ must also be a multiple root of $f_{d,c}^N(X)-X$, so

$$0=(f_{d,c}^N)'(x)-1=d^N\prod_{k=0}^{N-1}f_{d,c}^k(x)-1. \tag{2.2}$$

Since $c\in\overline{\mathbb{Z}}$ if and only if $x\in\overline{\mathbb{Z}}$ (Φ_N is monic in both variables), and since the rightmost expression of (2.2) cannot vanish if $x,c\in\overline{\mathbb{Z}}$ (the expression is congruent to -1 modulo $d\overline{\mathbb{Z}}$), we must have $x,c\notin\overline{\mathbb{Z}}$.

Finally, for an application in §3.1, we compare the degree of Φ_N to the degrees of the polynomials Φ_n with n properly dividing N. Let

$$D(N):=\sum_{n\mid N}\mu(N/n)\cdot d^n$$

denote the degree of Φ_N in X. Note that Φ_N has degree $D(N)/d$ in C.

Lemma 2.2. *Let $N\in\mathbb{N}$ be a positive integer. Then*

$$D(N)>\sum_{\substack{n\mid N\\ n<N}}D(n),$$

unless $N=d=2$, in which case equality holds.

Proof. If $N=1$, then the statement is trivial. We therefore assume $N\geq 2$.

Since the polynomial $f_{d,C}^N(X)-X$ has degree d^N in X, we can see from (2.1) that the sum appearing in the lemma is actually equal to $d^N-D(N)$. Hence, it suffices to prove the equivalent inequality

$$D(N)>\frac{1}{2}\cdot d^N. \tag{2.3}$$

We first obtain a rough lower bound for $D(N)$, using the fact that the largest proper divisor of N has size at most $\lfloor N/2\rfloor$:

$$\begin{aligned}D(N)=\sum_{n\mid N}\mu(N/n)d^n\geq d^N-\sum_{\substack{n\mid N\\ n<N}}d^n\geq d^N-\sum_{n=1}^{\lfloor N/2\rfloor}d^n&=d^N-\frac{d}{d-1}\cdot\left(d^{\lfloor N/2\rfloor}-1\right)\\ &>d^N-\frac{d}{d-1}\cdot d^{N/2}.\end{aligned}$$

It therefore suffices to show that

$$\frac{d}{d-1}\cdot d^{N/2}\leq\frac{1}{2}\cdot d^N,$$

which we can rearrange to become

$$d^{N/2-1}\geq\frac{2}{d-1}. \tag{2.4}$$

First, suppose $d=2$. Then (2.4) becomes

$$2^{N/2-1}\geq 2,$$

which is satisfied for $N \geq 4$. For $N = 2$, we have $D(2) = 2 = D(1)$, which gives us the desired equality in this case. For $N = 3$, we have $D(3) = 6 > D(1)$.

Finally, when $d \geq 3$, we observe that the right hand side of (2.4) is at most 1, while the left hand side is at least 1 when $N \geq 2$. Therefore (2.4) is satisfied whenever $d \geq 3$ and $N \geq 2$, completing the proof. □

2.2. **Generalized dynatomic polynomials.** To say that a point $x \in K$ has portrait (M, N) for $f_{d,c}$ is to say that $f_{d,c}^M(x)$ has period N but $f_{d,c}^{M-1}(x)$ does not. For this reason, if M and N are positive integers, we define the **generalized dynatomic polynomial** $\Phi_{M,N}(X, C)$ to be the polynomial

$$\Phi_{M,N}(X,C) := \frac{\Phi_N(f_{d,C}^M(X), C)}{\Phi_N(f_{d,C}^{M-1}(X), C)} \in \mathbb{Z}[X, C]. \tag{2.5}$$

For convenience, we set $\Phi_{0,N} := \Phi_N$, and we again omit the dependence on d. That $\Phi_{M,N}$ is a polynomial is shown in [8, Thm. 1]. If $(x, c) \in K^2$ satisfies $\Phi_{M,N}(x, c) = 0$, we will say that x has **formal portrait** (M, N) for $f_{d,c}$, and we similarly attach "formal" to the terms "preperiod" and "eventual period" in this case. As in the periodic case, every point with exact portrait (M, N) has formal portrait (M, N), but a point with formal portrait (M, N) may have exact portrait (m, n) with $m < M$ or n a proper divisor of N. It is again not difficult to see that $\Phi_{M,N}$ is monic in both X and C, and that, when $M \geq 1$, $\Phi_{M,N}$ has degree $(d-1)d^{M-1}D(N)$ in X and degree $(d-1)d^{M-2}D(N)$ in C.

