

ON 2-BRIDGE KNOTS WITH DIFFERING SMOOTH AND TOPOLOGICAL SLICE GENERA

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ABSTRACT. We give infinitely many examples of 2-bridge knots for which the topological and smooth slice genera differ. The smallest of these is the 12-crossing knot $12a255$. These also provide the first known examples of alternating knots for which the smooth and topological genera differ.

1. INTRODUCTION

For a knot $K \subset S^3$, the smooth (resp. topological) slice genus, $g_4(K)$ (resp. $g_4^{\text{top}}(K)$), is defined to be the minimal possible genus of a properly embedded smooth (resp. locally flat) surface in B^4 with boundary K . The slice genera generalize the notion of sliceness in that, by definition, a knot K is smoothly (resp. topologically) slice if and only if $g_4(K)$ (resp. $g_4^{\text{top}}(K)$) is zero.

Since any smoothly embedded surface is also locally flat, we always have the inequality

$$g_4^{\text{top}}(K) \leq g_4(K).$$

It is a well-known phenomenon of knot theory and 4-dimensional topology that this inequality can be strict. For example, the $(-3, 5, 7)$ -pretzel knot is topologically, but not smoothly, slice. The difference between the two genera can even be made arbitrarily large [Rud84, KM93]. However, many of the invariants with which one could hope to prove $g_4^{\text{top}}(K) < g_4(K)$ cannot be applied when K is 2-bridge. For example, when K is alternating, the Rasmussen s -invariant and the Ozsváth and Szabó τ -invariant both give the same lower bound for $g_4(K)$ as the signature $\sigma(K)$, and, hence, also give an identical lower bound for $g_4^{\text{top}}(K)$ [Ras10, OS03]. In fact, conjecturally, the topological and smooth notion of sliceness are the same for 2-bridge knots.

Conjecture 1 ([Mil15, EL09, CG86]). *For all 2-bridge knots K , $g_4(K) = 0$ if and only if $g_4^{\text{top}}(K) = 0$.*

The aim of this paper is to show that for 2-bridge knots the smooth and topological genera are different. In other words, the natural generalization of Conjecture 1 to higher genera is false.

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Let $\mathcal{K}_{m,n}$ be, as depicted in Figure 1, the 2-bridge knot corresponding to the rational number¹

$$[2m + 3, 1, 2n + 4, 1, 1, 2]^+ = \frac{20mn + 56m + 40n + 107}{10n + 28}.$$

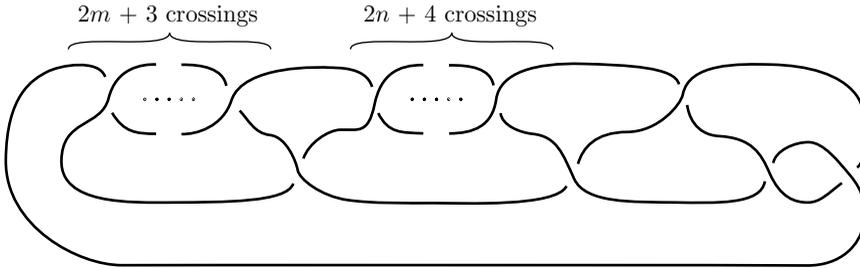


FIGURE 1. The knot $\mathcal{K}_{m,n}$.

The main result of this paper is that for many choices of m and n , the $\mathcal{K}_{m,n}$ give examples of 2-bridge knots with differing topological and slice genera. To the best of the authors’ knowledge, these are also the first known instances of alternating knots for which the smooth and topological genera differ.

Theorem 2. *For all $m, n \geq 0$, the smooth slice genus of $\mathcal{K}_{m,n}$ is $g_4(\mathcal{K}_{m,n}) = 2$. Furthermore, if any of the following hold:*

- (i) $m = n = 0$,
- (ii) $m + 3$ is a perfect square or
- (iii) $n + 2$ is a perfect square,

then $g_4^{\text{top}}(\mathcal{K}_{m,n}) = 1$.

