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ISOLATED POINT THEOREMS FOR UNIFORM ALGEBRAS ON TWO- AND THREE-MANIFOLDS

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Abstract. In 1957, Andrew Gleason conjectured that if A is a uniform algebra on its maximal ideal space X and every point of X is a one-point Gleason part for A, then A must contain all continuous functions on X. However, in 1968, Brian Cole produced a counterexample to disprove Gleason's conjecture. In this paper, we establish that Gleason's conjecture still holds for two important classes of uniform algebras considered by John Anderson, Alexander Izzo and John Wermer in connection with the peak point conjecture. In fact, we prove stronger results by weakening the hypothesis of Gleason's conjecture for those two classes of uniform algebras.

1. Introduction

Suppose X is a compact Hausdorff space and $C(X)$ is the algebra of all complex-valued continuous functions on X with the supremum norm (defined as $\|f\| = \sup\{|f(x)| : x \in X\}$ for f in $C(X)$). A *uniform algebra* A on X is a uniformly closed subalgebra of $C(X)$ that separates the points of X and contains all the constant functions on X. The *maximal ideal space* $\mathfrak{M}_A$ of A is the collection of all maximal ideals of A. In fact, $\mathfrak{M}_A$ can be viewed as the collection of all multiplicative linear functionals on A. By identifying each point x of X with the point evaluation functional on A at x, we can regard X as a closed subset of $\mathfrak{M}_A$. A well-known necessary (but not sufficient) condition for A to be $C(X)$ is that $\mathfrak{M}_A$ be X (that is, the only nonzero multiplicative linear functionals on A be the point evaluations at elements of X). There are other necessary conditions for a uniform algebra to be $C(X)$ involving peak points, point derivations and isolated points (in the Gleason metric). A point x in X is called a *peak point* for A if there exists f in A such that $f(x) = 1$ and $|f(y)| < 1$ for all y in $X \setminus \{x\}$. A *point derivation* at a point x in X is a linear functional $\psi\colon A \to \mathbb{C}$ ($\mathbb{C}$ denotes the complex plane) that satisfies the Leibniz rule: $\psi(fg) = \psi(f)g(x) + f(x)\psi(g)$, for f, g in A. For ϕ, ψ in $\mathfrak{M}_A$, the formula

$$\|\phi - \psi\|_A = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \le 1\}$$

defines a metric on $\mathfrak{M}_A$, called the *Gleason metric* on $\mathfrak{M}_A$. In fact, the Gleason metric on $\mathfrak{M}_A$ is the restriction of the dual metric on A^* to $\mathfrak{M}_A$. Using this metric, Gleason [20] introduced a nontrivial equivalence relation $\sim$ on $\mathfrak{M}_A$ defined by $\phi \sim \psi$ if and only if $\|\phi - \psi\|_A < 2$ (see [13, Theorem 2.6.3]). The equivalence classes of $\mathfrak{M}_A$ under the equivalence relation $\sim$ are called the *Gleason parts* for A.

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For an arbitrary point p in X, consider the following four statements:

(a) p is a peak point for A;
(b) there is no nonzero point derivation on A at p;
(c) p is a one-point Gleason part for A;
(d) p is an isolated point in the Gleason metric for A.

We can show that (a) $\Rightarrow$ (b) $\Rightarrow$ (d) (using [13, Corollary 1.6.7] and [13, Theorem 1.6.2] respectively) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d) (the first implication follows easily and the second one is obvious). The reverse implications in both cases are not true in general (for counterexamples see [14], [26, Example 5.13], [18] and [27, §18]). It easily follows that if $A = C(X)$, then each of the statements (b), (c) and (d) holds for all points p in X. In addition, if X is metrizable, then $A = C(X)$ implies that the statement (a) holds for all points p in X. In 1957, Gleason [20] conjectured that if the maximal ideal space of A is X and the statement (c) holds for all points p in X, then A must be $C(X)$. More explicitly, the following conjecture was made.

Conjecture 1.1 (Gleason's conjecture)**.** *If the maximal ideal space of A is X and every point of X is a one-point Gleason part for A, then $A = C(X)$.*

Subsequently, the following two conjectures related to Gleason's conjecture were considered.

Conjecture 1.2 (Peak point conjecture)**.** *If the maximal ideal space of A is X and every point of X is a peak point for A, then $A = C(X)$.*

Conjecture 1.3 (Point derivation conjecture)**.** *If the maximal ideal space of A is X and there is no nonzero point derivation for A, then $A = C(X)$.*

In this paper, we also consider the following stronger conjecture.

Conjecture 1.4 (Isolated point conjecture)**.** *If the maximal ideal space of A is X and every point of X is isolated in the Gleason metric for A, then $A = C(X)$.*

In 1959, Bishop [11] showed that if X is a compact subset of $\mathbb{C}$, the peak point conjecture is true for $A = R(X)$, the uniform closure of the collection of all rational functions with no poles on X. Since statements (a), (b), (c) and (d) are equivalent for $A = R(X)$ [13, Corollary 3.3.10], the other three conjectures are also true for $R(X)$ by Bishop's result. However, all of the four conjectures fail in general by a counterexample produced by Cole in 1968 [14] (or see [13, Appendix], [27, Section 19]). A few years later a simpler counterexample was given by Basener [9] (or see [27, Example 19.8]). Nevertheless, in 2001, Anderson, Izzo and Wermer showed that the peak point conjecture is true for two important classes of uniform algebras ([5, Theorem 4.1] and [8, Theorem 1.1]).

In this paper, we will establish each of the other three conjectures, namely, Gleason's conjecture, the point derivation conjecture and the isolated point conjecture for the same classes of uniform algebras considered by Anderson, Izzo and Wermer. In view of the relations of isolated point (in the Gleason metric) with Gleason part, point derivation and peak point, it is sufficient to consider the isolated point conjecture, the strongest of all the four conjectures. Hence, in particular, our results will contain the corresponding results proved by Anderson, Izzo and Wermer. In Section 2, we will establish the isolated point conjecture for uniform algebras generated by collections of C^1 functions on a compact two-dimensional real manifold-with-boundary of class C^1. In Section 3, we will establish the isolated point conjecture for uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of $\mathbb{C}^n$. The techniques used in proving the main results in Sections 2 and 3 are inspired by works of Anderson, Izzo and Wermer.