Let $Y_1(M, N)$ denote the affine plane curve defined by $\Phi_{M,N}(X, C) = 0$. We call a curve defined in this way a **dynamical modular curve**. We summarize the relevant properties of $Y_1(M, N)$ in the following lemma:

Lemma 2.3. *Let K be an algebraically closed field of characteristic zero, and let $M \geq 0$ and $N \geq 1$ be integers.*

(A) *If $M = 0$, then the curve $Y_1(0, N)$ is nonsingular and irreducible over K.*

(B) *If $M \geq 1$, then for each dth root of unity ζ, define*

$$\Psi^{\zeta}_{M,N}(X, C) := \Phi_N(\zeta f_{d,C}^{M-1}(X), C). \tag{2.6}$$

Then

$$\Phi_{M,N}(X, C) = \prod_{\substack{\zeta^d=1 \\ \zeta \neq 1}} \Psi^{\zeta}_{M,N}(X, C). \tag{2.7}$$

Each of the polynomials $\Psi^{\zeta}_{M,N}(X, C)$ is irreducible over K, so $Y_1(M, N)$ has exactly $(d-1)$ irreducible components. Each of the components is smooth, and the points of intersection of the components are precisely those points (x, c) with $f_{d,c}^{M-1}(x) = 0$.

Part (A) was originally proven in the $d = 2$ case by Douady and Hubbard (smoothness; [4, §XIV]), and Bousch (irreducibility; [2, Thm. 1 (§3)]). A subsequent proof of (A) in the $d = 2$ case was later given by Buff and Lei [3, Thm. 3.1]. For $d \geq 2$, irreducibility was proven by Lau and Schleicher [10, Thm. 4.1] using analytic methods and by Morton [11, Cor. 2] using algebraic methods, and both irreducibility and smoothness were later proven by Gao and Ou [6, Thms. 1.1, 1.2] using the methods of Buff-Lei. Part (B) is due to Gao [5, Thm. 1.2]. The lemma was originally proven over $\mathbb{C}$, but the Lefschetz principle allows us to extend the result to arbitrary fields of characteristic zero: since the curves $Y_1(M,N)$ are all defined over $\mathbb{Z}$, any singular points and irreducible components would be defined over a finitely generated extension of $\mathbb{Q}$, which could then be embedded into $\mathbb{C}$.

Finally, we briefly explain the factorization in (2.7). If x has portrait (M,N) for $f_{d,c}$, then $f_{d,c}^M(x)$ is periodic of period N, so precisely one preimage of $f_{d,c}^M(x)$ is also periodic. The periodic preimage cannot be $f_{d,c}^{M-1}(x)$, since this would imply that x has portrait (m,N) for some $m \le M-1$. Since any two preimages of a given point under $f_{d,c}$ differ by a dth root of unity, this implies that $\zeta f_{d,c}^{M-1}(x)$ is periodic for some dth root of unity $\zeta \ne 1$, and therefore $\Psi_{M,N}^{\zeta}(x,c) = 0$ for that particular value of ζ.

3. Formal portraits and exact portraits

In order to prove Theorem 1.5, we must describe those conditions under which a point may have formal portrait different from its exact portrait under the map $f_{d,c}$. We begin by giving a necessary and sufficient condition for the exact preperiod of a point to be strictly less than its formal preperiod.

Lemma 3.1. *Let $M, N \in \mathbb{N}$, and suppose x has formal portrait (M,N) for $f_{d,c}$. Then x has exact preperiod strictly less than M if and only if $f_{d,c}^{M-1}(x) = 0$. In this case, both x and c are algebraic integers and 0 is periodic of period equal to N (hence x has eventual period N).*

Proof. First, suppose x has exact preperiod $m < M$ for $f_{d,c}$. By Lemma 2.3, since x has formal portrait (M,N) for $f_{d,c}$, we must have

$$\Phi_N(\zeta f_{d,c}^{M-1}(x), c) = 0$$

for some dth root of unity $\zeta \ne 1$. Hence $\zeta f_{d,c}^{M-1}(x)$ is periodic. On the other hand, $f_{d,c}^{M-1}(x)$ is also periodic, since $f_{d,c}^{m}(x)$ is periodic and $m \le M-1$. Both $\zeta f_{d,c}^{M-1}(x)$ and $f_{d,c}^{M-1}(x)$ are preimages of $f_{d,c}^{M}(x)$; since a point can only have a single periodic preimage, it follows that $\zeta f_{d,c}^{M-1}(x) = f_{d,c}^{M-1}(x)$, which then implies that $f_{d,c}^{M-1}(x) = 0$.