Given that all our examples satisfy $g_4(K) - g_4^{\text{top}}(K) \leq 1$, it seems natural to ask the following.

Question 3. Are there 2-bridge knots for which the difference $g_4(K) - g_4^{\text{top}}(K)$ is arbitrarily large?

Although it seems likely that this has an affirmative answer, proving this would require exhibiting 2-bridge knots for which $2g_4(K) - |\sigma(K)|$ is arbitrarily large. In particular, addressing Question 3 would require an approach to the smooth slice genus differing from the one taken in this paper: the best our method can achieve is to show $2g_4(K) - |\sigma(K)| \neq 0$.

Remark 4. The knot $\mathcal{K}_{0,0}$ appears in the tables as 12a255 [CC16]. With the possible exception of 11a211, for which $g_4(11a211) = 2$, but the authors are unable to determine the topological slice genus, there are no alternating knots with 11 or fewer crossings for which the smooth and topological slice genus differ [McC15].

¹Here the 2-bridge knot corresponding to p/q means the knot whose double branched cover is the lens space $L(p, q)$. By taking a continued fraction $p/q = [a_0, \dots, a_n]^+$ in which all the coefficients are positive, one can obtain an alternating 2-bridge diagram for such a knot in which the a_i count the number of crossings in each twist region.

Remark 5. Consider $\mathcal{K}_{m,n}$ such that $g_4^{\text{top}}(\mathcal{K}_{m,n}) = 1$. Since $\sigma(\mathcal{K}_{m,n}) = -2$, the double branched cover $\Sigma(\mathcal{K}_{m,n})$ is a lens space bounding a negative-definite topological manifold with $b_2 = 2$. However, the proof of Theorem 2 implies that this 4-manifold cannot be smooth. In fact, it follows from the proof of Proposition 10 that if $m > 0$, then $\Sigma(\mathcal{K}_{m,n})$ cannot bound a smooth negative-definite manifold with $b_2 < 4$.

2. PROOF OF THEOREM 2

In this section, we prove the main result of this paper. Since $\mathcal{K}_{m,n}$ possesses a genus two Seifert surface, as shown in Figure 2, and has signature $\sigma(\mathcal{K}_{m,n}) = -2$, we always have

$$1 \leq g_4^{\text{top}}(\mathcal{K}_{m,n}) \leq g_4(\mathcal{K}_{m,n}) \leq 2.$$

The statements concerning the topological genus in Theorem 2 are established by using Freedman’s disk theorem to construct a locally flat genus one surface bounding $\mathcal{K}_{m,n}$. The smooth part of Theorem 2 is established by using Donaldson’s diagonalisation theorem to show $2g_4(\mathcal{K}_{m,n}) \neq |\sigma(\mathcal{K}_{m,n})|$. These steps are carried out in the following two sections.

2.1. The topological genus.

Proposition 6. *If $n = m = 0$, $m + 3$ is a perfect square, or $n + 2$ is a perfect square, then the topological slice genus of $\mathcal{K}_{m,n}$ is 1.*

The strategy of proof for Proposition 6 is to find an essential separating curve L on a minimal genus Seifert surface S for K with trivial Alexander polynomial. This allows us to establish that the topological slice genus is strictly smaller than the genus of S by invoking Freedman’s disk theorem [Fre82].

This idea was first used by Rudolph to show that the $g_4^{\text{top}}(K) < g_4(K)$ for most torus knots [Rud84]. The curve L is found by considering the Seifert form; compare also [Fel15] and [BL15] for this type of argument.

