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THE TRIANGULAR SPECTRUM OF MATRIX FACTORIZATIONS IS THE SINGULAR LOCUS

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ABSTRACT. The singularity category of a ring/scheme is a triangulated category defined to capture the singularities of the ring/scheme. In the case of a hypersurface R/f, it is given by the homotopy category of matrix factorizations $[MF(R,f)]$. In this paper, we apply Balmer's theory of tensor triangular geometry to matrix factorizations by taking into consideration their tensor product. We show that the underlying topological space of the triangular spectrum of $[MF(R,f)]$ is the singular locus of the hypersurface by using a support theory developed by M. Walker.

1. INTRODUCTION

The singularity category carries information about the singularities of a ring/scheme, however, how exactly the singularities are characterized requires further study. It is known that triangulated categories alone are not enough for the purpose of geometry (for example, Mukai's classic result states that the Poincaré bundle induces an equivalence between the derived category of an abelian variety A and its dual $\hat{A}$) so extra information must be considered.

One way of doing so is Balmer's theory of tensor triangular geometry [3] where a tensor product of the category is included in the discussion. It is shown that one can reconstruct a scheme X by applying Balmer's theory to the category of perfect complexes of X and the usual tensor product $\otimes^L_{\mathcal{O}_X}$. Studies of Balmer's theory seem to be concentrated on the perfect complexes/derived category side so far. We would like to apply Balmer's theory to study singularity categories in this paper. In particular, we focus our attention to the singularity category of a hypersurface. This is the same as the homotopy category of matrix factorizations by a famous theorem of Buchweitz [4] and Orlov [8]:

Theorem 1.1. *For R a regular commutative ring, and $f \in R$ a non-zero-divisor, one has the following equivalence of triangulated categories:*

$$\mathcal{D}_{Sg}(R/f) \cong [MF(R,f)]$$

where $\mathcal{D}_{Sg}(R/f)$ is the singularity category of R/f and the category $[MF(R,f)]$ on the right hand side is the homotopy category of matrix factorizations.

Matrix factorizations were introduced by Eisenbud in the 1980s [7]. It has found lots of applications in recent years, both from the point of view of the above theorem and its relation to Kontsevich's homological mirror symmetry.

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The main theorem of this paper is the following:

Theorem 1.2 (Corollary 4.5)**.** *For R regular of finite Krull dimension, $2 \in R$ is invertible, f a non-zero-divisor, then the underlying topological space of the triangular spectrum is the singular locus, i.e.,*

$$Spc([MF(R,f)], \otimes^{\frac{1}{2}}) \cong Sing(R/f)$$

where $Sing(R/f)$ means the singular locus of the hypersurface R/f and $\otimes^{\frac{1}{2}}$ the modified tensor product (Definition-Lemma 2.6).

This result is a quick application of Balmer's theory. We show it by proving that a support theory of matrix factorizations, developed by M. Walker [11], is in fact classifying support data (Remark 4.2).

As in [2], the space produced by Balmer's theory is immediate once a classification of thick tensor ideal is given and such a classification has already been proved by Stevenson [9] and Takahashi [10]. The referee pointed out to the author that Stevenson and Takahashi's classification, together with Stone duality, can also produce the singular locus from categorical information so the real "new" part of our result is the usage of a tensor product for the singularity category. One can therefore also regard the main input of our theorem to produce a suitable tensor product giving support data (See Remark 4.2) on the singularity category (which agrees with the support used in the existing classification).

From now on, we fix a commutative ring R and an element $f \in R$.

2. Background

We will give some basic definitions and results below, mostly to fix notation. For unexplained terminologies, one can consult papers [5], [6], [12], [13] for matrix factorizations and [2] for tensor triangular geometry. In particular, our notation and terminologies are mostly the same as in the author's thesis [12].

2.1. Matrix factorizations.

Definition 2.1. A *matrix factorization* of $f \in R$ is a $\mathbb{Z}/2$-graded R-module $M = M_0 \oplus M_1$, where M is a finitely generated projective R-module, together with a degree one endomorphism

$$d = \begin{bmatrix} 0 & d_1 \\ d_0 & 0 \end{bmatrix}$$

such that $d^2 = f \cdot 1_M$.

We denote matrix factorizations as

$$\mathcal{M} = (M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0) \quad \text{or} \quad (M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1)$$

with M_i the degree i component.

