

REMARKS ON THE HOROCYCLE FLOWS FOR FOLIATIONS BY HYPERBOLIC SURFACES

SHIGENORI MATSUMOTO

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ABSTRACT. We show that the horocycle flow associated with a foliation on a compact manifold by hyperbolic surfaces is minimal under certain conditions.

1. INTRODUCTION

In the 1936 paper [3], G. A. Hedlund showed the minimality of the horocycle flows associated to closed oriented hyperbolic surfaces. In [4], the authors consider a problem of generalizing this fact to compact laminations by hyperbolic surfaces.

Throughout this paper, we work under the following assumption.

Assumption 1.1. *M is a closed smooth¹ manifold, and \mathcal{F} a codimension q minimal foliation on M by hyperbolic surfaces*

The latter condition means that \mathcal{F} is a 2-dimensional smooth foliation equipped with a continuous leafwise metric of curvature -1 and that all the leaves of \mathcal{F} are dense in M . Let $\Pi : \hat{M} \rightarrow M$ be the unit tangent bundle of the foliation \mathcal{F} . The total space \hat{M} admits a locally free action of $PSL(2, \mathbb{R})$. Denote the orbit foliation by \mathcal{A} . See Section 2 of [4] for more detail. (The foliation \mathcal{A} is denoted by $T^1\mathcal{F}$ in [4].) In other words, we have the geodesic flow g^t , the stable horocycle flow h_+^t and the unstable horocycle flow h_-^t . They preserve leaves of \mathcal{A} and satisfy

$$(1.1) \quad g^t \circ h_+^s \circ g^{-t} = h_+^{se^{-t}}, \quad g^t \circ h_-^s \circ g^{-t} = h_-^{se^t}.$$

This says that the flow g^t is uniformly hyperbolic along the leaves of \mathcal{A} . The flows g^t and h_\pm^t jointly define an action of a closed subgroup B_\pm of $PSL(2, \mathbb{R})$, whose orbit foliation is denoted by \mathcal{B}_\pm . They are subfoliations of \mathcal{A} transverse to each other in a leaf of \mathcal{A} and the intersection is the orbit foliation of g^t . The group B_\pm is isomorphic to the group of the orientation preserving affine transformations on the real line. There is an involution $J : \hat{M} \rightarrow \hat{M}$ sending a leafwise unit tangent vector $\zeta \in \hat{M}$ to $-\zeta$. The involution J maps the flow g^t to g^{-t} ; $Jg^tJ = g^{-t}$, and the flow h_\pm^t to h_\mp^{-t} ; $Jh_\pm^tJ = h_\mp^{-t}$. Therefore the minimality of \mathcal{B}_+ (resp. h_+^t) is equivalent to the minimality of \mathcal{B}_- (resp. h_-^t). Notice also that the minimality of \mathcal{F} immediately implies the minimality of \mathcal{A} . On the other hand, there are examples of minimal \mathcal{F} for which the foliations \mathcal{B}_+ are not minimal [4]. The purpose of this paper is to study the following question which arises naturally from the result of Hedlund [3].

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¹In this paper, smooth means C^∞ .

Question 1.2. Does the minimality of the foliation \mathcal{B}_+ imply the minimality of the flow h_+^t ?

So far no counterexamples are known. There are some positive partial answers connected with this question. (Below (M, \mathcal{F}) is assumed to satisfy Assumption 1.1.)

- (A) If \mathcal{F} is a homogeneous Lie foliation, then the flow h_+^t is minimal [1].
- (B) If a leaf of \mathcal{F} admits a simple closed geodesic with trivial holonomy, then the flow h_+^t is minimal [2].
- (C) If the foliation \mathcal{B}_+ is minimal and if there are nonplanar leaf in \mathcal{F} , then the flow h_+^t admits a dense orbit [4].

Assume there is a simple closed oriented geodesic c in some leaf of \mathcal{F} . Let $D' \subset D$ be smooth closed q -disks in M transverse to \mathcal{F} such that $D' \cap c = D \cap c = \{z_0\}$. (Recall that \mathcal{F} is of codimension q .) Let $f : D' \rightarrow D$ be the holonomy map of \mathcal{F} along the curve c . Thus $f(z_0) = z_0$. Let us consider the following condition. See Figure 1.