Conversely, suppose that $f_{d,c}^{M-1}(x) = 0$. Since x has formal portrait (M,N) for $f_{d,c}$, the factorization in Lemma 2.3 implies that $\Phi_N(0,c) = 0$, so 0 is periodic for $f_{d,c}$. In particular, this means that the preperiod of x is at most $M-1$.

The fact that $\Phi_N(0,c) = 0$ implies that 0 is periodic for $f_{d,c}$ and, since $\Phi_N(0,C)$ is monic in C, that $c \in \overline{\mathbb{Z}}$. Moreover, since $f_{d,c}^{M-1}(x) = 0$, we also conclude that $x \in \overline{\mathbb{Z}}$. The final claim — that the period of 0 (and hence the eventual period of x) is equal to N — follows from Lemma 2.1. ☐

As a consequence of Lemma 3.1, we see that if x has formal portrait (M,N) and exact portrait (m,n) for $f_{d,c}$, then either $m = M$ or $n = N$. We can actually say a bit more, using the fact that if x is preperiodic for $f_{d,c}$ — which is necessarily the case if $\Phi_{M,N}(x,c) = 0$ — then $x \in \overline{\mathbb{Z}}$ if and only if $c \in \overline{\mathbb{Z}}$.

Lemma 3.2. *Let $x \in K$, and let $c_1, \ldots, c_n$ be the roots of $\Phi_{M,N}(x,C) \in K[C]$. Then one of the following must be true:*

(A) *for all $i \in \{1,\ldots,n\}$, x has preperiod equal to M for f_{d,c_i}; or*
(B) *for all $i \in \{1,\ldots,n\}$, x has eventual period equal to N for f_{d,c_i}.*

Proof. Let $i \in \{1, \dots, n\}$ be arbitrary. If $x \in \overline{\mathbb{Z}}$, then $c_i \in \overline{\mathbb{Z}}$, and therefore $f_{d,c_i}^M(x) \in \overline{\mathbb{Z}}$. Since $f_{d,c_i}^M(x)$ has formal period N, Lemma 2.1 implies that $f_{d,c_i}^M(x)$ must have *exact* period N, and therefore x has eventual period N for f_{d,c_i}. On the other hand, if $x \notin \overline{\mathbb{Z}}$, then it follows from Lemma 3.1 that x must have preperiod equal to M for f_{d,c_i}. $\square$

Now let $x \in K$ be such that x does not realize portrait (M, N) in degree d. It follows from Lemma 3.2 that either x has preperiod strictly less than M for $f_{d,c}$ for every root c of $\Phi_{M,N}(x, C)$, or x has eventual period strictly less than N for all such maps $f_{d,c}$. We handle these two cases separately.

3.1. **Eventual period less than formal eventual period.** Throughout this section, we suppose the tuple $(x, M, N, d) \in K \times \mathbb{Z}^3$, with $M \geq 0$, $N \geq 1$, and $d \geq 2$, satisfies the following condition:

$$(*) \qquad \text{For all roots } c \text{ of } \Phi_{M,N}(x, C) \in K[C],\ x \text{ has eventual period strictly less than } N \text{ for } f_{d,c}.$$

Now fix one such root $c \in K$, and assume for the moment that $M \geq 1$. By Lemma 2.3, $\zeta f_{d,c}^{M-1}(x)$ is periodic for some root of unity $\zeta \neq 1$. The period of $\zeta f_{d,c}^{M-1}(x)$ is equal to the period of $f_{d,c}(\zeta f_{d,c}^{M-1}(x)) = f_{d,c}^M(x)$, which is less than N by $(*)$. Lemma 2.1 then implies that

$$\left.\frac{\partial \Phi_N(Z, C)}{\partial Z}\right|_{(\zeta f_{d,c}^{M-1}(x), c)} = 0.$$

Therefore, using the factorization appearing in Lemma 2.3 and applying the chain rule, we have

$$\left.\frac{\partial \Phi_{M,N}(X, C)}{\partial X}\right|_{(x,c)} = 0. \tag{3.1}$$