Proof of Proposition 6. We establish Proposition 6 by showing that $\mathcal{K}_{m,n}$ has a genus two Seifert surface S containing a simple closed curve L such that $\Delta_L = 1$ and L separates S into two connected components of genus one. Here, Δ_L denotes the Alexander polynomial of a knot $L \subset S^3$. Indeed, this suffices to show $g_4^{\text{top}}(\mathcal{K}_{m,n}) \leq 1$ by the following argument: removing the connected component C of $S \setminus L$ that does not contain $\mathcal{K}_{m,n}$, one obtains a genus one surface $\overline{S \setminus C} \subset S^3$ with boundary $\mathcal{K}_{m,n} \sqcup L$. By Freedman’s disk theorem [Fre82, Theorem 1.13], L bounds a topological locally flat disk B^2 in B^4 . Gluing $\overline{S \setminus C}$ and B^2 along L yields a locally flat surface in B^4 of genus one with boundary $\mathcal{K}_{m,n} \subset S^3 = \partial B^4$.

We orient $\mathcal{K}_{m,n}$ and choose S to be the genus two Seifert surface for $\mathcal{K}_{m,n}$ depicted in Figure 2.

A Seifert matrix for S is

$$(1) \quad M = \begin{bmatrix} -m-2 & 1 & 0 & 0 \\ 0 & -n-3 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Indeed, M is the matrix with entries

$$m_{ij} = \text{lk}(\delta_i, \delta_j^+),$$

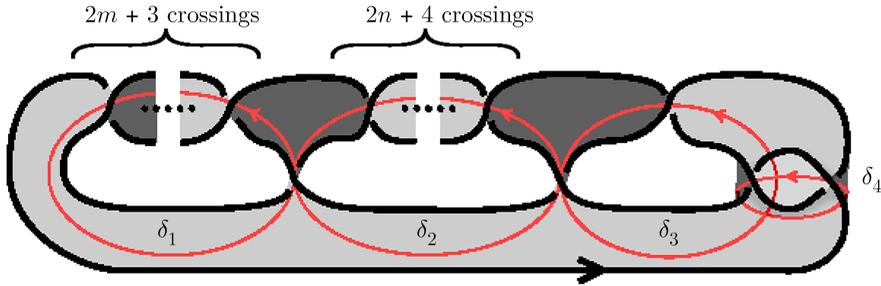


FIGURE 2. A Seifert surface S (gray) for the knot $\mathcal{K}_{m,n}$ (black) with 4 simple closed curves (red) such that their homology classes provide a basis for $H_1(S, \mathbb{Z})$. The orientation of S is chosen such that its boundary induces the indicated orientation of $\mathcal{K}_{m,n}$.

where the δ_i are the curves indicated in Figure 2, δ_j^+ denotes the curve in $S^3 \setminus S$ obtained by a small perturbation of δ_j in the positive normal direction to S , and $\text{lk}(\cdot, \cdot)$ denotes the linking number between disjoint curves in S^3 . In other words, M is the Seifert matrix obtained by writing the Seifert form with respect to the basis of $H_1(S, \mathbb{Z})$ given by $([\delta_1], [\delta_2], [\delta_3], [\delta_4])$. Note that the signature of the symmetrization of M is -2 , which establishes $\sigma(\mathcal{K}_{m,n}) = -2$ as claimed earlier.

Next we pick two elements a and b in $H_1(S, \mathbb{Z})$ such that the Seifert form restricted to the span of a and b is the same as the Seifert form on a genus one Seifert surface of the unknot. We first consider the case where $m + 2$ is a perfect square. The Seifert form on a genus one Seifert surface of the unknot is (for an appropriately chosen basis) given by a Seifert matrix of the form $\begin{bmatrix} 0 & 1 \\ 0 & * \end{bmatrix}$. Therefore, we look for elements $a \in H_1(S, \mathbb{Z})$ such that $\text{lk}(a, a^+) = 0$. Inspection of M reveals that $a = [\delta_1] + \sqrt{m + 2}[\delta_4]$ is such an element. Setting $b = [\delta_2]$, yields