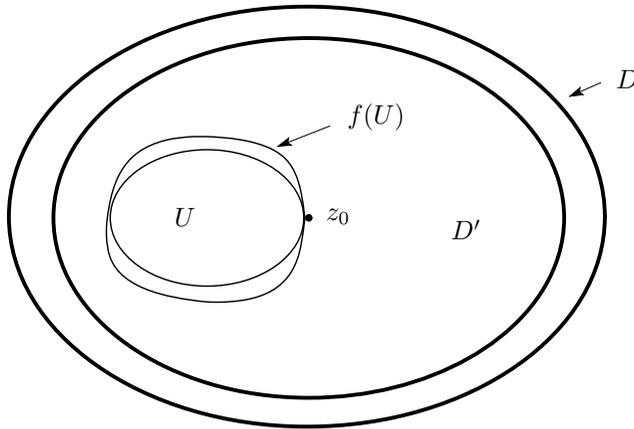


FIGURE 1

(*) There exists an open subset $U \subset D' \subset D$ such that $z_0 \in \text{Cl}(U)$, $f(U) \supseteq U$, and for any $z \in \text{Cl}(U)$, $d(f(z), z_0) \geq d(z, z_0)$.

The main result of this paper is the following.

Theorem 1.3. *If there is a leafwise simple closed geodesic which satisfies (*), then Question 1.2 has a positive answer.*

Notice that any leafwise simple closed geodesic with trivial holonomy satisfies (*). So our result overlaps with the result (B) above. Our proof, quite different from that of [2], uses only the leafwise hyperbolicity (1.1) of the flow g^t . More generally, any leafwise simple closed geodesic of a Riemannian foliation satisfies (*). In fact, one can take as U in (*) a transverse metric ball centered at z_0 . On the other hand, it is shown (Theorem 6, [4]) that if \mathcal{F} admits a holonomy invariant transverse measure, then the foliation \mathcal{B}_+ is minimal. Of course a Riemannian foliation satisfies this condition. Thus we have the following corollary.

Corollary 1.4. *If \mathcal{F} is a Riemannian foliation which admits a nonplanar leaf, then the flow h_{\pm}^t is minimal.*

Unfortunately, this result, even combined with result (A), is not sufficient to solve the following conjecture.

Conjecture 1.5. *For any Riemannian foliation \mathcal{F} , the flow h_{\pm}^t is minimal.*

Next consider the case where \mathcal{F} is of codimension one, that is, $\dim(M) = 3$. We have a satisfactory answer in this case. As will be shown in Section 4, a codimension one foliation \mathcal{F} by hyperbolic surfaces must have a nonplanar leaf. Let c be a leafwise simple closed geodesic. Consider the holonomy map f along c on a transverse open interval I which intersects c at a point z_1 . If the fixed point set $\text{Fix}(f)$ has a nonempty interior, then a point z_0 from the interior satisfies (*). Otherwise an endpoint z_0 of a connected component $I \setminus \text{Fix}(f)$ satisfies (*). That is, if there is a nonplanar leaf, then (*) is always satisfied. See Figure 2.

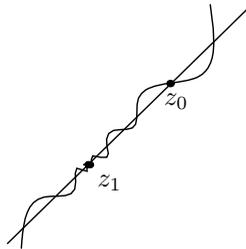


FIGURE 2. Although z_1 does not satisfy (*), some point $z_0 \in I$ satisfies (*).

Therefore we have:

Corollary 1.6. *If \mathcal{F} is of codimension one, then Question 1.2 has an affirmative answer.*

Notice that there are many examples of the codimension one foliations \mathcal{F} for which \mathcal{B}_{\pm} are not minimal [4].