Note that if $M = 0$, then (3.1) holds immediately by Lemma 2.1. In this case, since $Y_1(0, N)$ is nonsingular for all $N \geq 1$, we conclude that $\frac{\partial}{\partial C}\Phi_{0,N}(X, C)$ does not vanish at (x, c). The same is true for $M \geq 1$: Indeed, by Lemma 3.2, x must have preperiod equal to M for $f_{d,c}$, and therefore $f_{d,c}^{M-1}(x) \neq 0$ by Lemma 3.1. It then follows from Lemma 2.3 that (x, c) is a nonsingular point on $Y_1(M, N)$, so the C-partial of $\Phi_{M,N}(X, C)$ cannot vanish at (x, c).

In any case, we have shown that each root of $\Phi_{M,N}(x, C) \in K[C]$ is a *simple* root, so the number of distinct roots of $\Phi_{M,N}(x, C)$ is precisely

$$\deg_C \Phi_{M,N} = \begin{cases} \dfrac{1}{d} D(N), & \text{if } M = 0; \\ (d-1)d^{M-2} D(N), & \text{if } M \geq 1. \end{cases}$$

On the other hand, since every root satisfies $\Phi_{M,n}(x, c) = 0$ for some n strictly dividing N, the number of roots of $\Phi_{M,N}(x, C)$ can be at most

$$\sum_{\substack{n \mid N \\ n < N}} \deg_C \Phi_{M,n} = \begin{cases} \dfrac{1}{d} \sum\limits_{\substack{n \mid N \\ n < N}} D(n), & \text{if } M = 0; \\ (d-1)d^{M-2} \sum\limits_{\substack{n \mid N \\ n < N}} D(n), & \text{if } M \geq 1. \end{cases}$$

In particular, this means that

$$D(N) \leq \sum_{\substack{n|N \\ n<N}} D(n),$$

which implies that $N = d = 2$ by Lemma 2.2. We assume henceforth that $(N, d) = (2, 2)$.

Suppose $M = 0$. In this case, $(*)$ says that for every $c \in K$ with $\Phi_2(x, c) = 0$ we also have $\Phi_1(x, c) = 0$. In the $d = 2$ case, we have

$$\Phi_1(X, C) = X^2 - X + C \ , \ \Phi_2(X, C) = X^2 + X + C + 1.$$

The condition $\Phi_2(x, c) = \Phi_1(x, c) = 0$ implies that $(x, c) = (-1/2, -3/4)$. Therefore, if $x \neq -1/2$ and $\Phi_2(x, c) = 0$, then x has exact period 2 for $f_{2,c}$.

Now suppose $M = 1$, and let $x \in K$ with $x \neq 1/2$. By the previous paragraph, there exists $c \in K$ for which $\Phi_2(-x, c) = 0$ and $-x$ has period 2 under $f_{2,c}$. Since $d = 2$, Lemma 2.3 yields

$$\Phi_{1,2}(X, C) = \Phi_2(-X, C),$$

so for this particular value of c we have $\Phi_{1,2}(x, c) = 0$. Moreover, since $f_{2,c}(x) = f_{2,c}(-x)$ has period 2, x has eventual period 2 for $f_{2,c}$.

Finally, consider the case $M \geq 2$. Let $c \in K$ satisfy $\Phi_{M,2}(x, c) = 0$. By hypothesis, x has portrait $(M, 1)$ for $f_{2,c}$, which implies that

$$\Phi_2(f_{2,c}^M(x), c) = \Phi_1(f_{2,c}^M(x), c) = 0.$$

As explained above, this means that $c = -3/4$; in particular, the polynomial $\Phi_{M,2}(x, C)$ has only the single root $c = -3/4$. Since $\Phi_{M,2}(X, C)$ has degree $2^{M-2}D(2) \geq 2$ in C, the root $c = -3/4$ must be a multiple root of $\Phi_{M,2}(x, C)$, contradicting our previous assertion that $\Phi_{M,N}(x, C)$ has only simple roots.

We have shown that if (x, M, N, d) satisfies $(*)$, then $(N, d) = (2, 2)$ and $(x, M) \in \{(-1/2, 0), (1/2, 1)\}$. From this, we draw the following conclusion:

Proposition 3.3. *Let $(x, M, N, d) \in K \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. Suppose that*

$$(x, M, N, d) \notin \left\{ \left(-\frac{1}{2}, 0, 2, 2\right), \left(\frac{1}{2}, 1, 2, 2\right) \right\}.$$

Then there exists $c \in K$ with $\Phi_{M,N}(x, c) = 0$ for which x has eventual period equal to N for $f_{d,c}$.