A *strict morphism* of matrix factorizations from $\mathcal{M}$ to $\mathcal{N}$ is a pair of R-linear maps $\alpha = (\alpha_1, \alpha_0)$ with $\alpha_i : M_i \to N_i$, such that the two evident squares commute.

The *category of matrix factorizations* $MF(R,f)$ associated to $f \in R$, is the exact category with object matrix factorizations of $f \in R$ and strict morphisms. The *homotopy category of matrix factorizations*, $[MF(R,f)]$, is the triangulated category obtained from $MF(R,f)$ by taking morphisms to be homotopy classes of strict morphisms.

Definition 2.2 ([6])**.** Given $f, g \in R$ and matrix factorizations $\mathcal{M} \in MF(R,f), \mathcal{N} \in MF(R,g)$, the *tensor product* of $\mathcal{M}$ and $\mathcal{N}$ is

$$\mathcal{M} \otimes_{mf} \mathcal{N} := ((M_1 \otimes_R N_0) \oplus (M_0 \otimes_R N_1) \underset{d_{\mathcal{M}\otimes\mathcal{N}}}{\overset{d_{\mathcal{M}\otimes\mathcal{N}}}{\rightleftarrows}} (M_0 \otimes_R N_0) \oplus (M_1 \otimes_R N_1))$$

where $d_{\mathcal{M}\otimes\mathcal{N}} = d_{\mathcal{M}} \otimes 1 + (-1)^{|-|} \otimes d_{\mathcal{N}}$. The tensor product $\mathcal{M} \otimes_{mf} \mathcal{N}$ is a matrix factorization of $f + g \in R$.

2.2. **Tensor triangular geometry.** For the theory of Balmer's tensor triangular geometry, one can look at [3]. However, it's enough to focus on an earlier paper [2] of Balmer for our purpose.

Remark 2.3. Balmer's theory requires the existence of a tensor identity $1_{\otimes}$ for the tensor product. There is such an identity for a tensor product of matrix factorizations: $1_{\otimes_{mf}} = (0 \rightleftarrows R)$. However, the tensor identity is not a matrix factorization of $f \in R$ but rather a matrix factorization of $0 \in R$, i.e., $1_{\otimes_{mf}} \in [MF(R,0)]$. This is not a big problem since one checks very easily that most of Balmer's theory, except that related to the tensor identity, is still valid (by Balmer's own proofs). Please look at Section 4.1.2 of [12] for the complete list of theorems (lemmas, etc.) that we need. We still refer to tensor triangulated categories without tensor identity as tensor triangulated categories.

Another problem is that the tensor product is not closed, i.e., for $\mathcal{M}, \mathcal{N} \in MF(R,f)$, $\mathcal{M} \otimes_{mf} \mathcal{N} \in MF(R, f+f)$. One can modify the tensor product a bit using the following

Definition 2.4. For $\lambda \in R^{\times}$, one can define a functor

$$\lambda : MF(R,f) \to MF(R,\lambda f).$$

λ sends an object $\mathcal{M} = (M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0)$ to the object $(M_1 \underset{\lambda d_0}{\overset{d_1}{\rightleftarrows}} M_0)$; a strict morphism $\alpha = (\alpha_1, \alpha_0) : \mathcal{M} \to \mathcal{N}$ to the strict morphism $\lambda(\alpha) = (\alpha_1, \alpha_0)$, i.e.,

$$\begin{tikzcd}
M_1 \arrow[r, "d^{\mathcal{M}}"] \arrow[d, "\alpha_1"'] & M_0 \arrow[r, "d^{\mathcal{M}}"] \arrow[d, "\alpha_0"] & M_1 \arrow[d, "\alpha_1"] \\
N_1 \arrow[r, "d^{\mathcal{N}}"'] & N_0 \arrow[r, "d^{\mathcal{N}}"'] & N_1
\end{tikzcd}$$

gets sent to

$$\begin{tikzcd}
M_1 \arrow[r, "d^{\mathcal{M}}"] \arrow[d, "\alpha_1"'] & M_0 \arrow[r, "\lambda d^{\mathcal{M}}"] \arrow[d, "\alpha_0"] & M_1 \arrow[d, "\alpha_1"] \\
N_1 \arrow[r, "d^{\mathcal{N}}"'] & N_0 \arrow[r, "\lambda d^{\mathcal{N}}"'] & N_1
\end{tikzcd}$$