2. PROOF OF THEOREM 1.3

Let c be a leafwise simple closed geodesic in M which satisfies (*). Associated to c , there is a periodic orbit \hat{c} of the leafwise geodesic flow g^t on \hat{M} , defined by $\hat{c} = (c, c')$, where c' stands for the derivative. Our overall strategy is to show that if \mathcal{B}_{+} is minimal, any minimal set \mathcal{M} of h_{+}^t intersects \hat{c} : $\mathcal{M} \cap \hat{c} \neq \emptyset$. We shall divide the proof of Theorem 1.3 into 6 steps.

Step 1. We construct a coordinate neighbourhood in a transversal of \hat{c} for which the dynamics of the first return map of g^t is well described.

For a point z_0 in (*), let $\zeta_0 \in \hat{M}$ be the tangent vector of c at z_0 . Thus $\zeta_0 \in \hat{c}$ and $\Pi(\zeta_0) = z_0$. Let \hat{E} be a smooth closed $(q + 2)$ -disk in the $(q + 3)$ -dimensional manifold \hat{M} transverse to g^t centered at ζ_0 . Let \hat{D} be a q -disk centered at ζ_0 contained in $\text{Int}(\hat{E})$ and transverse to the foliation \mathcal{A} . If \hat{D} is small enough, the projection Π yields a diffeomorphism from \hat{D} to its image $\Pi(\hat{D})$, which is a q -disk in

M transverse to \mathcal{F} . Here notice that the open set U of (*) can be chosen arbitrarily small (close to z_0). In fact, if we replace U by its intersection with the metric disk centered at z_0 of small radius in the transverse q -disk, condition (*) is still satisfied. This shows that the q -disks D and D' in (*) can also be chosen arbitrarily small. Therefore we may assume that $D = \Pi(\hat{D})$ is the disk in (*).

The disk \hat{E} , being transverse to the flow g^t , is transverse to the foliation \mathcal{B}_\pm and \mathcal{A} . Let β_\pm and α be the restriction of the foliation \mathcal{B}_\pm and \mathcal{A} to \hat{E} . See Figure 3.

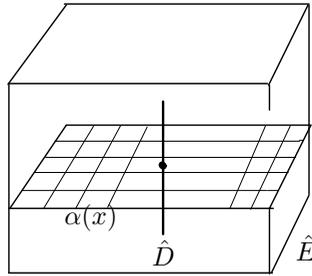


FIGURE 3

The 1-dimensional foliations β_+ and β_- are subfoliations of the 2-dimensional foliation α , transverse to each other in a leaf of α . Given a point $x \in \hat{E}$, let us denote by $\beta_\pm(x)$ and $\alpha(x)$ the leaves of the corresponding foliations which pass through x . Given $\zeta \in \hat{D}$ and small $r > 0$, let $\iota_{\pm, \zeta} : [-r, r] \rightarrow \beta_\pm(\zeta)$ be the isometric embedding such that $\iota_{\pm, \zeta}(0) = \zeta$. For $\xi, \eta \in [-r, r]$ and $\zeta \in \hat{D}$, let us denote by $[\xi, \eta, \zeta]$ the unique point of the intersection of $\beta_+(\iota_{-, \zeta}(\xi))$ and $\beta_-(\iota_{+, \zeta}(\eta))$. See Figure 4.

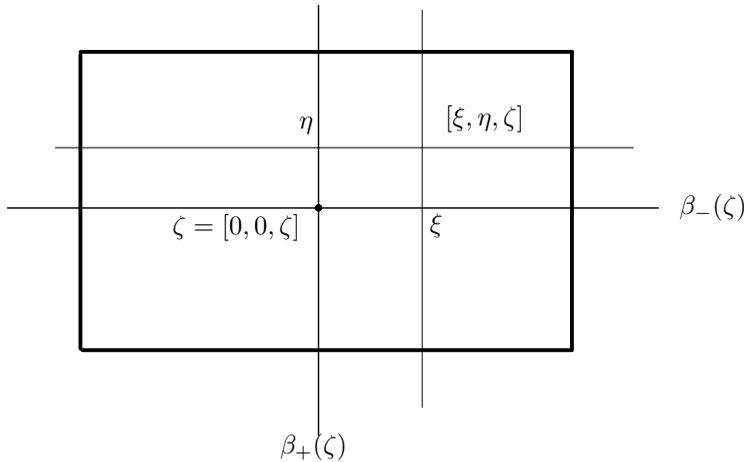


FIGURE 4. For each $\zeta \in \hat{D}$, the set $\{[\xi, \eta, \zeta] \mid |\xi| \leq r, |\eta| \leq r\}$ is a rectangle in $\alpha(\zeta)$.