If (x, M, N, d) is any exception to Theorem 1.5 not appearing in Proposition 3.3, then for every root c of $\Phi_{M,N}(x, C)$, x must have exact preperiod less than M for $f_{d,c}$. We now consider this situation.

3.2. **Preperiod less than formal preperiod.** Suppose now that $(x, M, N, d) \in K \times \mathbb{Z}^3$, with $M \geq 0$, $N \geq 1$, $d \geq 2$, satisfies the following condition:

$(**)$ For all roots c of $\Phi_{M,N}(x, C) \in K[C]$,
x has preperiod strictly less than M for $f_{d,c}$.

For all such roots c, Lemma 3.1 implies that $f_{d,c}^{M-1}(x) = 0$ is periodic of period N, and therefore x must have eventual period equal to N for $f_{d,c}$.

If $M = 1$, then $f_{d,c}^{M-1}(x) = 0$ means precisely that $x = 0$, and we have already seen that 0 cannot have portrait $(1, N)$ for $f_{d,c}$ for any $N \geq 1$ and $c \in K$. We will therefore assume that $M \geq 2$.

We first prove an elementary lemma.

Lemma 3.4. *Suppose* $(**)$ *is satisfied, and let* ζ *be a dth root of unity. If* $M \geq 2$, *then the polynomial* $\Psi_{M,N}^{\zeta}(x, C) \in K[C]$ *has a multiple root.*

Proof. Let c be any root of $\Psi_{M,N}^{\zeta}(x, C)$. By Lemma 2.3, this implies that $\Phi_{M,N}(x, c) = 0$, so we have $f_{d,c}^{M-1}(x) = 0$ by the assumption in $(**)$. Therefore $\Psi_{M,N}^{\zeta}(x, C)$ has at most

$$\deg_C f_{d,C}^{M-1}(X) = d^{M-2}$$

distinct roots c. On the other hand, the degree (in C) of $\Psi_{M,N}^{\zeta}(x, C)$ satisfies

$$\deg_C \Psi_{M,N}^{\zeta}(X, C) = \deg_C \Phi_N(\zeta f_{d,C}^{M-1}(X), C) = d^{M-2} D(N) > d^{M-2},$$

so $\Psi_{M,N}^{\zeta}(X, C)$ must have a multiple root. $\square$

We now show that, in *most* cases, if $f_{d,c}^{M-1}(x) = 0$ and $\Phi_{M,N}(x, c) = 0$, then c must actually be a simple root of the polynomial $\Psi_{M,N}^{\zeta}(x, C)$ when ζ is a *primitive* dth root of unity. Such cases contradict Lemma 3.4, and therefore $(**)$ must fail in these cases.

Lemma 3.5. *Let* $(M, N, d) \in \mathbb{Z}^3$ *with* $N \geq 1$; $M, d \geq 2$; *and* $(M, N, d) \neq (2, 2, 2)$. *Let* ζ *be a primitive dth root of unity, and suppose* $(x, c) \in K^2$ *satisfies* $\Psi_{M,N}^{\zeta}(x, c) = 0 = f_{d,c}^{M-1}(x)$. *Then* c *is a simple root of* $\Psi_{M,N}^{\zeta}(x, C) \in K[C]$.

Remark 3.6. Lemma 3.5 actually holds if ζ is any dth root of unity different from 1, though the proof is somewhat more involved and we do not require this level of generality. We also note that $\zeta \neq 1$ is necessary: for example, if we take $x = 0$, $N = 1$, and let $d, M \geq 2$ be arbitrary, then $c = 0$ satisfies $f_{d,c}^{M-1}(0) = 0$, and one can check that 0 is a multiple root of

$$\Psi_{M,1}^{1}(0, C) = \Phi_1(f_{d,C}^{M-1}(0), C) = \Phi_1(f_{d,C}^{M-2}(C), C) = \left(f_{d,C}^{M-2}(C)\right)^2 - f_{d,C}^{M-2}(C) + C.$$

In order to prove Lemma 3.5, we require the following description of the C-partials of the iterates of $f_{d,C}$. We omit the relatively simple proof by induction, but mention that the proof of the case $d = 2$ may be found in [3, Lem. 3.3].