2. Two-dimensional isolated point theorem

Suppose M is a compact two-dimensional real manifold-with-boundary of class C^1. Let A be a uniform algebra on M generated by a collection of C^1 functions. Also, assume that the maximal ideal space of A is M. In [5], Anderson and Izzo proved that if every point of M is a peak point for A, then A is the trivial uniform algebra $C(M)$. In this section, we will establish that their result remains true even if we replace the condition "every point of M is a peak point for A" by the weaker condition "every point of M is isolated in the Gleason metric for A". More precisely, the following result will be established.

Theorem 2.1 (Two-dimensional isolated point theorem)**.** *Suppose M is a compact two-dimensional real manifold-with-boundary of class C^1. Let A be a uniform algebra on M generated by a collection of C^1 functions. If*

(i) *the maximal ideal space of A is M and*
(ii) *every point of M is isolated in the Gleason metric for A,*

then $A = C(M)$.

Theorem 2.1 will then be extended to a class of uniform algebras on compact subsets of smooth two-dimensional manifolds. In fact, we will prove the following result.

Theorem 2.2. *Suppose X is a compact subset of M, a two-dimensional real manifold-with-boundary of class C^1, and A is a uniform algebra on X generated by continuous functions that extend to be C^1 on a neighborhood of X. If*

(i) *the maximal ideal space of A is X and*
(ii) *every point of X is isolated in the Gleason metric for A,*

then $A = C(X)$.

We need some preliminaries in order to prove the two-dimensional isolated point theorem. We begin with the following elementary result whose proof is omitted.

Lemma 2.3. *Suppose X and Y are compact Hausdorff spaces and $f\colon X \to Y$ is a homeomorphism. If A is a uniform algebra on X with maximal ideal space $\mathfrak{M}_A$, then the following hold:*

(i) *$B = \{g \circ f^{-1} : g \in A\}$ is a uniform algebra on Y,*
(ii) *$F\colon A \to B$ defined by $F(g) = g \circ f^{-1}$ is an isomorphism, and*
(iii) *if φ is in $\mathfrak{M}_B$, the maximal ideal space of B, then $\varphi \circ F$ is in $\mathfrak{M}_A$. Also, for ϕ, ψ in $\mathfrak{M}_B$, $\|\phi \circ F - \psi \circ F\|_A = \|\phi - \psi\|_B$.*

Lemma 2.4. *Suppose A is a uniform algebra on X and Y is a closed subset of X.*

(i) *If a point in Y is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for $\overline{A|Y}$, the uniform closure of the algebra $A|Y = \{f|Y \in C(Y) : f \in A\}$.*
(ii) *Let $\tilde{B}$ be a uniform algebra on Y containing $A|Y$ and with maximal ideal space Y. If a point in Y is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for $\tilde{B}$.*

Proof. Write $B = \overline{A|Y}$. Note that the map $T\colon A \to B$ defined by $T(f) = f|Y$ is a multiplicative linear operator. In fact, T is bounded with $\|T\| = 1$. So, $\|T^*\| = \|T\| = 1$, where $T^*\colon B^* \to A^*$, the adjoint of T, is given by $T^*(\phi) = \phi \circ T$. Clearly T has range $A|Y$ that is dense in B. So, by applying duality, we see that T^* is injective. Hence, for ϕ, ψ in B^* with $\phi \neq \psi$, we obtain

$$(*) \qquad 0 < \|\phi \circ T - \psi \circ T\| = \|T^*(\phi - \psi)\| \leq \|\phi - \psi\|.$$

For the proof of part (i), fix p in Y. If ϕ is in $\mathfrak{M}_B$, then $\phi \circ T$ is in $\mathfrak{M}_A$. Note that, at p, if ϕ_p is the point evaluation functional on B, then $\phi_p \circ T$ is the point evaluation functional on A. Hence, by taking ϕ in $\mathfrak{M}_B$ and $\psi = \phi_p$, we see from the inequality (*) that if p is isolated in the Gleason metric for A, then p is also isolated in the Gleason metric for B.

For the proof of part (ii), suppose that p in Y is isolated in the Gleason metric for A. By part (i), p is also isolated in the Gleason metric for B. So, there exists $\delta > 0$ such that $\|\phi - \phi_p\|_B \geq \delta$ for all ϕ in $\mathfrak{M}_B$ with $\phi \neq \phi_p$, where ϕ_p is the point evaluation functional on B at p. Since $B \subseteq \tilde{B}$ and A separates the points of X, for all q in $Y \setminus \{p\}$ we obtain $\|p-q\|_{\tilde{B}} \geq \|p-q\|_B \geq \delta$ (here we identify the points with the corresponding point evaluations on the respective uniform algebras). Since the maximal ideal space of $\tilde{B}$ is Y, the preceding inequality shows that p is isolated in the Gleason metric for $\tilde{B}$. $\square$

The *essential set* for A, a notion introduced by Bear, is the unique minimal closed subset E of X with the property that A contains every continuous function on X that vanishes on E [10, §2] (or see [13, Theorem 2.8.1]). Bear proved that the restriction of a uniform algebra to its essential set is uniformly closed (and hence forms a uniform algebra) [10, Theorem 2]. More importantly, he showed that the restricted uniform algebra is defined on its maximal ideal space if and only if the original uniform algebra is defined on its maximal ideal space [10, Theorem 4]. We now strengthen these results in the form of the following theorem.

Theorem 2.5. *Suppose A is a uniform algebra on X, and L is a closed subset of X containing the essential set E for A. Then $A|L$ is uniformly closed in $C(L)$. Moreover, the maximal ideal space of $A|L$ is L if and only if the maximal ideal space of A is X.*

Proof. To show that $A|L$ is uniformly closed in $C(L)$, fix f in $\overline{A|L}$. Then, there is a sequence $\{f_n\}_{n=1}^{\infty}$ in A such that $f = \lim_{n\to\infty} f_n|L$. Since $E \subseteq L$ and $A|E$ is uniformly closed in $C(E)$, it follows that $f|E$ is in $A|E$. Next, by the Tietze extension theorem, f can be extended continuously to a function $\tilde{f}$ on X. Clearly $\tilde{f}|E = f|E$ is in $A|E$, which then implies $\tilde{f}$ is in A as E is the essential set for A. Hence, $f = \tilde{f}|L$ is in $A|L$. This shows that $A|L$ is uniformly closed in $C(L)$.