Recall the open subset U of D' in condition (*). For small $r > 0$ (much smaller than the diameter of U), define

$$U_r = \{z \in U \mid d(z, z_0) < r\}.$$

We have $f(U_r) \supseteq U_r$. Let

$$\hat{U}_r = \Pi^{-1}(U_r) \cap \hat{D} \subset \hat{M}$$

and for $0 < r' \leq r$, define

$$\hat{V}_{r,r'} = \{[\xi, \eta, \zeta] \mid |\xi| \leq r, |\eta| \leq r, \zeta \in \text{Cl}(\hat{U}_{r'})\}.$$

The first return map $F : \hat{V}_{r,r'} \rightarrow \hat{E}$ of the flow g^t preserves the foliations α and β_{\pm} , and therefore can be written as

$$F[\xi, \eta, \zeta] = [\phi_{\zeta}(\xi), \psi_{\zeta}(\eta), f(\zeta)].$$

By some abuse, the conjugate of the map f in (*) by $\Pi|_{\hat{D}} : \hat{D} \rightarrow D$ is denoted here by f . It satisfies $f(\zeta_0) = \zeta_0$, $f(\hat{U}_{r'}) \supseteq \hat{U}_{r'}$ and

$$d(f(\zeta), f(\zeta_0)) \geq d(\zeta, \zeta_0), \quad \forall \zeta \in \text{Cl}(\hat{U}_{r'}),$$

for an appropriate metric d .

Step 2. We shall describe (a part of) the unstable set $K_{r,r'}$ (defined below) of the point ζ_0 with respect to the map F .

By the hyperbolicity (1.1) of the leafwise geodesic flow g^t , there is $\lambda \in (0, 1)$ such that

$$d(\phi_{\zeta}(\xi), \phi_{\zeta}(\xi')) \geq \lambda^{-1}d(\xi, \xi'), \quad \forall \xi, \xi' \in [-r, r], \quad \forall \zeta \in \text{Cl}(\hat{U}_{r'}),$$

$$(2.1) \quad d(\psi_{\zeta}(\eta), \psi_{\zeta}(\eta')) \leq \lambda d(\eta, \eta'), \quad \forall \eta, \eta' \in [-r, r], \quad \forall \zeta \in \text{Cl}(\hat{U}_{r'}).$$

On the other hand, since $\zeta_0 = [0, 0, \zeta_0]$ is a fixed point of F , we have for small $r > 0$ and even smaller $r' = r'(r) > 0$, if $\zeta \in \text{Cl}(\hat{U}_{r'})$,

$$\phi_{f^{-1}(\zeta)}(-r) < -r < r < \phi_{f^{-1}(\zeta)}(r)$$

and

$$(2.2) \quad -r < \psi_{f^{-1}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}(r) < r.$$

Therefore

$$F(\hat{V}_{r,r'}) \cap \hat{V}_{r,r'} = \{[\xi, \eta, \zeta] \mid |\xi| \leq r, \psi_{f^{-1}(\zeta)}(-r) \leq \eta \leq \psi_{f^{-1}(\zeta)}(r), \zeta \in \hat{U}_{r'}\}.$$

Replacing $\zeta \in \text{Cl}(\hat{U}_{r'})$ by $f^{-1}(\zeta) \in \text{Cl}(\hat{U}_{r'})$ in (2.2), we get

$$-r < \psi_{f^{-2}(\zeta)}(-r) < \psi_{f^{-2}(\zeta)}(r) < r.$$

Applying $\psi_{f^{-1}(\zeta)}$, we have

$$\psi_{f^{-1}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}(r) < \psi_{f^{-1}(\zeta)}(r).$$

