

A GEOMETRY WHERE EVERYTHING IS BETTER THAN NICE

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ABSTRACT. We present a geometry in the disk whose metric truth is curiously arithmetic.

1. THE BETTER-THAN-NICE METRIC

Consider the metric

$$g = \frac{4}{1-r^2}(dx^2 + dy^2)$$

in the unit disk. Here $r^2 = x^2 + y^2$ as usual. Everything of interest can be computed explicitly, and with surprising results.

1.1. Hypocycloids in the disk. Consider the curve

$$c(t) = (1-a)e^{i\theta(t)} + ae^{-i\phi(t)}, \quad 0 < a < 1,$$

thought of as a point on a circle of radius a turning at a rate $\dot{\phi}$ in the clockwise direction as the centre of the circle rotates on a circle of radius $1-a$ rotating at a rate $\dot{\theta}$ in the counterclockwise direction.

For the small circle of radius a to roll without slipping on the inside of the circle of radius 1 requires the point $c(t)$ to have velocity 0 when $|c(t)| = 1$, which is the relation

$$a\dot{\phi} = (1-a)\dot{\theta}.$$

Because g is rotationally invariant, without loss of generality we may take

$$\theta(t) = \frac{at}{2\sqrt{a(1-a)}}, \quad \phi(t) = \frac{(1-a)t}{2\sqrt{a(1-a)}}.$$

Theorem 1.1. *The curve $c(t)$ is a geodesic for the metric g , parameterized by arclength.*

Proof. For our g , the equations $\ddot{u}^i + \Gamma_{jk}^i \dot{u}^j \dot{u}^k = 0$ determining geodesics $u(t) = (x(t), y(t))$ parameterized proportional to arclength are

$$(1) \quad \begin{aligned} (1-x^2-y^2)\ddot{x} + x(\dot{x}^2 - \dot{y}^2) + 2y\dot{x}\dot{y} &= 0 \quad \text{and} \\ (1-x^2-y^2)\ddot{y} - y(\dot{x}^2 - \dot{y}^2) + 2x\dot{x}\dot{y} &= 0, \end{aligned}$$

which can be expressed in terms of $z = x + iy$, as the single equation

$$(2) \quad (1 - z\bar{z})\ddot{z} + \bar{z}\dot{z}^2 = 0.$$

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Given the formula for $c(t)$ it is straightforward to verify that $c(t)$ satisfies (2) and moreover that $|\dot{c}| = \sqrt{1 - |c|^2}/2$, from which it follows that $\|\dot{c}\|_g = 1$, meaning that the parameterization is by arclength. \square

Theorem 1.2. *The closed geodesics (i.e. keep rolling the generating circle of the hypocycloid until it closes up) have length $4\pi\sqrt{n}$, and the number of geometrically distinct geodesics of length $4\pi\sqrt{n}$ is given by the arithmetic function $\psi(n)$.*

The function $\psi(n)$ counts the number of different ways that the integer n may be written as a product $n = pq$, with $p \leq q$, $(p, q) = 1$. Values of this function are tabulated in sequence A007875 in the online encyclopedia of integer sequences [1].

Proof. Beginning at $c(0) = 1$, the geodesic $c(t)$ first returns to the boundary circle at

$$c\left(4\pi\sqrt{a(1-a)}\right) = e^{2\pi i(1-a)},$$

returning again at points of the form $e^{2\pi im(1-a)}$ ($m \in \mathbf{Z}_+$). The corresponding succession of cycloidal geodesic arcs winds clockwise around the origin if $0 < a < 1/2$ and counterclockwise if $1/2 < a < 1$; when $a = 1/2$, $c(t)$ traverses back and forth along the x -axis. Thus $c(t)$ forms a once-covered closed geodesic precisely when $2\pi m(1-a) = q2\pi$ for some relatively prime pair of positive integers $q < m$, in which case the geodesic has length

$$4\pi m\sqrt{a(1-a)} = 4\pi\sqrt{(m-q)q}.$$

To count geometrically distinct closed geodesics we restrict to $0 < a \leq 1/2$, in which case $p := m - q = ma \leq m(1-a) = q$. Given any relatively prime positive integers $p \leq q$ the geodesic of length $4\pi\sqrt{pq}$ occurs when $a = p/(p+q)$. \square

1.2. Eigenfunctions and eigenvalues of the Laplacian. Set the Laplacian Δ to be

$$\Delta = -g^{ab}\nabla_a\nabla_b$$

where ∇_a is the covariant derivative operator associated to the metric g via the Levi-Civita connection. Consider the eigenvalue problem

$$\Delta u = \lambda u$$

for functions u with the boundary value $u(r = 1) = 0$. This problem has a number of remarkable features.