$$(2) \quad \begin{bmatrix} \text{lk}(a, a^+) & \text{lk}(a, b^+) \\ \text{lk}(b, a^+) & \text{lk}(b, b^+) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -n - 3 \end{bmatrix}$$

as wanted. Similar choices for a and b are possible in the cases where $n + 3$ is a perfect square or $n = m = 0$. Concretely, if we choose a and b in $H_1(S, \mathbb{Z})$ as

$$\begin{aligned} a &= [\delta_1], & b &= [\delta_2] + \sqrt{n + 3}[\delta_4] & \text{if } n + 3 \text{ is a square,} & \text{and} \\ a &= [\delta_1] + [\delta_4], & b &= [\delta_1] + [\delta_2] + 2[\delta_4] & \text{if } m = n = 0, \end{aligned}$$

respectively; then a calculation shows that we have

$$(3) \quad \begin{bmatrix} \text{lk}(a, a^+) & \text{lk}(a, b^+) \\ \text{lk}(b, a^+) & \text{lk}(b, b^+) \end{bmatrix} = \begin{bmatrix} * & 1 \\ 0 & 0 \end{bmatrix}.$$

Our choices of a and b ensures that a intersects b once algebraically on S (for this recall that the algebraic intersection form on $H_1(S, \mathbb{Z})$ is equal to the antisymmetrization of the Seifert form). Therefore, the classes a and b are represented by simple closed curves α and β in S that intersect once transversally; see e.g. [FM12, Third proof of Theorem 6.4]. Let C be a closed neighborhood of the union of α and β and let L be the boundary of C . By construction, L separates $S \setminus L$ into two surfaces of genus one. Furthermore, $\Delta_L = 1$ since the surface C

is a Seifert surface for L and, therefore, (2) respectively (3) is a Seifert matrix for L . □

Remark 7. The above proof yields the curve L explicitly. Indeed, the only non-explicit step in the above proof—the realization of the homology classes a and b by once intersecting simple closed curves α and β —can be accomplished by a geometric version of the Euclidian algorithm; compare [FM12, Third proof of Theorem 6.4].

2.2. The smooth genus. Let K be a knot with reduced alternating diagram D . Let us take a chessboard colouring of this diagram such that every crossing has incidence number $\mu = -1$, as according to the convention in Figure 3. If we label

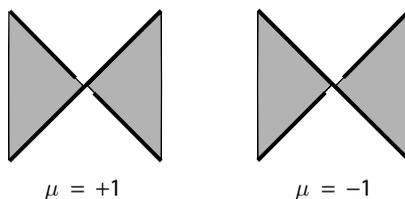


FIGURE 3. The incidence number of a crossing.

the white regions v_1, \dots, v_{r+1} , then we can define the Goeritz form G_D on $\oplus_{i=1}^r v_i \mathbb{Z}$ by

$$v_i \cdot v_j = \begin{cases} \# \text{ crossings around } v_i & \text{if } i = j \text{ and} \\ -\# \text{ crossings between } v_i \text{ and } v_j & \text{if } i \neq j. \end{cases}$$

As D is a reduced alternating diagram, this is positive definite. The signature of K can be computed by the following formula [GL78]:

$$(4) \quad \sigma(K) = \text{rank}(G_D) - n_+,$$

where n_+ denotes the number of positive crossings in D . We will use the following obstruction, obtained by combining the work of Gordon and Litherland with Donaldson’s diagonalisation theorem, to determine $g_4(\mathcal{K}_{m,n})$.

