PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 2, February 2017, Pages 853–860
http://dx.doi.org/10.1090/proc/13237
Article electronically published on July 28, 2016

A SYMMETRY RESULT FOR AN ELLIPTIC PROBLEM ARISING FROM THE 2-D THIN FILM EQUATION

KA-LUEN CHEUNG AND KAI-SENG CHOU

(Communicated by Joachim Krieger)

ABSTRACT. It is shown that every positive, stable H_0^2-solution to $\Delta u+f(u)=c$ in $\mathbb{R}^2$ is radially symmetric. This problem arises from the study of the steady states for the two dimensional thin film equation.

1. STATEMENT OF RESULT

Let Ω be a bounded domain in $\mathbb{R}^2$ and $f \in C([0,\infty))$. In this note we are concerned with the symmetry of positive functions which satisfy the equation

$$\Delta u + f(u) = c \ , \tag{1}$$

for some constant c in Ω.

Here the elliptic equation admits a variational structure. Under suitable conditions on f, it is the critical point of the energy

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u) \ ,$$

where F is the primitive of f with $F(0)=0$ in the set

$$\mathcal{A} = \left\{ u \in H_0^1(\Omega) : \ \int_\Omega u = A, \ u \geq 0 \ \right\} \ ,$$

where $A>0$ is fixed. Indeed, any positive, continuous critical point u of the energy in $\mathcal{A}$ satisfies

$$\int_\Omega (\nabla u \cdot \nabla \varphi - f(u)\varphi) = 0, \quad \forall \varphi \in H_0^1(\Omega), \int_\Omega \varphi = 0 \ ,$$

so there is some constant c such that $\Delta u + f(u) = c$. A critical point u in $\mathcal{A}$ is called linearly stable if

$$\int |\nabla \varphi|^2 - \int f'(u)\varphi^2 \geq 0, \quad \forall \varphi \in H_0^1(\Omega), \int_\Omega \varphi = 0 \ ,$$

holds. A solution of (1) is called a linearly stable solution if it is a linearly stable critical point of the associated energy.

In this paper we will prove

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and f in $C^1([0,\infty)) \cap C^2((0,\infty))$. Every positive, linearly stable solution of (1) in $H_0^2(\Omega)$ is radially symmetric and decreasing.*

Received by the editors January 23, 2016 and, in revised form, April 9, 2016.
2010 *Mathematics Subject Classification.* Primary 76A20; Secondary 35B35, 35K55.

Theorem 1 bears some resemblance to the symmetry result in [24]. In this celebrated work C^2-regularity of the domain and the solution being in $C^2(\overline{\Omega})$ are assumed. Recently, in [1], a radially symmetry result was obtained without the regularity of the boundary based on continuous Steiner symmetrization, where it is required that the modulus of the gradient of the solution is close to being radially symmetric near the boundary in a certain way. Here we work on a bounded domain in $\mathbb{R}^2$ without any regularity assumption, and, to compensate, the energy stability condition is used in an essential way. As linearly stable solutions are very often the preferred ones in application, the study of a radially symmetry condition for these solutions may be a direction for further pursuit. It is interesting to investigate whether Theorem 1 still holds without this condition. We should point out that the method of moving planes, as developed in [16] (see more in [14]), is an effective approach to radial symmetry for solutions of elliptic equations. However, as some convexity property of the domain is usually needed for the arguments, it cannot be applied directly to the present situation. In [17], some interesting elliptic problems in $\mathbb{R}^2$ with overdetermined boundary conditions are studied by complex function theory where it is shown that the domain must be an ellipse.

Our symmetry result was motivated by the study of the steady states of the thin film equation. A positive solution of (1) which belongs to $H_0^2(\Omega)$ may be called a droplet with zero contact angle. Thus our result asserts that all 2-D energy stable droplets with zero contact angle are radially symmetric. We will discuss its background and the difference between linear stability and energy stability in Section 3 after the proof of this theorem.

2. Proof of the theorem

Lemma 2. *Let Ω be a bounded domain and $u \in H_0^1(\Omega) \cap C^2(\Omega)$. Let D be a connected domain of $\{x \in \Omega : \ u(x) \neq 0\}$. The function*

$$U(x) = \begin{cases} u(x), & x \in D, \\ 0\ , & x \in \Omega \setminus D, \end{cases}$$

belongs to $H_0^1(\Omega)$.