To prove the second assertion, let E' be the essential set for $A|L$. We claim that $E' = E$. Let $g \in C(L)$ with $g|E = 0$. Then g can be extended to a function $\tilde{g}$ in $C(X)$ by the Tietze extension theorem. Since $\tilde{g}|E = g|E = 0$, and E is the essential set for A, we obtain $\tilde{g} \in A$. Consequently, $g = \tilde{g}|L$ is in $A|L$. This shows that $E' \subseteq E$. Now let $h \in C(X)$ with $h|E' = 0$. Then $h|L$ is in $A|L$ since E' is the essential set for $A|L$. Since $E \subseteq L$, and E is the essential set for A, it follows that h is in A. This shows that $E \subseteq E'$. Hence our claim is true, that is, E is the essential set for $A|L$. Then, by Bear's result [10, Theorem 4], the maximal ideal space of $A|L$ is L if and only if the maximal ideal space of $A|E$ is E if and only if the maximal ideal space of A is X. □

Now we provide a partial converse of part (i) of Lemma 2.4.

Lemma 2.6. *Suppose Y is a compact Hausdorff space, and X is a closed subset of Y. Let A be a uniform algebra on X with maximal ideal space X. Then, $B = \{f \in C(Y) : f|X \in A\}$ is a uniform algebra on Y with maximal ideal space Y. If a point in X is isolated in the Gleason metric for A, then it is also isolated in the Gleason metric for B. Moreover, each point in $Y \setminus X$ is a one-point Gleason part for B.*

Proof. It easily follows that B is a uniform algebra on Y from the fact that A is a uniform algebra on X. To see that the maximal ideal space of B is Y, first note that both A and B have same essential set, say E, and also $A|E = B|E$. Then, from the hypothesis that the maximal ideal space of A is X, we obtain that the maximal ideal space of B is Y, by Bear's result [10, Theorem 4].

To verify the second assertion, let p in X be isolated in the Gleason metric for A. Then, there exists $\delta > 0$ such that $\|p - q\|_A \geq \delta$ for all q in $X \setminus \{p\}$. Note that $\|r-s\|_A = \|r-s\|_B$, for r, s in X. So, in particular, $\|p-q\|_B \geq \delta$ for all q in $X \setminus \{p\}$. Next, for q in $Y \setminus X$, by Urysohn's lemma (see [22, Theorem 33.1]), there exists h in $C(Y)$ with $0 \leq h \leq 1$ such that $h(X) = \{0\}$ and $h(q) = 1$. Then, h is in B, and $\|p - q\|_B \geq |h(p) - h(q)| = 1$ for q in $Y \setminus X$. Hence, $\|p - q\|_B \geq \delta_0 = \min(\delta, 1) > 0$ for all q in $Y \setminus \{p\}$. Therefore, p is isolated in the Gleason metric for B.

The last assertion easily follows from Urysohn's lemma (see [22, Theorem 33.1]). □

An *annihilating measure* for A is a regular complex Borel measure μ on X so that $\int f\, d\mu = 0$ for each f in A. The collection of all annihilating measures for A is denoted by $A^\perp$.

Given a regular complex Borel measure μ on X and f in $C(X)$, we define the *push-forward measure* $f_*(\mu)$ on $\mathbb{C}$ by $f_*(\mu)(K) = \mu(f^{-1}(K))$ for each Borel subset K of $\mathbb{C}$. If g is an $f_*(\mu)$-integrable function, then it follows that

$$\int_{\mathbb{C}} g\, d(f_*(\mu)) = \int_X g \circ f\, d\mu.$$

The following sufficient condition for a uniform algebra to be the collection of all continuous functions is due to Anderson and Izzo.

Lemma 2.7 ([5, Lemma 2.1])**.** *Suppose A is a uniform algebra on X and A_0 is a dense subset of A. If $f_*(\mu) = 0$ for each f in A_0 and each measure μ in $A^\perp$, then $A = C(X)$.*

Let μ be a complex Borel measure on $\mathbb{C}$ with compact support. Then, the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}$$

for all z in $\mathbb{C}$ such that the integral converges absolutely.

Let M be a compact n-dimensional real manifold-with-boundary of class C^1, and $\mathcal{F}$ be a collection of complex-valued C^1 functions on M. Then, the set

$$E = \{p \in M : df_1 \wedge \ldots \wedge df_n(p) = 0 \text{ for each } n\text{-tuple } f_1, \ldots, f_n \text{ in } \mathcal{F}\}$$

is called the *exceptional set* of $\mathcal{F}$. If M is two-dimensional, following Freeman [16, Section 1] (or see [5, Section 2]), we obtain another way of describing the exceptional set E of M. If U is an open subset of M, and if x, y are local coordinates on U, then by setting $z = x + iy$ we can expand a 2-form $df \wedge dg$ in terms of the differentials $dz = dx + idy$ and $d\bar{z} = dx - idy$ to obtain

$$df \wedge dg = (\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z})\, dz \wedge d\bar{z}.$$

If some function f in $\mathcal{F}$ agrees with z on U, then it follows that

$$E \cap U = \{p \in U : \frac{\partial g}{\partial \bar{z}}(p) = 0 \text{ for every } g \in \mathcal{F}\}.$$

Next we state a result of Freeman that plays a crucial role in proving the main result of the section.

Theorem 2.8 ([16, Theorem 3.2]). *Let M be a compact two-dimensional real manifold-with-boundary of class C^1, and A be a uniform algebra on M generated by a collection $\mathcal{F}$ of C^1 functions with exceptional set E. Suppose that the maximal ideal space of A is M. If f is in $A \cap C^1(M)$, and μ is in $A^\perp$, then $\widehat{f_*(\mu)} = 0$ almost everywhere on $\mathbb{C} \setminus f(E)$ with respect to the Lebesgue measure.*

The proof of our main result in this section makes repeated use of the following theorem of Alexander [1] (or see [27, Theorem 26.4]).