Proposition 8. *Let K be a knot with a reduced alternating diagram D and $\sigma(K) \leq 0$. If $g_4(K) = -\sigma(K)/2$, then there is a lattice embedding*

$$G_D \hookrightarrow \mathbb{Z}^{\text{rank}(G_D) - \sigma(K)}.$$

Proof. Let X be the 4-manifold obtained by taking a double cover of B^4 branched over the (not necessarily orientable) surface given by the shaded regions of D . This has intersection form isomorphic to G_D and boundary $\Sigma(K)$ [GL78]. On the other hand, taking a double cover of B^4 branched over a minimal genus smooth surface with boundary K gives a 4-manifold Y with signature $\sigma(K)$ and $b_2(Y) = 2g_4(K)$ [GL78, KT76]. Consider the smooth closed 4-manifold $Z := X \cup_{\Sigma(K)} -Y$. If $g_4(K) = -\sigma(K)/2$, then Z is positive definite and $b_2(Z) = \text{rank}(G_D) - \sigma(K)$. Since this implies that the intersection form of Z must be diagonalisable [Don87], the inclusion $X \subset Z$ induces the required embedding of lattices. □

Remark 9. A good knot with which to test that Proposition 8 is compatible with the shading conventions in use is the positive clasp knot 5_2 . This has signature -2 according to (4), and $g_4(5_2) = g(5_2) = 1$. It can easily be verified that its positive

Goeritz form, which has rank 3, embeds into \mathbb{Z}^5 , but the positive Goeritz form for its mirror, which has rank 2, does not embed into \mathbb{Z}^4 .

For $\mathcal{K}_{m,n}$, the Goeritz form is isomorphic to $Q_{m,n}$, the lattice generated by vectors $v_1, \dots, v_{2m+2n+8}$, with pairing given by

$$v_i \cdot v_j = \begin{cases} 3 & \text{if } i = j \in \{2m + 3, 2n + 2m + 7, 2n + 2m + 8\}, \\ 2 & \text{if } i = j \notin \{2m + 3, 2n + 2m + 7, 2n + 2m + 8\}, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

In fact, $Q_{m,n}$ is isomorphic to the intersection form of the positive definite plumbing $P_{m,n}$, as shown in Figure 4. In this case, the manifold X arising in the proof of Proposition 8 is in fact diffeomorphic to $P_{m,n}$.

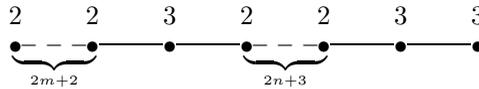


FIGURE 4. The plumbing $P_{m,n}$.

The following proposition completes the proof of Theorem 2.

Proposition 10. *For all $m, n \geq 0$, the smooth genus of $\mathcal{K}_{m,n}$ is $g_4(\mathcal{K}_{m,n}) = 2$.*

Proof. It is easy to verify, either by using (4) or computing directly from the Seifert matrix (1), that $\sigma(\mathcal{K}_{m,n}) = -2$ for all $m, n \geq 0$. Since $\mathcal{K}_{m,n}$ has a genus two Seifert surface, this shows $1 \leq g_4(\mathcal{K}_{m,n}) \leq 2$. Thus the proposition follows from Proposition 8 combined with the following claim.

Claim. Suppose that we have an embedding $Q_{m,n} \hookrightarrow \mathbb{Z}^M$; then

$$M \geq \begin{cases} \text{rank}(Q_{m,n}) + 3 & \text{if } m = 0, \\ \text{rank}(Q_{m,n}) + 4 & \text{if } m \geq 1. \end{cases}$$

Proof of the Claim. Suppose that \mathbb{Z}^M contains $Q_{m,n}$ as a sublattice. Throughout this proof, we will use e_i and f_j to denote unit vectors in \mathbb{Z}^M with the assumption that $e_i \cdot e_j = f_i \cdot f_j = 0$ whenever $i \neq j$.