This way, inductive use of (2.2) shows

$$\begin{aligned} -r &< \psi_{f^{-1}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}\psi_{f^{-3}(\zeta)}(-r) \\ &< \dots < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}\psi_{f^{-3}(\zeta)}(r) < \psi_{f^{-1}(\zeta)}\psi_{f^{-2}(\zeta)}(r) < \psi_{f^{-1}(\zeta)}(r) < r \end{aligned}$$

and by the hyperbolicity (2.1)

$$\lim_{n \rightarrow \infty} \psi_{f^{-1}(\zeta)} \cdots \psi_{f^{-n}(\zeta)}(-r) = \lim_{n \rightarrow \infty} \psi_{f^{-1}(\zeta)} \cdots \psi_{f^{-n}(\zeta)}(r) =: \Psi(\zeta).$$

See Figure 5. Therefore for any $n > 0$,

$$\bigcap_{0 \leq i \leq n} F^i(\hat{V}_{r,r'}) = \{[\xi, \eta, \zeta] \mid |\xi| \leq r,$$

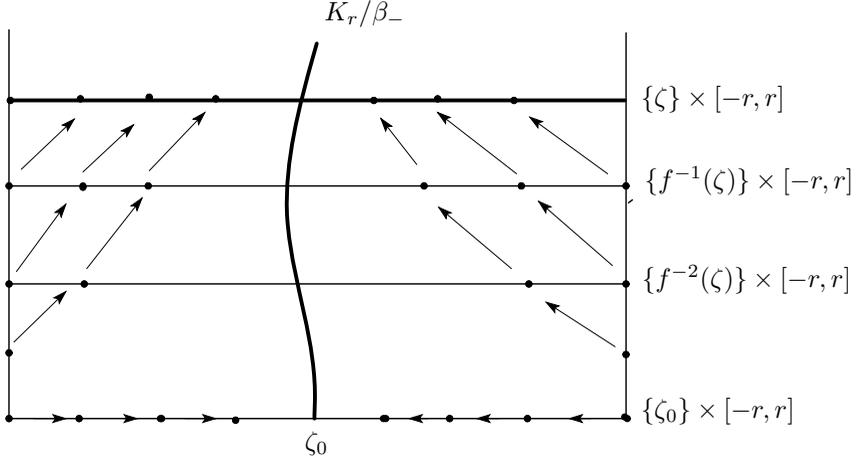


FIGURE 5. The map induced by F on the quotient space \hat{V}_r/β_- .

$$\psi_{f^{-1}(\zeta)} \cdots \psi_{f^{-n}(\zeta)}(-r) \leq \eta \leq \psi_{f^{-1}(\zeta)} \cdots \psi_{f^{-n}(\zeta)}(r), \zeta \in \text{Cl}(\hat{U}_{r'})$$

and

$$K_{r,r'} := \bigcap_{n \geq 0} F^n(\hat{V}_{r,r'}) = \{[\xi, \Psi(\zeta), \zeta] \mid |\xi| \leq r, \zeta \in \text{Cl}(\hat{U}_{r'})\}.$$

The subset $K_{r,r'}$ is closed, and by the closed graph theorem, the function $\Psi : \text{Cl}(\hat{U}_{r'}) \rightarrow [-r, r]$ is continuous. A crucial fact is that if $x \in K_{r,r'}$ and $n > 0$, then $F^{-n}(x) \in \hat{V}_{r,r'}$. (In fact, we have $F^{-n}(x) \in \hat{K}_{r,r'}$. But we do not use this.)

Step 3. Using the above result, we shall describe the dynamics of the geodesic flow g^t near the closed orbit \hat{c} .