Theorem 2.9 ([1]). *Let $\{X_n\}_{n=1}^\infty$ be a sequence of compact subsets of $\mathbb{C}$ with compact union X. If $R(X_n) = C(X_n)$ for all n, then $R(X) = C(X)$.*

Finally, we present a proof of the two-dimensional isolated point theorem.

Proof of Theorem 2.1. Let A_0 denote the collection of all C^1 functions in A, and E be the exceptional set of A_0. Since A_0 is dense in A, by Lemma 2.7, it is sufficient to show that $f_*(\mu) = 0$ for each f in A_0 and each μ in $A^\perp$. So, fix f in A_0, and μ in $A^\perp$. Then, by Theorem 2.8, $\widehat{f_*(\mu)} = 0$ almost everywhere on $\mathbb{C} \setminus f(E)$. Consequently, $f_*(\mu)$ is supported on $f(E)$ (by [17, Chapter II, Theorem 8.2]) and $f_*(\mu) \in R(f(E))^\perp$ (by [17, Chapter II, Theorem 8.1] or [27, Lemma 26.2]). Hence, it is sufficient to show that $R(f(E)) = C(f(E))$.

Note that the set of all critical values of f, denoted by S, is compact. Also, by Sard's theorem [24] (or see [21, §7]), $m(S) = 0$, where m denotes the two-dimensional Lebesgue measure on $\mathbb{C}$. Moreover, if ∂M is the boundary of M, then $m(f(\partial M)) = 0$. Hence, $m(S \cup f(\partial M)) = 0$. In particular, $m(f(E) \cap (S \cup f(\partial M))) = 0$. So, by the Hartogs-Rosenthal theorem (see [13, Theorem 3.2.4]), we obtain $R(f(E) \cap (S \cup f(\partial M))) = C(f(E) \cap (S \cup f(\partial M)))$.

Now, we claim the following: For each z_0 in $f(E) \setminus (S \cup f(\partial M))$, there is a closed disc D centered at z_0 such that $R(f(E) \cap D) = C(f(E) \cap D)$.

If the preceding claim holds, we obtain a sequence $\{D_n\}_{n=1}^\infty$ of closed discs with $R(f(E) \cap D_n) = C(f(E) \cap D_n)$ for each $n \in \mathbb{N}$ such that

$$f(E) \setminus (S \cup f(\partial M)) = \bigcup_{n=1}^{\infty} D_n.$$

Then, $\{f(E) \cap D_n : n \in \mathbb{N}\} \cup \{f(E) \cap (S \cup f(\partial M))\}$ is a countable collection of compact sets with union $f(E)$ which is compact. By Theorem 2.9, we conclude that $R(f(E)) = C(f(E))$.

To prove the claim, fix z_0 in $f(E) \setminus (S \cup f(\partial M))$. Applying a corollary of the inverse function theorem, there is a closed disc D, centered at z_0, such that $f^{-1}(D)$ is a disjoint union of finitely many compact subsets $U_1, U_2, \ldots, U_t$ of M, such that f maps each U_j $(j = 1, 2, \ldots, t)$ diffeomorphically onto D.

Write $A_j = \overline{A|U_j}$ for $j = 1, 2, \ldots, t$. Since $f|U_j$ is a diffeomorphism of U_j onto D, by part (i) of Lemma 2.3, $B_j = \{h \circ (f|U_j)^{-1} : h \in A_j\}$ is a uniform algebra on D. Moreover, by part (ii) of Lemma 2.3, $f|U_j$ induces an isomorphism from A_j onto B_j, given by $h \mapsto h \circ (f|U_j)^{-1}$.

Let $E_j = E \cap U_j$ for $j = 1, 2, \ldots, t$. Note that for each $j = 1, 2, \ldots, t$, the uniform algebra B_j is generated by the collection $\mathcal{F}_j = \{h \circ (f|U_j)^{-1} : h \in A_0\}$ of C^1 functions. Since $f \circ (f|U_j)^{-1} \in \mathcal{F}_j$ is the identity function $z \mapsto z$ on D, by the remarks following the definition of the exceptional set, we obtain $f(E_j) = \{w \in D : \frac{\partial g}{\partial \bar{z}}(w) = 0 \text{ for all } g \text{ in } \mathcal{F}_j\}$. Therefore, by [13, Corollary 3.2.2], $k|f(E_j)$ is in $R(f(E_j))$ for k in B_j, and consequently, $B_j|f(E_j) \subseteq R(f(E_j))$ $(j = 1, 2, \ldots, t)$. Since every point of M is isolated in the Gleason metric for A, by part (i) of Lemma 2.4, we obtain that every point of U_j is also isolated in the Gleason metric for A_j $(j = 1, 2, \ldots, t)$. So, by part (iii) of Lemma 2.3, every point of D is isolated in the Gleason metric for B_j $(j = 1, 2, \ldots, t)$. Note $B_j|f(E_j) \subseteq R(f(E_j))$ and $f(E_j)$ is the maximal ideal space of $R(f(E_j))$ (see [17, Chapter III, Lemma 2.1, Lemma 2.2]). So, by part (ii) of Lemma 2.4, every point of $f(E_j)$ is isolated in the Gleason metric for $R(f(E_j))$ $(j = 1, 2, \ldots, t)$. Consequently, for all $j = 1, 2, \ldots, t$, each point of $f(E_j)$ is a peak point for $R(f(E_j))$ because the peak points for $R(f(E_j))$ are precisely the isolated points (in the Gleason metric) for $R(f(E_j))$ (see [13, Corollary 3.3.10]). Therefore, $R(f(E_j)) = C(f(E_j))$ for all $j = 1, 2, \ldots, t$, by Bishop's peak point theorem for rational approximation (see [13, Theorem 3.3.3]). Finally, since $f(E_j)$ $(j = 1, 2, \ldots, t)$ and $f(E) \cap D$ are compact subsets of $\mathbb{C}$, and $\bigcup_{j=1}^{t} f(E_j) = f(E) \cap D$, applying Theorem 2.9 again, we obtain $R(f(E) \cap D) = C(f(E) \cap D)$. This proves the claim and, hence, the theorem. $\square$