First consider the vectors of norm 2 in \mathbb{Z}^M given by v_1, \dots, v_{2m+2} . Since v_1 and v_2 both have norm 2 and $v_1 \cdot v_2 = -1$, there must be unit vectors $e_1, e_2, e_3 \in \mathbb{Z}^M$ such that $v_1 = e_1 - e_2$ and $v_2 = e_2 - e_3$. Suppose now that $m \geq 1$. In this case, v_3 also has $\|v_3\|^2 = 2$, and additionally satisfies $v_3 \cdot v_2 = -1$ and $v_3 \cdot v_1 = 0$. Thus we see that either (1) $v_3 = -e_1 - e_2$ or (2) there is $e_4 \in \mathbb{Z}^M$ such that $v_3 = e_3 - e_4$. However, if (1) holds, then any vector x satisfying $x \cdot v_1 = 0$, must also satisfy $x \cdot v_3 = 2x \cdot e_1 \equiv 0 \pmod{2}$, meaning that we obtain a contradiction when we consider v_4 . Thus we must have $v_3 = e_3 - e_4$. Continuing in this way, we see that there are $e_1, \dots, e_{2m+3} \in \mathbb{Z}^M$ such that $v_i = e_i - e_{i+1}$ for $i = 1, \dots, 2n + 2$.

The vectors $v_{2m+4}, \dots, v_{2m+2n+6}$ also form a chain of vectors of norm 2 in \mathbb{Z}^M , so by an identical argument to the preceding paragraph, there must exist $f_1, \dots, f_{2n+4} \in \mathbb{Z}^M$ such that for $i = 1, \dots, 2n + 3$, we have $v_{2m+3+i} = f_i - f_{i+1}$. Furthermore, since $v_i \cdot v_j = 0$ for all $i < 2m + 3$ and $j > 2m + 3$, these must satisfy $e_i \cdot f_j = 0$ for all i and all j .

Now consider $v_{2m+2n+7}$. Since $v_{2m+2n+7} \cdot v_i = 0$ for $i \leq 2m + 2n + 5$ and $v_{2m+2n+7} \cdot v_{2m+2n+6} = -1$, either (1) $n = 0$ and $v_{2m+2n+7} = -(f_1 + f_2 + f_3)$ or (2) $v_{2m+2n+7} = f_{2n+4} + f_{2n+5} + f_{2n+6}$, where f_{2n+5} and f_{2n+6} are orthogonal to the e_i . Since there is no vector $v_{2m+2n+8}$ of norm 3 with $v_{2m+2n+8} \cdot (f_1 + f_2 + f_3) = 1$ and $v_{2m+2n+8} \cdot v_i = 0$ for $i \leq 2m + 2n + 6$, it follows that (2) must always hold and $v_{2m+2n+7} = f_{2n+4} + f_{2n+5} + f_{2n+6}$. By similar considerations, we see that there must be f_{2n+7} and f_{2n+8} which are orthogonal to the e_i and such that $v_{2m+2n+8} = -f_{2n+5} + f_{2n+7} + f_{2n+8}$ or $v_{2m+2n+8} = -f_{2n+6} + f_{2n+7} + f_{2n+8}$.

The set $\{e_1, \dots, e_{2m+3}, f_1, \dots, f_{2n+8}\}$ gives a set of $2n + 2m + 11$ linearly independent unit vectors in \mathbb{Z}^M , showing that $M \geq 2n + 2m + 11 = \text{rank}(Q_{m,n}) + 3$. In fact, if $m = 0$, then $\mathbb{Z}^{2m+2n+11}$ contains a sublattice isomorphic to $Q_{m,n}$ (we can take $v_3 = e_1 + e_2 - f_1$).

It remains only to consider v_{2m+3} when $m \geq 1$. Similarly, we can deduce that v_{2m+3} takes the form $v_{2m+3} = e_{2m+3} + e_{2m+4} - f_1$, where $e_{2m+4} \in \mathbb{Z}^M$ is a unit vector which is orthogonal to the f_i . This shows that if $m \geq 1$, then there are at least $2m + 2n + 12$ linearly independent unit vectors in \mathbb{Z}^M . Thus we can conclude that $M \geq \text{rank}(Q_{m,n}) + 4$, completing the proof of the claim. \square

This completes the proof of the proposition. \square

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