Lemma 3.7. *For* $k \in \mathbb{N}$,

$$\frac{\partial}{\partial C} f_{d,C}^{k}(X) = 1 + \sum_{j=1}^{k-1} d^j \cdot \prod_{i=1}^{j} f_{d,C}^{k-i}(X)^{d-1}.$$

We also require the following special case of a result due to Morton and Silverman [12, Thm. 1.1]. For a number field F, we will denote by $\mathcal{O}_F$ the ring of integers of F.

Lemma 3.8. *Let* F *be a number field, and let* $c \in \mathcal{O}_F$. *Let* $p \in \mathbb{Z}$ *be prime, let* $\mathfrak{p} \subset \mathcal{O}_F$ *be a prime ideal lying above* p, *and let* $k_{\mathfrak{p}} := \mathcal{O}_F/\mathfrak{p}$ *be the residue field of*

$\mathfrak{p}$. *Suppose $P \in \mathcal{O}_F$ has exact period N for $f_{d,c}$, and suppose the reduction $\widetilde{P} \in k_{\mathfrak{p}}$ of P has exact period N' for $\widetilde{f_{d,c}} \in k_{\mathfrak{p}}[z]$. Then*

$$N = N' \quad or \quad N = N' r p^e,$$

where r is the multiplicative order of $\left(\widetilde{f_{d,c}}^{N'}\right)'\left(\widetilde{P}\right)$ in $k_{\mathfrak{p}}$ and $e \in \mathbb{Z}_{\geq 0}$. In particular, if $\left(\widetilde{f_{d,c}}^{N'}\right)'\left(\widetilde{P}\right) = \widetilde{0}$, then $N = N'$.

Proof of Lemma 3.5. Since $\Psi^{\zeta}_{M,N}(X,C) = \Phi_N(\zeta f_{d,C}^{M-1}(X), C)$, and since $\Phi_N(X,C)$ divides $f_{d,C}^N(X) - X$, it suffices to show that c is a simple root of the polynomial

$$f_{d,C}^N(\zeta f_{d,C}^{M-1}(x)) - \zeta f_{d,C}^{M-1}(x) = f_{d,C}^{M+N-1}(x) - \zeta f_{d,C}^{M-1}(x),$$

which is equivalent to showing that

$$\frac{d}{dC}\left(f_{d,C}^{M+N-1}(x) - \zeta f_{d,C}^{M-1}(x)\right)$$

does not vanish at $C = c$. By Lemma 3.7, this is equivalent to showing that

$$\delta := 1 - \zeta + \sum_{j=1}^{M+N-2} d^j \cdot \prod_{i=1}^{j} f_{d,c}^{N+M-1-i}(x)^{d-1} - \zeta \cdot \sum_{j=1}^{M-2} d^j \cdot \prod_{i=1}^{j} f_{d,c}^{M-1-i}(x)^{d-1} \tag{3.2}$$

is nonzero. The conditions of the lemma imply that $\Phi_N(0,c) = 0$, so that $c \in \overline{\mathbb{Z}}$, and therefore the condition $f_{d,c}^{M-1}(x) = 0$ implies that $x \in \overline{\mathbb{Z}}$ as well. Thus, δ is an algebraic integer; let $F := \mathbb{Q}(x,c,\zeta)$, so that $\delta \in \mathcal{O}_F$.

Suppose first that d is not a prime power. Then $1-\zeta$ is an algebraic unit. Since $\delta = 1 - \zeta + d\alpha$ for some $\alpha \in \mathcal{O}_F$, we have

$$\delta \equiv 1 - \zeta \not\equiv 0 \mod d\mathcal{O}_F.$$

In particular, $\delta \neq 0$.

Now suppose that $d = p^k$ is a prime power, in which case $1-\zeta$ is no longer an algebraic unit. Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal lying above $p \in \mathbb{Z}$. Then $\mathfrak{p} \cap \mathbb{Z}[\zeta] = (1-\zeta)$ and $p\mathbb{Z}[\zeta] = (1-\zeta)^r$, where $r = \varphi(d) = p^{k-1}(p-1)$. Therefore,

$$\mathrm{ord}_{\mathfrak{p}}(d) = k \cdot \mathrm{ord}_{\mathfrak{p}}(p) = kp^{k-1}(p-1) \cdot \mathrm{ord}_{\mathfrak{p}}(1-\zeta),$$

which is strictly greater than $\mathrm{ord}_{\mathfrak{p}}(1-\zeta)$ unless $k=1$ and $p=2$; that is, unless $d=2$.