Proof of Theorem 2.2. First, we choose a compact submanifold-with-boundary N of M containing X. Next, define $B = \{f \in C(N) : f|X \in A\}$. Then, using condition (i) and by applying Lemma 2.6, we see that B is a uniform algebra on N with maximal ideal space N. Also, it easily follows that B is generated by C^1 functions from the fact that A is generated by continuous functions that extend to be C^1 on a neighborhood of X. Moreover, from condition (ii), each point of X is isolated in the Gleason metric for B by Lemma 2.6. Since a one-point Gleason part is, in particular, an isolated point in the Gleason metric, again from Lemma 2.6, we see that each point in $N \setminus X$ is also isolated in the Gleason metric for B. Thus, by Theorem 2.1, $B = C(N)$ and consequently, $A = C(X)$. $\square$

The condition (ii) in Theorem 2.1 can be weakened by assuming "almost every point of M is isolated in the Gleason metric for A". In fact, the same proof remains valid in the weakened condition case too because $R(f(E_j)) = C(f(E_j))$ even when almost every point of $f(E_j)$ is isolated in the Gleason metric for $R(f(E_j))$. Consequently, the condition (ii) of Theorem 2.2 can be weakened by assuming "almost every point of X is isolated in the Gleason metric for A".

3. Three-dimensional isolated point theorem

In this section, we establish an isolated point theorem for uniform algebras generated by polynomials on a compact subset of a three-dimensional real-analytic manifold-with-boundary embedded in $\mathbb{C}^n$.

Let X be a compact subset of $\mathbb{C}^n$. We denote the uniform closure of the collection of all polynomials on X by $P(X)$. Also, the uniform closure of the collection of all continuous functions that are holomorphic in a neighborhood (dependent on the function) of X is denoted by $O(X)$. It is easy to see that both $P(X)$ and $O(X)$ are uniform algebras on X, and that $P(X) \subseteq O(X) \subseteq C(X)$. The *polynomial convex hull* $\hat{X}$ (or $X^\frown$) of X is defined as the set

$$\hat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x \in X} |p(x)| \text{ for all polynomials } p\}.$$

In fact, the polynomial convex hull of X can be naturally identified with the maximal ideal space of $P(X)$ [17, Chapter III, Theorem 1.2]. The set X is said to be *polynomially convex* if $\hat{X} = X$, that is, if the maximal ideal space of $P(X)$ is X. Similarly, X is said to be *holomorphically convex* if the maximal ideal space of $O(X)$ is X.

Suppose M is a real-analytic three-dimensional submanifold of $\mathbb{C}^n$. For a compact subset X of M, by the boundary of X relative to M we mean the union of the topological boundary of X relative to M and the set $X \cap \partial M$, and is denoted by ∂X. Now assume that X is a polynomially convex compact subset of M such that ∂X is a two-dimensional submanifold of class C^1. In [8], Anderson, Izzo and Wermer proved that if every point of X is a peak point for $P(X)$, then $P(X) = C(X)$. In this section, we will establish that their result remains true even if the condition "every point of X is a peak point for $P(X)$" is replaced by the weaker condition "every point of X is isolated in the Gleason metric for $P(X)$". More precisely, the following result will be established.

Theorem 3.1 (Three-dimensional isolated point theorem)**.** *Suppose M is a real-analytic three-dimensional submanifold of $\mathbb{C}^n$. Assume that X is a compact subset of M such that the boundary ∂X of X relative to M is a two-dimensional submanifold of class C^1. If*

(i) *X is polynomially convex and*
(ii) *every point of X is isolated in the Gleason metric for $P(X)$,*

then $P(X) = C(X)$.

Suppose M is a real submanifold of $\mathbb{C}^n$, of class C^1, and p is a point in M. In general, the real tangent space $T_p(M)$ of M at p is not a complex vector subspace of $T_p(\mathbb{C}^n) \simeq T_p(\mathbb{R}^{2n})$. The largest complex vector subspace of $T_p(M)$, denoted by $H_p(M)$, is called the *holomorphic tangent space* of M at p and the complex dimension of it is called the *Cauchy-Riemann rank* (in short, *CR rank*) of M at p. If $H_p(M)$ is nontrivial, then M is said to have a *complex tangent* at p. On the contrary, M is called *totally real* if it has no complex tangent at any point. The following lemma, a proof of which is in [8], characterizes the points where a real submanifold of $\mathbb{C}^n$ has a complex tangent.

Lemma 3.2 ([8, Lemma 2.5])**.** *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$. Then M has a complex tangent at p if and only if $dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_m}(p) = 0$ as a form on M, for all m-tuples $(i_1, i_2, \ldots, i_m)$ with $i_1 \leq i_2 \leq \ldots \leq i_m$.*

The following result due to O'Farrell, Preskenis and Walsh [23] gives a necessary and sufficient condition for continuous functions on K to be in $O(K)$, where K is a holomorphically convex compact subset of $\mathbb{C}^n$.

Theorem 3.3 ([23, Theorem 2]). *Suppose K is a holomorphically convex compact set and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. Then, a continuous function f is in $O(K)$ if and only if there exists g in $O(K)$ with $f = g$ on K_0.*

For convenience we state two corollaries of Theorem 3.3 that will be used. The first corollary and its proof is in [8, Corollary 2.4]. The proof of the second one follows from the fact that a polynomially convex set is also holomorphically convex and from the definition of essential set.

Corollary 3.4 ([8, Corollary 2.4]). *Suppose K is a polynomially convex compact set and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. If $P(K_0) = C(K_0)$, then $P(K) = C(K)$.*

Corollary 3.5. *Suppose K is a polynomially convex compact set and K_0 is a closed subset of K such that $K \setminus K_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1. Then, K_0 contains the essential set E for $P(K)$.*

For certain polynomially convex compact subset K of $\mathbb{C}^n$, the following result of Anderson, Izzo and Wermer provides a sufficient condition for $P(K)$ to be $C(K)$. The n-dimensional Hausdorff measure is denoted by $\mathcal{H}^n$.

Lemma 3.6 ([8, Lemma 3.1]). *Suppose Y is a compact subset of $\mathbb{C}^n$ with $P(Y) = C(Y)$ and S is a subset of $\mathbb{C}^n$ with $\mathcal{H}^2(S) = 0$. If $Y \cup S$ is compact and polynomially convex, then $P(Y \cup S) = C(Y \cup S)$.*

We will prove the following lemma.