λ induces a functor on the homotopy category of matrix factorizations via the following lemma

Lemma 2.5. *The functor λ maps a homotopy (h_0, h_1) to the homotopy $(h_0, \lambda^{-1} h_1)$.*

Thus for any $\lambda \in R^{\times}$, we have functors

$$\lambda : [MF(R,f)] \rightleftarrows [MF(R,\lambda f)] : \lambda^{-1}.$$

They are obviously inverse to each other so the two categories $[MF(R,f)]$ and $[MF(R,\lambda f)]$ are equivalent. More importantly, they are equivalent as triangulated categories because λ and λ^{-1} are triangulated functors (Prop 4.1.19 of [12]).

Definition-Lemma 2.6 (Prop 4.1.22 of [12])**.** Suppose 2 is invertible in R; the category $[MF(R,f)]$, together with

$$\otimes^{\frac{1}{2}} := \frac{1}{2} \circ \otimes_{mf} : [MF(R,f)] \times [MF(R,f)] \xrightarrow{\otimes_{mf}} [MF(R,2f)] \xrightarrow{\frac{1}{2}} [MF(R,f)]$$

is a tensor triangulated category without unit.

Remark 2.7. In fact, an earlier paper of Balmer [1] gives a theory about tensor triangulated categories without the tensor identity. However, after Balmer's series of papers, the existence of tensor identity is now standard for the theory. (The tensor identity is important to the construction of the structure sheaf on the triangular spectrum.) Therefore, instead of following Balmer's old terminologies and notation in [1], we choose to use the language in [2] by checking that everything (not related to the tensor identity) is still valid (by Balmer's original proofs).

3. Support theory for matrix factorization

There is a well-developed support theory for matrix factorizations due to many people in different contexts. In particular, the author learned a version from Walker [11], which develops support theory in the language of matrix factorizations.

Everything in this section belongs to Walker [11] so we skip all the proofs. One can find all the details in [11]. The author is grateful to Professor Walker for sharing his work. Let's recall everything we need from the support theory in this section.

Definition 3.1. A matrix factorization $\mathcal{M}$ is *contractible* if the identity map $1_{\mathcal{M}}$ is homotopic to the zero map, i.e., there is a degree one map from the $\mathbb{Z}/2$-graded module M to itself (that is, R-linear maps $h_0 : M_0 \to M_1$ and $h_1 : M_1 \to M_0$) such that $h_1 \circ d_0 + d_1 \circ h_0 = 1_{M_0}$ and $h_0 \circ d_1 + d_0 \circ h_1 = 1_{M_1}$.

When the ring R is regular of finite Krull dimension, the *support* of a matrix factorization is

$$supp_{mf}(\mathcal{M}) := \{\mathfrak{p} \in Spec(R) | \mathcal{M}_{\mathfrak{p}} \text{ is not contractible}\}.$$

Equivalently, the above is the same as

$$supp_{mf}(\mathcal{M}) = \{\mathfrak{p} \in Spec(R) | \mathcal{M}_{\mathfrak{p}} = \mathcal{M} \otimes_{mf} R_{\mathfrak{p}} \neq 0 \text{ in } [MF(R_{\mathfrak{p}}, f)]\}.$$

Note that the support here is the same as the one used by Stevenson in Lemma 5.12 of [9]. From now on, we will assume that the ring R is regular of finite Krull dimension and f a non-zero-divisor.

Proposition 3.2. *For $\mathcal{M},\mathcal{N} \in MF(R,f)$, $\alpha : \mathcal{M} \to \mathcal{N}$ is a homotopy equivalence. One gets $supp_{mf}(\mathcal{M}) = supp_{mf}(\mathcal{N})$.*

This means Walker's notion of support, $supp_{mf}$, is well defined on the homotopy category $[MF(R,f)]$ of matrix factorizations.