Let $\tau : \hat{V}_{r,r'} \rightarrow (0, \infty)$ be the first return time to \hat{E} of the flow g^t and let

$$\check{V}_{r,r'} = \{g^t(x) \mid x \in \hat{V}_{r,r'}, t \in [0, \tau(x)]\} \quad \text{and} \quad \check{K}_{r,r'} = \{g^t(x) \mid x \in K_{r,r'}, t \in [0, \tau(x)]\}.$$

Both are compact sets, and we have:

$$(**) \text{ If } x \in \check{K}_{r,r'} \text{ and } t > 0, \text{ then } g^{-t}(x) \in \check{V}_{r,r'}.$$

Step 4. We assume henceforth that the foliation \mathcal{B}_+ is minimal. Let \mathcal{M} be any minimal set of the flow h_+^t . Then we have

$$(2.3) \quad \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^t(\mathcal{M})} = \hat{M}.$$

This is because the LHS is \mathcal{B}_+ -invariant, closed and nonempty. For the sake of completeness, we include the proof of the \mathcal{B}_+ -invariance, that is, the g^s -invariance and the h_+^s -invariance of the LHS. For the former, we have

$$\begin{aligned} g^s \left(\bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^t(\mathcal{M})} \right) &= \bigcap_{t_0 \in \mathbb{R}} \overline{g^s \bigcup_{t \geq t_0} g^t(\mathcal{M})} = \bigcap_{t_0 \in \mathbb{R}} \overline{g^s \left(\bigcup_{t \geq t_0} g^t(\mathcal{M}) \right)} \\ &= \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^{s+t}(\mathcal{M})} = \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0+s} g^t(\mathcal{M})} = \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^t(\mathcal{M})}. \end{aligned}$$

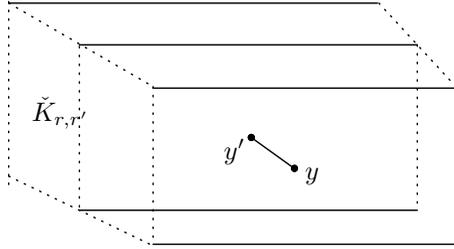


FIGURE 6

For the latter, notice that $g^t(\mathcal{M})$ is invariant by h_+^s and therefore

$$h_+^s\left(\bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^t(\mathcal{M})}\right) = \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} h_+^s(g^t(\mathcal{M}))} = \bigcap_{t_0 \in \mathbb{R}} \overline{\bigcup_{t \geq t_0} g^t(\mathcal{M})}.$$

Step 5. We shall show $\mathcal{M} \cap \hat{c} \neq \emptyset$.

Now (2.3) implies in particular $\overline{\bigcup_{t \geq 0} g^t(\mathcal{M})} = \hat{M}$. Since $\check{V}_{r,r'}$ has nonempty interior, we have $\bigcup_{t \geq 0} g^t(\mathcal{M}) \cap \check{V}_{r,r'} \neq \emptyset$. That is, there is $x \in \mathcal{M}$ and $t \geq 0$ such that $y = g^t(x) \in \check{V}_{r,r'}$. Then an orbit segment of h_+^t through y intersects $\check{K}_{r,r'}$, say at a point y' ; $y' = h_+^s(y) \in \check{K}_{r,r'}$. This is true for any $y \in \hat{V}_{r,r'}$ (Figure 5), and for any $y \in \check{V}_{r,r'}$ by (1.1). See Figure 6. Let $x' = g^{-t}(y')$. Then we have

$$x' = g^{-t}(y') = g^{-t}h_+^s(y) = g^{-t}h_+^s g^t(x) = h_+^{se^t}(x).$$

Hence $x' \in \mathcal{M}$. On the other hand, since $y' \in \check{K}_{r,r'}$, we have $x' \in \check{V}_{r,r'}$ by (**). That is, $\mathcal{M} \cap \check{V}_{r,r'} \neq \emptyset$. Since $\bigcap_{r>0} \check{V}_{r,r'}(r) = \hat{c}$, we get by the finite intersection property that

(***) $\mathcal{M} \cap \hat{c} \neq \emptyset$.

Step 6. We shall finish the proof of Theorem 1.3.