Lemma 3.7. *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$, of class C^2. Also, assume that*

(i) *K is polynomially convex and*
(ii) *every point of K is isolated in the Gleason metric for $P(K)$.*

Then $E \cap K$ has empty interior in M, where E is the set of all points at which M has a complex tangent.

For class C^1 submanifolds of $\mathbb{C}^n$, Anderson, Izzo and Wermer [8, Lemma 3.2] showed that the same conclusion is true under the stronger condition "every point of K is a peak point for $P(K)$" in place of the condition (ii). In addition, they mentioned that for class C^2 submanifolds of $\mathbb{C}^n$ their result can be proved using different techniques.

An *analytic disc* is a one-to-one continuous map $\Phi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ which is holomorphic in $\mathbb{D}$, the open unit disc in the complex plane. By the boundary of the analytic disc Φ, we will mean the map $\Phi|\partial\mathbb{D}$, the restriction of Φ to the unit circle $\partial\mathbb{D}$. Often in the literature, the analytic disc and its boundary are identified with their images in $\mathbb{C}^n$.

In order to prove Lemma 3.7, we first need the following lemma. The proof of this result is due to John Wermer (obtained via a personal communication). Also, it was brought to the author's attention that the following lemma is implicit in [4, Theorem 3.2] by John Anderson.

Lemma 3.8. *Suppose M is a real m-dimensional submanifold of $\mathbb{C}^n$, of class C^2. Let E be the set of points at which M has a complex tangent. Assume that U is an open subset of $\mathbb{C}^n$ so that $M \cap U$ is a nonempty subset of E. Then $M \cap U$ contains the boundary of an analytic disc.*

Proof. Denote the CR rank of M at q by $r(q)$, and put $r_0 = \min\{r(q) : q \in M \cap U\}$. Note that $\{q \in M \cap U : r(q) = r_0\}$ is a relatively open subset of $M \cap U$. Then, there is a nonempty open subset $V \subseteq \mathbb{C}^n$ with $V \subseteq U$ such that $M \cap V$ is nonempty and $r(q) = r_0$ for all q in $M \cap V$.

By definition of CR rank, $2r_0 \leq m$. Also, $r_0 \geq 1$ as $M \cap U \subseteq E$. Then, (by a result in [12, §12.5]) without loss of generality we can find a generic m-dimensional CR submanifold M_0 of $\mathbb{C}^{m-r_0}$, of class C^2 and a CR map $g\colon M_0 \to \mathbb{C}^{n-m+r_0}$ so that $M \cap V = \{(\zeta, g(\zeta)) : \zeta \in M_0\}$. Note that g, being a CR map, can be written as $g = (g_1, \ldots, g_{n-m+r_0})$, where each $g_i\colon M_0 \to \mathbb{C}$ is a CR function for $i = 1, \ldots, n-m+r_0$. Also, note that $m - r_0 < m \leq 2(m - r_0)$. So, by an approximation theorem of Baouendi and Treves (see [12, §13, Theorem 1]), there exists an open subset N_0 of M_0 such that for each $i = 1, \ldots, n-m+r_0$, there is a sequence of polynomials $\{p_n^i\}_{n=1}^{\infty}$ on $\mathbb{C}^{m-r_0}$ that converges uniformly to g_i on N_0. Since M_0 is a generic CR manifold with CR rank r_0, by a result of Bishop [3, Theorem 18.7], there is an analytic disc Δ in $\mathbb{C}^{m-r_0}$ with boundary $\partial\Delta$ contained in N_0. So, in particular, $\{p_n^i\}_{n=1}^{\infty}$ converges uniformly to g_i on the boundary $\partial\Delta$ of Δ, for $i = 1, \ldots, n-m+r_0$. Also, for each $i = 1, \ldots, n-m+r_0$, the maximum modulus principle on Δ implies that $\{p_n^i\}_{n=1}^{\infty}$ converges uniformly on Δ to a function, say, G_i which is analytic in the interior of Δ and $G_i = g_i$ on $\partial\Delta$. Define a map $G\colon \Delta \to \mathbb{C}^{n-m+r_0}$ by $G = (G_1, \ldots, G_{n-m+r_0})$. Note that G is a continuous map that is analytic in the interior of Δ and agrees with g on $\partial\Delta$. Then, $\Phi\colon \Delta \to \mathbb{C}^n$ given by $\Phi(\zeta) = (\zeta, G(\zeta))$ is an analytic map with $\Phi(\partial\Delta) = \{(\zeta, g(\zeta)) : \zeta \in \partial\Delta\} \subseteq M \cap V$. Thus, the image $\Phi(\Delta)$ of Φ is an analytic disc in $\mathbb{C}^n$ whose boundary lies in $M \cap V \subseteq M \cap U$. □

Proof of Lemma 3.7. We claim that K contains no analytic disc. Suppose that the claim is not true, that is, there is an analytic disc $\Phi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ with $\Phi(\overline{\mathbb{D}}) \subseteq K$. Then, for z in $\mathbb{D} \setminus \{0\}$, we obtain

$$\begin{aligned} \|z - 0\|_{P(\overline{\mathbb{D}})} &\geq \sup\{|f(\Phi(z)) - f(\Phi(0))| : f \in P(K), \|f\| \leq 1\} \\ &= \|\Phi(z) - \Phi(0)\|_{P(K)}. \end{aligned}$$

Since Φ is one-to-one, by condition (ii), there exists $\delta > 0$ such that $\|\Phi(z) - \Phi(0)\|_{P(K)} \geq \delta$ for all z in $\mathbb{D} \setminus \{0\}$. Therefore, $\|z - 0\|_{P(\overline{\mathbb{D}})} \geq \delta$, for all z in $\mathbb{D} \setminus \{0\}$. Hence, 0 is an isolated point of $\overline{\mathbb{D}}$ in the Gleason metric for $P(\overline{\mathbb{D}})$, a contradiction. So, the claim is true, that is, K does not contain any analytic disc.