If $\mathrm{ord}_{\mathfrak{p}}(d) > \mathrm{ord}_{\mathfrak{p}}(1-\zeta)$, then we again write $\delta = 1 - \zeta + d\alpha$ for some $\alpha \in \mathcal{O}_F$ and find that $\mathrm{ord}_{\mathfrak{p}}(\delta) = \mathrm{ord}_{\mathfrak{p}}(1-\zeta)$ is finite, hence $\delta \neq 0$. For the remainder of the proof, we take $d = 2$ and, therefore, $\zeta = -1$. Observe that the second sum appearing in (3.2) is empty if $M = 2$. We therefore consider the cases $M = 2$ and $M > 2$ separately.

Case 1. $M > 2$. In this case, we have

$$\begin{aligned} \delta &= 2 + \sum_{j=1}^{M+N-2} 2^j \cdot \prod_{i=1}^{j} f_{2,c}^{N+M-1-i}(x) + \sum_{j=1}^{M-2} 2^j \cdot \prod_{i=1}^{j} f_{2,c}^{M-1-i}(x) \\ &= 2\left(1 + f_{2,c}^{N+M-2}(x) + f_{2,c}^{M-2}(x) + 2\alpha\right) \end{aligned}$$

for some $\alpha \in \mathcal{O}_F$. To show that $\delta \neq 0$, it suffices to show that

$$\beta := 1 + f_{2,c}^{N+M-2}(x) + f_{2,c}^{M-2}(x) \notin 2\mathcal{O}_F.$$

We are assuming that $f_{2,c}^{M-1}(x) = 0$ has period N for $f_{2,c}$, so also $f_{2,c}^{N+M-1}(x) = 0$. Hence

$$f_{2,c}(f_{2,c}^{M-2}(x)) = 0 = f_{2,c}(f_{2,c}^{N+M-2}(x)).$$

Since $f_{2,c}^{M-2}(x)$ and $f_{2,c}^{N+M-2}(x)$ are preimages of a common point — namely, 0 — under $f_{2,c}$, we must have

$$f_{2,c}^{N+M-2}(x) = \pm f_{2,c}^{M-2}(x).$$

This means that $\beta - 1 = f_{2,c}^{N+M-2}(x) + f_{2,c}^{M-2}(x) \in 2\mathcal{O}_F$, and therefore $\beta \notin 2\mathcal{O}_F$, as desired.

Case 2. $M = 2$. Since the second sum appearing in (3.2) is empty, we may write

$$\delta = 2\Big(\big(1 + f_{2,c}^N(x)\big) + 2\alpha\Big)$$

for some $\alpha \in \mathcal{O}_F$. Let $\mathfrak{p} \subset \mathcal{O}_F$ be any prime lying above 2, and let $k_{\mathfrak{p}}$ denote the residue field of $\mathfrak{p}$.

Now suppose that $\delta = 0$. We will show that we must have $N = 2$, which yields precisely the exception $(M, N, d) = (2, 2, 2)$ in the statement of the lemma and completes the proof.

Since $\delta = 0$, we must have $1 + f_{2,c}^N(x) \in \mathfrak{p}$; that is, in $k_{\mathfrak{p}}$ we have $\widetilde{f_{2,c}^N(x)} = \widetilde{-1}$. Since $f_{2,c}^{N+1}(x) = 0 = f_{2,c}(x)$ by hypothesis, we have

$$\widetilde{0} = \widetilde{f_{2,c}^{N+1}(x)} = \left(\widetilde{f_{2,c}^N(x)}\right)^2 + \widetilde{c} = \widetilde{1 + c},$$

so $\widetilde{c} = \widetilde{-1}$. Therefore the period of $\widetilde{0}$ under $\widetilde{f_{2,c}}$ is equal to 2, since

$$\widetilde{f_{2,-1}}(\widetilde{0}) = \widetilde{0^2 - 1} = \widetilde{-1} \quad \text{and} \quad \widetilde{f_{2,-1}}(\widetilde{-1}) = (\widetilde{-1)^2 - 1} = \widetilde{0}.$$

Since $\left(\widetilde{f_{2,-1}^2}\right)'(\widetilde{0}) = \widetilde{0}$, it follows from Lemma 3.8 that 0 must have period $N = 2$ for $f_{2,c}$, as claimed. $\square$

Combining Lemmas 3.4 and 3.5 yields the following:

Proposition 3.9. *Let $(x, M, N, d) \in K \times \mathbb{Z}^3$ with $M, N \geq 1$ and $d \geq 2$. Suppose that*

$$(x, M) \neq (0, 1) \quad \text{and} \quad (x, M, N, d) \notin \{(\pm 1, 2, 2, 2)\}.$$

Then there exists $c \in K$ with $\Phi_{M,N}(x, c) = 0$ for which x has preperiod equal to M for $f_{d,c}$.