Proposition 3.3. *For any $\mathcal{M} \in MF(R,f)$, $supp_{mf}(\mathcal{M})$ is contained in $Spec(R/f)$ and is specialization closed. In particular, when $f \in R$ is a non-zero-divisor, $supp_{mf}(\mathcal{M}) \subset Sing(R/f)$, the singular locus of R/f.*

Because we already assumed f to be a non-zero-divisor, the support of matrix factorizations is in fact

$$supp_{mf}(\mathcal{M}) = \{\mathfrak{p} \in Sing(R/f) | \mathcal{M}_{\mathfrak{p}} \neq 0 \text{ in } [MF(R_{\mathfrak{p}}, f)]\}.$$

We will use this as the definition of support from now on.

Proposition 3.4. *For every closed subset Z of $Sing(R/f)$, there is a matrix factorization $\mathcal{M} \in MF(R, f)$ such that $supp_{mf}(\mathcal{M}) = Z$.*

Theorem 3.5. *There exists a bijective correspondence*

$$\{\textit{specialization closed subsets of } Sing(R/f)\} \longleftrightarrow \{\textit{thick subcategories of } [MF(R,f)]\}$$

given by

$$Z \longmapsto \{\mathcal{M} \in [MF(R,f)] | supp_{mf}(\mathcal{M}) \subset Z\}$$

and

$$\bigcup_{\mathcal{M} \in T} supp_{mf}(\mathcal{M}) \leftarrow\!\shortmid T.$$

Remark 3.6. This theorem is also proved by Stevenson [9] and Takahashi [10] in a different context.

4. Proof of the main theorem

We will show that the support $supp_{mf}$ discussed in the last section, together with the topological space $Sing(R/f)$, is classifying data for the tensor triangulated category $K = ([MF(R,f)], \otimes^{\frac{1}{2}})$ (where $\otimes^{\frac{1}{2}}$ is the modified tensor product defined earlier). This gives a reconstruction of $Sing(R/f)$ by Theorem 5.2 of [2]. We require $2 \in R$ to be invertible in order to define the modified tensor product $\otimes^{\frac{1}{2}}$.

Lemma 4.1. *For any $\mathcal{M}, \mathcal{N}, \mathcal{L} \in MF(R,f)$, $supp_{mf}$ has the following properties:*

1. $supp_{mf}(0) = \varnothing$.
2. $supp_{mf}(\mathcal{M} \oplus \mathcal{N}) = supp_{mf}(\mathcal{M}) \cup supp_{mf}(\mathcal{N})$.
3. $supp_{mf}(\mathcal{M}[1]) = supp_{mf}(\mathcal{M})$.
4. $supp_{mf}(\mathcal{M}) \subset supp_{mf}(\mathcal{N}) \cup supp_{mf}(\mathcal{L})$ *for any distinguished triangle* $\mathcal{M} \to \mathcal{N} \to \mathcal{L} \to \mathcal{M}[1]$.
5. $supp_{mf}(\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N}) = supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N})$.

Proof. Details can be found in the author's thesis [12]. □

Remark 4.2. For a tensor triangulated category, with an identity, recall from Definition 3.1 of [2] that *support data* is a pair (X, σ) with X a topological space and σ an assignment which associates a closed subset $\sigma(a) \subset X$ to any object a, subject to a series of rules.

The rules are exactly the ones listed in Lemma 4.1, plus a condition on the tensor identity: $\sigma(1) = X$. We replace this condition by a new one: $X = \bigcup_{a \in T} \sigma(a)$ because of the lack of tensor identity in our situation. Everything else is still exactly the same as [2], for example, we still call support data *classifying support data* once the two extra conditions in Definition 5.1 of [2] are satisfied. For details please see Section 4.1.2 of [12].

Corollary 4.3. *$(Sing(R/f), supp_{mf})$ is support data for K.*

Proof. We will start by proving that $supp_{mf}$ is well defined on K, i.e.,

(1) $Sing(R/f)$ is a topological space.
(2) Given $\alpha : \mathcal{M} \to \mathcal{N}$ a homotopy equivalence, $supp_{mf}(\mathcal{M}) = supp_{mf}(\mathcal{N})$.
(3) For any $\mathcal{M} \in [MF(R,f)]$, $supp_{mf}(\mathcal{M})$ is a closed subset of $Sing(R/f)$.

We know all of these hold. In particular, (1) is trivial, (2) is true by Proposition 3.2 and (3) is Proposition 3.3.