Theorem 1.3. *The eigenfunctions and eigenvalues satisfy*

- (1) *The eigenvalues λ_n are precisely the positive integers $n = 1, 2, 3, \dots$*
- (2) *The eigenfunctions are polynomials.*
- (3) *The dimension of the eigenspace for eigenvalue n is the number of divisors of n . (The number of divisors function is denoted by $\tau(n)$.)*

Proof. Since the operator

$$-\Delta + \lambda = \frac{1-r^2}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \lambda$$

is analytic hypoelliptic, distributional solutions to the eigenvalue equation $\Delta u = \lambda u$ are necessarily real analytic, and representable near $(x, y) = (0, 0)$ by absolutely convergent Fourier series

$$u(r, \theta) = \sum_{n \in \mathbf{Z}} a_n(r) e^{in\theta}.$$

Observing that $(-\Delta + \lambda)(a_n(r)e^{in\theta})$ has the form $A_n(r)e^{in\theta}$ for some A_n , it follows that u is an eigenfunction of Δ only if each summand $a_n(r)e^{in\theta}$ is, so it suffices to consider just products of the form

$$u(r, \theta) = f(r)e^{in\theta}.$$

Expressing $-\Delta + \lambda$ in polar coordinates yields

$$(-\Delta + \lambda)(f(r)e^{in\theta}) = \frac{1 - r^2}{4} \left(f''(r) + \frac{1}{r}f'(r) + \left(\frac{4\lambda}{1 - r^2} - \frac{n^2}{r^2} \right) f(r) \right) e^{in\theta}.$$

Therefore $u(r, \theta) = f(r)e^{in\theta}$ is an eigenfunction of Δ only if f satisfies

$$r^2(1 - r^2)f'' + r(1 - r^2)f' + (4\lambda r^2 - n^2(1 - r^2))f = 0.$$

Assume for definiteness that $n \geq 0$ and write $f(r) = r^n g(r^2)$, so that g satisfies the hypergeometric equation

$$r(1 - r)g'' + (c - (a + b + 1)r)g' - abg = 0,$$

where

$$a = \left(n + \sqrt{n^2 + 4\lambda} \right) / 2, \quad b = \left(n - \sqrt{n^2 + 4\lambda} \right) / 2, \quad \text{and } c = n + 1.$$

This has a unique non-singular solution (up to scalar multiplication), the hypergeometric function

$$g(r) = {}_2F_1(a, b; n + 1; r) \quad \text{where} \quad g(1) = \frac{n!}{\Gamma(1 + a)\Gamma(1 + b)},$$

which is zero at $r = 1$ if and only if $b = -m$ for some integer $m \geq 1$. This implies $\lambda = m(m + n)$ is a positive integer, proving part (1) of the theorem. If $n > 0$ there are two corresponding eigenfunctions $u(r, \theta) = r^n g(r^2)e^{\pm in\theta}$, making a total of $\tau(\lambda)$ eigenfunctions for each positive integer eigenvalue λ . That these are linearly independent (part (3)), and that the eigenfunctions are polynomials (part (2)) can be verified using an explicit formula for the eigenfunctions, as follows.

Using the formulation $\Delta = -(1 - z\bar{z})\frac{\partial^2}{\partial\bar{z}\partial z}$, where $z = x + iy$, one can check directly that the Rodrigues-type formula

$$(3) \quad u^{(p,q)}(z) = \frac{(-1)^p}{q(p + q - 1)!} (1 - z\bar{z}) \frac{\partial^{p+q}}{\partial\bar{z}^p \partial z^q} (1 - z\bar{z})^{p+q-1}$$

represents eigenfunctions corresponding to $\lambda = pq$, i.e., $\Delta u^{(p,q)} = pq u^{(p,q)}$. □

1.3. Two corollaries.

Corollary 1.4. *The spectral function is precisely the square of the Riemann zeta function*

$$\sum_n \frac{1}{(\lambda_n)^s} = \sum_n \frac{\tau(n)}{n^s} = (\zeta(s))^2.$$

In a more applied vein, consider a unit radius circular membrane fixed at the boundary (i.e. a drumhead) having constant tensile force per unit length S and radially varying density $\rho(r) = 4S/(1 - r^2)$. Small transverse displacements of the membrane $w(x, y, t)$ are governed by the equation

$$(4) \quad w_{tt} + \Delta w = 0,$$

so squared eigenfrequencies of the membrane correspond to eigenvalues of Δ .