Next suppose, on the contrary to our assertion, that $E \cap K$ has nonempty interior in M. Then, there is a nonempty open subset U of $\mathbb{C}^n$ with $M \cap U \subseteq E \cap K$. Therefore, by Lemma 3.8, there is an analytic disc $\Psi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ whose boundary lies in $M \cap U \subseteq K$. Moreover, by condition (i), the analytic disc $\Psi(\overline{\mathbb{D}})$ is contained in K. However, this contradicts the fact that K does not contain any analytic disc. Consequently, $E \cap K$ has empty interior. □

The following lemma is a corollary of results due independently to Alexander [2, Theorem 1] and Sibony [25, Theorem 17]. This result was brought to the author's attention by Edgar Lee Stout.

Lemma 3.9 ([3, Corollary 21.10]). *Suppose K is a compact subset of $\mathbb{C}^n$ with polynomial convex hull $\hat{K}$. Then, $\mathcal{H}^2((\hat{K} \setminus K) \cap \mathbb{B}_n(z;r)) > 0$, for z in $\hat{K} \setminus K$ and $r > 0$, where $\mathbb{B}_n(z;r)$ is the open ball in $\mathbb{C}^n$ with center z and radius r.*

Let U be an open subset of $\mathbb{C}^n$. A closed subset V of U is a *real-analytic subvariety* of U if for each point p in V, there exists a neighborhood $W \subseteq U$ of p in $\mathbb{C}^n$ and a finite collection $\mathcal{F}$ of real-valued functions that are real-analytic in W, so that

$$V \cap W = \{q \in W : f(q) = 0 \text{ for all } f \text{ in } \mathcal{F}\}.$$

A point p in V is a *regular point* (of *dimension* d) of V if there is a neighborhood O of p in $\mathbb{C}^n$ such that $V \cap O$ is a real-analytic submanifold (of dimension d) of O. A point of V that is not a regular point is a *singular point* of V. The set of all regular points of V is denoted by V_{reg}, whereas the set of all singular points of V is denoted by V_{sing}. The *dimension* of V is the largest integer d such that V has regular points of dimension d. The following result is regarding the Hausdorff measure of the singular set of a real-analytic subvariety of $\mathbb{C}^n$.

Lemma 3.10 ([15, Section 3.4.10]). *Suppose V is an m-dimensional real-analytic subvariety of an open subset U of $\mathbb{C}^n$. Then $\mathcal{H}^{m-1}(V_{\text{sing}} \cap C)$ is finite for each compact subset C of U.*

Finally, we present a proof of the three-dimensional isolated point theorem.

Proof of Theorem 3.1. Let E be the set of all points at which M has a complex tangent. Also, let X_0 be the interior of X relative to M, and Ω_0 be an open subset of $\mathbb{C}^n$ with $X_0 = X \cap \Omega_0$. Define $\tilde{E} = E \cap X_0$ and $K_0 = \partial X \cup \tilde{E}$. Note that K_0 is compact because each limit point of $\tilde{E}$, that is not in $\tilde{E}$, belongs to ∂X. Also, note X is polynomially convex by assumption, and $X \setminus K_0$ is a totally real submanifold of Ω_0. Hence, by Corollary 3.4, to show $P(X) = C(X)$ it is sufficient to prove that $P(K_0) = C(K_0)$. Moreover, by Corollary 3.5, K_0 contains the essential set for $P(X)$ and, hence, by Theorem 2.5, K_0 is polynomially convex.

Lemma 3.2 implies that $\tilde{E}$ is a real-analytic subvariety of Ω_0. Let $\tilde{E}_{\text{c}}$ be the set of all points at which $\tilde{E}_{\text{reg}}$ itself has complex tangent, and set $Z = \partial X \cup \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. It follows that Z is compact and that $K_0 \setminus Z$ $(= \tilde{E}_{\text{reg}} \setminus \tilde{E}_{\text{c}})$ is a totally real, real-analytic submanifold of Ω_0. So, again by Corollary 3.4, to show $P(K_0) = C(K_0)$ it is sufficient to prove that $P(Z) = C(Z)$. Moreover, by Corollary 3.5, Z contains the essential set for $P(K_0)$ and, hence, by Theorem 2.5, Z is polynomially convex.

Next, to prove $P(Z) = C(Z)$, we apply Lemma 3.6 with $Y = \partial X$ and $S = \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. First, we verify that $\mathcal{H}^2(S) = 0$, that is, $\mathcal{H}^2(\tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}) = 0$. By Lemma 3.7, $E \cap X$ and, hence, $\tilde{E}$ has no interior in M. Therefore, the dimension of $\tilde{E}$ is at most two. So, $\mathcal{H}^1(\tilde{E}_{\text{sing}} \cap C) < \infty$ for every compact subset C of Ω_0, by Lemma 3.10. Now, covering Ω_0 by countably many compact sets, we obtain $\mathcal{H}^2(\tilde{E}_{\text{sing}}) = 0$. Note that $\tilde{E}_c$ is a real-analytic subvariety of Ω_0 follows from Lemma 3.2. To show $\mathcal{H}^2(\tilde{E}_{\text{c}}) = 0$, fix a point p in $\tilde{E}_{\text{reg}}$. Since $\tilde{E}_{\text{reg}}$ is open in $\tilde{E}$ and $\tilde{E}$ is open in K_0, clearly $\tilde{E}_{\text{reg}}$ is open in K_0. Therefore, there is $r > 0$ such that $\overline{\mathbb{B}_n(p;r)} \cap K_0 \subseteq \tilde{E}_{\text{reg}}$. Denote $\overline{\mathbb{B}_n(p;r)} \cap K_0$ by K_p. Then, K_p is a compact subset of $\tilde{E}_{\text{reg}}$. Note K_p, being an intersection of two polynomially convex sets, is polynomially convex. Also, by applying Lemma 2.4 using condition (ii), note that every point of K_0 is an isolated point in the Gleason metric for $P(K_0)$. Therefore, Lemma 3.7 implies that $\tilde{E}_{\text{c}} \cap K_p$

has empty interior in $\tilde{E}_{\text{reg}}$. Since p in $\tilde{E}_{\text{reg}}$ is arbitrary, it follows that $\tilde{E}_c$ is a real-analytic subvariety of Ω_0 of dimension at most one. Consequently, $\mathcal{H}^2(\tilde{E}_{\text{c}}) = 0$.