We also have

$$Sing(R/f) = \bigcup_{\mathcal{M}\in[MF(R/f)]} supp_{mf}(\mathcal{M}).$$

The containment ($\supset$) is obvious from (3) mentioned at the beginning of the proof. The other containment is Proposition 3.4. Indeed, for any point $\mathfrak{p} \in Sing(R/f)$, its closure $\overline{\{\mathfrak{p}\}} \subset Sing(R/f)$. Then by Proposition 3.4, there is an object $\mathcal{M} \in K$ such that $supp_{mf}(\mathcal{M}) = \overline{\{\mathfrak{p}\}}$, so $\mathfrak{p} \in \overline{\{\mathfrak{p}\}} = supp_{mf}(\mathcal{M})$.

The remaining conditions are proved by the previous lemma. Therefore, $(Sing(R/f), supp_{mf})$ is support data. □

Proposition 4.4. *$(Sing(R/f), supp_{mf})$ is classifying support data for K.*

Proof. We will denote the tensor product $\otimes^{\frac{1}{2}}$ simply by $\otimes$ in the proof to avoid unclear notation like $x^{\otimes^{\frac{1}{2}n}}$ for $x \in [MF(R,f)]$. This is essentially Walker's theorem (Theorem 3.5).

(1) The fact that $Sing(R/f)$ is noetherian and any non-empty irreducible closed subset $Z \subset Sing(R/f)$ has a unique generic point comes from algebraic geometry.
(2) We need to show that there is a bijection

$$\theta : \{Y \subset Sing(R/f) |\ Y \text{ specialization closed}\} \longleftrightarrow \{J \subset [MF(R,f)] |\ J \text{ radical thick } \otimes\text{-ideal}\}$$

given by

$$Y \longmapsto \{\mathcal{E} \in [MF(R,f)] | supp_{mf}(\mathcal{E}) \subset Y\}$$

and

$$\bigcup_{\mathcal{E}\in J} supp_{mf}(\mathcal{E}) \leftarrow\!\shortmid J.$$

From Theorem 3.5, there is a bijective correspondence

$$\theta_W : \{\text{specialization closed subsets of } Sing(R/f)\} \longleftrightarrow \{\text{thick subcategories of } [MF(R,f)]\}$$

given by

$$Z \longmapsto \{\mathcal{E} \in [MF(R,f)] | supp_{mf}(\mathcal{E}) \subset Z\}$$

and

$$\bigcup_{\mathcal{E}\in T} supp_{mf}(\mathcal{E}) \leftarrow\!\shortmid T.$$

One can check very easily that θ_W is the required θ. All we need to show is that θ_W has the correct targets, for both directions. That is, we need to show

(a) $\theta_W(Z)$ is a radical thick $\otimes$-ideal.
(b) $\bigcup_{\mathcal{E}\in T} supp_{mf}(\mathcal{E})$ is specialization closed.

For (a), $\theta_W(Z)$ is thick by Theorem 3.5. It's radical, i.e., $\theta_W(Z) = \sqrt{\theta_W(Z)}$. Indeed, we always have $\theta_W(Z) \subset \sqrt{\theta_W(Z)}$; for the other direction, notice that if $x \in \sqrt{\theta_W(Z)}$, i.e., $x^{\otimes n} \in \theta_W(Z)$, then $supp_{mf}(x) = supp_{mf}(x^{\otimes n}) \subset Z$ (Lemma 4.1 (5)), then $x \in \theta_W(Z)$. The fact that $\theta_W(Z)$ is a $\otimes$-ideal is proven as follows: say $x \in \theta_W(Z)$, i.e., $supp_{mf}(x) \subset Z$. For any $a \in [MF(R,f)]$, we have $supp_{mf}(a \otimes x) = supp_{mf}(a) \cap supp_{mf}(x) \subset Z$.

It is clear that $\bigcup_{\mathcal{E} \in T} supp_{mf}(\mathcal{E})$ is specialization closed because each $supp_{mf}(\mathcal{E})$ is. □

Corollary 4.5. $Sing(R/f) \cong SpcK = Spc([MF(R,f)], \otimes^{\frac{1}{2}})$.

Remark 4.6. There is another slightly different modification by keeping the tensor product of matrix factorizations and enlarging the category $[MF(R,f)]$ into a "bigger" category $\coprod_{i\geq 0}[MF(R,if)]$ (so one does not need to assume 2 to be invertible). A similar reconstruction can also be obtained in this context using Walker's support theory. For details of this approach, one can consult [12].

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