Corollary 1.5. *Unlike the standard vibrating membrane whose eigenfrequencies are proportional to zeros of J_0 , for each eigenfrequency $\omega_n = \sqrt{n}$ of (4), all the higher harmonics $m\omega_n$ ($2 < m \in \mathbf{Z}_+$) are also eigenfrequencies.*

1.4. Acoustic imaging and combinatorics. Supplemented by the sequence of monomials $u^{(p,0)}(z) := z^p$ for $p \geq 0$, the eigenfunctions of Δ are the special functions suited to acoustic imaging of layered media. A layered medium consists of a stack of $n-1$ horizontal slabs between two semi-infinite half spaces, where each slab and half space has constant acoustic impedance. Thus impedance as a function of depth is a step function having jumps at the n interfaces. To image the layers, an impulsive horizontal plane wave is transmitted at time $t = 0$ from a reference plane in the upper half space down toward the stack of horizontal slabs, and the resulting echoes are recorded at the reference plane, producing a function $G(t)$ ($t > 0$) (the boundary Green’s function of the medium). Let L_1 denote the time required for acoustic waves to travel from the reference plane to the first interface and back, with L_2, \dots, L_n denoting two-way travel time within each successive layer. When a wave travels downward toward the j th interface, it is partly reflected back, with amplitude factor R_j , and partly transmitted, with transmission factor $1 - R_j$ ($1 \leq j \leq n$).

Theorem 1.6.

$$G(t) = \sum_{k \in \{1\} \times \mathbf{Z}_+^{n-1}} \left(\prod_{j=1}^n u^{(k_j, k_{j+1})}(R_j) \right) \delta(t - \langle L, k \rangle).$$

Here $L = (L_1, \dots, L_n)$, $u^{(0,q)} \equiv 0$ if $q \geq 1$ and $k_j = 0$ if $j > n$.

Proof. Expanding the binomial $(1 - z\bar{z})^{p+q-1}$ in the formula (3), and then applying the derivative $\partial^{p+q}/\partial \bar{z}^p \partial z^q$, yields

$$u^{(p,q)}(z) = \frac{(-1)^{q+\nu+1}}{q} (1 - z\bar{z}) z^{m+\nu-q+1} \bar{z}^{m+\nu-p+1} \sum_{j=0}^{\nu} (-1)^j \frac{(j + \nu + m + 1)!}{j!(j + m)!(\nu - j)!} (z\bar{z})^j,$$

where $m = |p - q|$ and $\nu = \min\{p, q\} - 1$. Switching to polar form $z = re^{i\theta}$, it follows that

$$u^{(p,q)}(re^{i\theta}) = e^{i(p-q)\theta} \frac{(-1)^{q+\nu+1}}{q} (1 - r^2) r^m \sum_{j=0}^{\nu} (-1)^j \frac{(j + \nu + m + 1)!}{j!(j + m)!(\nu - j)!} r^{2j}.$$

For $\theta = 0, \pi$, the latter coincide with the functions $f^{(p,q)}$ occurring in [2, Thm. 2.4, 4.3]. □

Each term $\left(\prod_{j=1}^n u^{(k_j, k_{j+1})}(R_j) \right) \delta(t - \langle L, k \rangle)$ corresponds to the set of all scattering sequences that have a common arrival time $t_i = \langle L, k \rangle$, with each individual scattering sequence weighted according to the corresponding succession of reflections and transmissions. A scattering sequence may be represented by a Dyck path as in Figure 1. The tensor products $\prod_{j=1}^n u^{(k_j, k_{j+1})}(R_j)$ thus have a combinatorial interpretation in that they count weighted Dyck paths having $2k_j$ edges at height j . See [2].

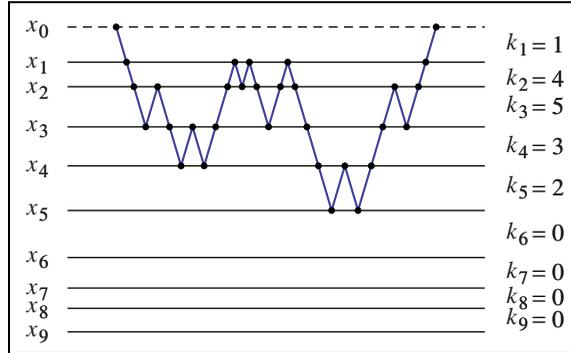


FIGURE 1. The Dyck path for a scattering sequence. Time increases to the right. Each node at depth x_j receives a weight according to the structure of the path at the node: weights $R_j, -R_j, 1 - R_j, 1 + R_j$ correspond respectively to down-up reflection, up-down reflection, downward transmission, upward transmission. The scattering sequence returns to the reference depth x_0 at time $t_i = \langle L, k \rangle$, where $k = (k_1, \dots, k_n)$. Its amplitude is the product of the weights.

REFERENCES

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