The argument is routine. For each $t \in \mathbb{R}$, we have either $\mathcal{M} \cap g^t(\mathcal{M}) = \emptyset$ or $\mathcal{M} = g^t(\mathcal{M})$ since both sets are minimal sets of h_+^t . Let

$$T = \{t \in \mathbb{R} \mid g^t(\mathcal{M}) = \mathcal{M}\}.$$

Then T is a closed subgroup of \mathbb{R} . The statement (***) shows that T is nontrivial. If $T = \mathbb{R}$, then \mathcal{M} is B_+ -invariant and we have $\mathcal{M} = \hat{M}$, as is required. Consider the remaining case where T is isomorphic to \mathbb{Z} . In this case, the minimal set \mathcal{M} is a global cross section of g^t . But this is impossible by Proposition 5 of [4]. For the sake of completeness, let us give an easy proof for the case where M is a manifold.

The closed graph theorem shows that \mathcal{M} must be a tamely embedded topological submanifold of codimension one. Thus the manifold \hat{M} must be a bundle over S^1 and admits a closed 1-form ω which takes positive value at $\frac{dg^t}{dt}(x)$ for any $x \in \hat{M}$. The closed geodesic c which we started with and the closed geodesic with the reverse orientation correspond to two periodic orbits \hat{c} and \hat{c}' of g^t and we must have $\int_{\hat{c}} \omega > 0$ and $\int_{\hat{c}'} \omega > 0$. However \hat{c}' is homotopic to $-\hat{c}$. A contradiction.

3. CODIMENSION ONE FOLIATIONS

Here we shall prove the following.

Theorem 3.1. *There is no foliation by hyperbolic disks on any closed 3-manifold M .*

Proof. Assume on the contrary that there is a smooth foliation \mathcal{F} by hyperbolic disks on closed 3-manifold M . H. Rosenberg [5] showed that the 3-torus T^3 is the only 3-manifold which admits a smooth foliation by planes. So we have $M = T^3$. According to W. Thurston [7], \mathcal{F} can be isotoped to be transverse to a fibration $S^1 \rightarrow T^3 \rightarrow T^2$. Let $h : \mathbb{Z}^2 = \pi_1(T^2) \rightarrow \text{Diff}_+^\infty(S^1)$ be the holonomy homomorphism of the foliated bundle \mathcal{F} . Then the associated \mathbb{Z}^2 -action on S^1 must be free, since all the leaves of \mathcal{F} are planar. Thus the action is topologically conjugate to an action by rigid rotations. In particular, there is an S^1 -action on S^1 which commutes with $h(\mathbb{Z}^2)$. This implies that there is an \mathcal{F} -preserving topological S^1 -action whose orbit foliation is the above smooth fibration. Consider the covering space $S^1 \times \mathbb{R}^2$ of T^3 , where the lifted foliation is $\{\{t\} \times \mathbb{R}^2\}$, and let $\{\phi^t\}_{t \in S^1}$ be the lifted S^1 -action. Each leaf $\{t\} \times \mathbb{R}^2$ is equipped with a hyperbolic metric. Fix one leaf $\{0\} \times \mathbb{R}^2$ which has a hyperbolic metric g_0 and replace the metric of the other leaf $\{t\} \times \mathbb{R}^2$ by $(\phi^{-t})^*g_0$. Then the new metric is K -quasiconformally equivalent to the old one with fixed constant K . The quotient space of $S^1 \times \mathbb{R}^2$ by the S^1 -action is identified with the Poincaré upper half plane \mathbb{H} . The covering transformation induces a K -quasiconformal action of \mathbb{Z}^2 on \mathbb{H} . Now a theorem of D. Sullivan [6] shows that such an action is topologically conjugate to an action of a subgroup of $PSL(2, \mathbb{R})$. Being a quotient action of a covering transformation, this action must be cocompact, that is, there is a compact subset of \mathbb{H} which intersects each orbit. But this is impossible since the group is \mathbb{Z}^2 , showing Theorem 3.1. \square

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DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, NIHON UNIVERSITY,
1-8-14 KANDA, SURUGADAI, CHIYODA-KU, TOKYO, 101-8308 JAPAN
E-mail address: matsumo@math.cst.nihon-u.ac.jp