Finally, we verify that $P(Y) = C(Y)$, that is, $P(\partial X) = C(\partial X)$. To do this we will first show that ∂X is polynomially convex. Because ∂X is a subset of the polynomially convex set Z, the polynomial convex hull $\widehat{\partial X}$ of ∂X is contained in Z. So, $\widehat{\partial X} \setminus \partial X \subseteq Z \setminus \partial X \subseteq \tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}$. However, $\mathcal{H}^2(\tilde{E}_{\text{sing}} \cup \tilde{E}_{\text{c}}) = 0$. Hence, by Lemma 3.9, $\widehat{\partial X} \setminus \partial X$ is empty, that is, ∂X is polynomially convex. Next, by applying Lemma 2.4 using condition (ii), we see that every point of ∂X is an isolated point in the Gleason metric for $P(\partial X)$. Hence, by the two-dimensional isolated point theorem (Theorem 2.1), $P(\partial X) = C(\partial X)$. $\square$

In 2009, Anderson and Izzo [6] established the peak point conjecture for abstract uniform algebras generated by real-analytic functions on a compact subset of real-analytic three-dimensional manifold-with-boundary, not necessarily embedded in a complex Euclidean space. Extending their work, the same authors have recently established the peak point conjecture for abstract uniform algebras generated by real-analytic functions on certain compact subsets of real-analytic varieties of arbitrary dimensions in [7]. A natural question here: Does the isolated point conjecture hold for these two classes of uniform algebras? Of course, an affirmative answer would generalize these works of Anderson and Izzo. The author plans to address this question in the future.

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References

[1] H. Alexander, *Polynomial approximation and analytic structure*, Duke Math. J. **38** (1971), 123–135. MR0283244 (44 #477)

[2] H. Alexander, *Structure of certain polynomial hulls*, Michigan Math. J. **24** (1977), no. 1, 7–12. MR0463500 (57 #3449)

[3] Herbert Alexander and John Wermer, *Several complex variables and Banach algebras*, 3rd ed., Graduate Texts in Mathematics, vol. 35, Springer-Verlag, New York, 1998. MR1482798 (98g:32002)

[4] John T. Anderson, *Finitely generated algebras and algebras of solutions to partial differential equations*, Pacific J. Math. **133** (1988), no. 1, 1–12. MR936353 (89d:46055)

[5] John T. Anderson and Alexander J. Izzo, *A peak point theorem for uniform algebras generated by smooth functions on two-manifolds*, Bull. London Math. Soc. **33** (2001), no. 2, 187–195, DOI 10.1112/blms/33.2.187. MR1815422 (2002j:32035)

[6] John T. Anderson and Alexander J. Izzo, *Peak point theorems for uniform algebras on smooth manifolds*, Math. Z. **261** (2009), no. 1, 65–71, DOI 10.1007/s00209-008-0313-x. MR2452637 (2009m:46076)

[7] J. T. Anderson and A. J. Izzo, *A peak point theorem for uniform algebras on real-analytic varieties*, Math. Ann. (to appear)

[8] John T. Anderson, Alexander J. Izzo, and John Wermer, *Polynomial approximation on three-dimensional real-analytic submanifolds of* $\mathbf{C}^n$, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2395–2402, DOI 10.1090/S0002-9939-01-05911-1. MR1823924 (2002d:32021)

[9] Richard F. Basener, *On rationally convex hulls*, Trans. Amer. Math. Soc. **182** (1973), 353–381. MR0379899 (52 #803)
[10] H. S. Bear, *Complex function algebras*, Trans. Amer. Math. Soc. **90** (1959), 383–393. MR0107164 (21 #5889)
[11] Errett Bishop, *A minimal boundary for function algebras*, Pacific J. Math. **9** (1959), 629–642. MR0109305 (22 #191)
[12] A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press, Inc., 1991.
[13] Andrew Browder, *Introduction to function algebras*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0246125 (39 #7431)
[14] Brian James Cole, *ONE-POINT PARTS AND THE PEAK POINT CONJECTURE*, ProQuest LLC, Ann Arbor, MI, 1968. Thesis (Ph.D.)–Yale University. MR2617861
[15] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325 (41 #1976)
[16] Michael Freeman, *Some conditions for uniform approximation on a manifold*, Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 42–60. MR0193538 (33 #1758)
[17] T. W. Gamelin, *Uniform Algebras*, 2nd ed., Chelsea Publishing Company, New York, NY, 1984.
[18] John Garnett, *A topological characterization of Gleason parts*, Pacific J. Math. **20** (1967), 59–63. MR0205107 (34 #4942)
[19] Swarup N. Ghosh, *Isolated point theorems for uniform algebras on manifolds*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–Bowling Green State University. MR3321927
[20] A. Gleason, *Function algebras*, Seminar on Analytic Functions, vol. II, Institute for Advanced Study, Princeton (1957), 213–226.
[21] Victor Guillemin and Alan Pollack, *Differential topology*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. MR0348781 (50 #1276)
[22] J. R. Munkres, *Topology*, 2nd ed., Prentice-Hall of India Private Limited, New Delhi, 2005.
[23] A. G. O'Farrell, K. J. Preskenis, and D. Walsh, *Holomorphic approximation in Lipschitz norms*, Proceedings of the conference on Banach algebras and several complex variables (New Haven, Conn., 1983), Contemp. Math., vol. 32, Amer. Math. Soc., Providence, RI, 1984, pp. 187–194, DOI 10.1090/conm/032/769507. MR769507 (86c:32015)
[24] Arthur Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890. MR0007523 (4,153c)
[25] Nessim Sibony, *Multi-dimensional analytic structure in the spectrum of a uniform algebra*, Spaces of analytic functions (Sem. Functional Anal. and Function Theory, Kristiansand, 1975), Springer, Berlin, 1976, pp. 139–165. Lecture Notes in Math., Vol. 512. MR0632106 (58 #30277)
[26] S. J. Sidney, *Properties of the sequence of closed powers of a maximal ideal in a sup-norm algebra*, Trans. Amer. Math. Soc. **131** (1968), 128–148. MR0222651 (36 #5701)
[27] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, New York, 1971.

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