Proof. We prove the converse, so assume that there is no $c \in K$ satisfying $\Phi_{M,N}(x, c) = 0$ such that x has preperiod equal to M for $f_{d,c}$ — that is, suppose (x, M, N, d) satisfies condition $(**)$. We have already seen that if $M = 1$, then this assumption implies that $x = 0$.

For $M \geq 2$, it follows from Lemmas 3.4 and 3.5 that $(M, N, d) = (2, 2, 2)$, so it remains only to show that $x \in \{\pm 1\}$. Let c be a root of $\Phi_{2,2}(x, C)$. The sentence following $(**)$ implies that

$$\Phi_2(0, c) = 0 = f_{2,c}(x).$$

Writing these expressions explicitly yields

$$c + 1 = 0 = x^2 + c,$$

and therefore $x = \pm 1$. □

3.3. **Proof of the main theorem.** We now combine the results of the previous sections to prove the main theorem.

Proof of Theorem 1.5. Let $(x, M, N, d) \in K \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. In the paragraphs immediately preceding the statement of Theorem 1.5, we verified that if $(x, M) = (0, 1)$ or

$$(x, M, N, d) \in \left\{ \left(-\frac{1}{2}, 0, 2, 2 \right), \left(\frac{1}{2}, 1, 2, 2 \right), (\pm 1, 2, 2, 2) \right\},$$

then x does not realize portrait (M, N) in degree d.

Conversely, suppose x does not realize portrait (M, N) in degree d. By Lemma 3.2, this means that one of the following must be true:

(A) for every root c of $\Phi_{M,N}(x, C)$, x has eventual period less than N; or
(B) for every root c of $\Phi_{M,N}(x, C)$, x has preperiod less than M.

If (A) is satisfied, then $(x, M, N, d) \in \{(-1/2, 0, 2, 2), (1/2, 1, 2, 2)\}$ by Proposition 3.3; if (B) is satisfied, then $(x, M) = (0, 1)$ or $(x, M, N, d) \in \{(\pm 1, 2, 2, 2)\}$ by Proposition 3.9. □

4. Further questions

One might ask the following more general question: Let K be an algebraically closed field of characteristic zero, let $\mathcal{K} := K(t)$ be the function field in one variable over K, and let $\varphi_d(z) := z^d + t \in \mathcal{K}[z]$. Let $(x, M, N, d) \in \mathcal{K} \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. Does there exist a prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{\mathcal{K}}$ such that, modulo $\mathfrak{p}$, $\widetilde{x}$ has portrait (M, N) for $\widetilde{\varphi_d}$? Theorem 1.5 answers this question when x is chosen to be a constant point (i.e., $x \in K$), since reducing modulo a place of $\mathcal{K}$ is equivalent to specializing t to a particular element of K.

There are at least two tuples (x, M, N, d) with $x \in \mathcal{K}$ nonconstant for which the answer to the above question is negative: one can show that if

$$(x, M, N, d) \in \{(-t, 1, 1, 2), (t + 1, 1, 2, 2)\},$$

then there is no place $\mathfrak{p}$ such that $\widetilde{x}$ has portrait (M, N) for $\widetilde{\varphi_d}$ (modulo $\mathfrak{p}$). We do not know if there are any other such examples; however, it follows from the results of [7] that, for a fixed $d \geq 2$, the set of remaining examples is finite and effectively (though perhaps not *practically*) computable.

Another direction one might pursue is to consider Question 1.4 with K an algebraically closed field of positive characteristic. In this case, the analogue of Baker's theorem (Theorem 1.2) was proven by Pezda [14–16]. Pezda's theorem is more complicated than that of Baker, so it seems that a proof of the positive characteristic analogue of Theorem 1.5 would also be considerably more involved. Another obstacle is the fact that the polynomials $\Phi_N(X, C)$ are not generally irreducible in positive characteristic, so the methods of this article would require significant modifications if they are to be used to prove a version of the main theorem in positive characteristic.

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