Proof. Let D be a connected component of the non-zero set of u on which u is positive and $D_\varepsilon = \{x \in D : \ u(x) > \varepsilon\}$. The proof is the same when u is negative on D. By Sard's theorem, we can find $\{\varepsilon_k\}, \varepsilon_k \to 0$, such that each D_{ε_k} is a domain with C^1-boundary. We define

$$U_k(x) = \begin{cases} u(x) - \varepsilon_k\ , & x \in D_{\varepsilon_k}, \\ 0\ , & x \in \Omega \setminus D_{\varepsilon_k}. \end{cases}$$

Then $U_k \in H_0^1(\Omega)$. (In fact, it is compactly supported in D.) We have

$$\begin{aligned} \int_\Omega |\nabla U_k|^2 &= \int_{D_{\varepsilon_k}} |\nabla(u - \varepsilon_k)|^2 \\ &\leq \int_\Omega |\nabla u|^2\ , \end{aligned}$$

for all k. So $\{U_k\}$ is uniformly bounded in $H_0^1(\Omega)$. By passing to a subsequence if necessary, we may assume $\{U_k\}$ converges weakly to some function $w \in H_0^1(\Omega)$.

Using Rellich's theorem, we may further assume that $\{U_k\}$ converges to w in $L^2(\Omega)$. Since U_k converges pointwisely to U, w is equal to U, so $U \in H_0^1(\Omega)$ as asserted. $\square$

Next, let V be a non-zero function in $C(\overline{\Omega}) \cap C^1(\Omega)$. We consider the minimization problem

$$\mu_0 = \inf \left\{ \frac{\displaystyle\int_\Omega |\nabla\varphi|^2}{\displaystyle\int_\Omega V\varphi^2} :\ \varphi \in H_0^1(\Omega),\ \varphi \neq 0, \int_\Omega \varphi = 0 \right\}. \tag{2}$$

Any critical point η of this problem satisfies

$$\int_\Omega (\nabla\eta \cdot \nabla\varphi - \mu_0 V\eta\varphi) = 0, \quad \forall \varphi \in H_0^1(\Omega),\ \int_\Omega \varphi = 0\ ,$$

so there is some constant c such that

$$\Delta\eta + \mu_0 V\eta = c\ ,$$

in Ω. By elliptic regularity, η belongs to $C^{2,\alpha}(\Omega)$ for all $\alpha \in (0,1)$.

Lemma 3. *Consider* (2) *where* Ω *is a bounded domain and* $V \in C(\overline{\Omega}) \cap C^1(\Omega)$. *The non-zero set of any minimizer of this problem with* $c = 0$ *must have exactly two connected components.*

Proof. Let η be such a minimizer. Since it has zero mean, it must change sign in Ω. Let us assume that there are three connected components in $\{\eta \neq 0\}$, namely, D_1, D_2 and D_3 and try to draw a contradiction. For $i = 1, 2$, define

$$\eta_i(x) = \begin{cases} \eta(x), & x \in D_i, \\ 0, & x \in \Omega \setminus D_i. \end{cases}$$

By Lemma 2, $\eta_i \in H_0^1(\Omega)$. Now consider $\psi \equiv \eta_1 + \gamma\eta_2 \in H_0^1(\Omega)$ where γ is chosen so that ψ has zero mean. We multiply $\Delta\eta_i + \mu_0\eta_i = 0$ in D_i with $\eta_i, i = 2$, and then integrate to get

$$\int_\Omega |\nabla\psi|^2 = \mu_0 \int_\Omega V\psi^2\ .$$

The vanishing of this expression implies that ψ is a constant. As it is of zero mean, it means that ψ is the zero function, which is impossible. Therefore, this expression cannot be equal to 0. We can write it as

$$\mu_0 = \frac{\displaystyle\int_\Omega |\nabla\psi|^2}{\displaystyle\int_\Omega V\psi^2},$$

so ψ is again a minimizer. Consequently it satisfies $\Delta\psi + \mu_0 V\psi = c_1$ for some constant c_1. (In fact, $c_1 = 0$, but we do not need this fact.) However, as it vanishes identically in D_3, by Carleman's unique continuation theorem ([8]) it must vanish everywhere in Ω, and the contradiction holds. We conclude that η has exactly two components. $\square$

Proof of Theorem 1. Let u be a positive, energy solution of (1) in $H_0^2(\Omega)$. By Sobolev embedding and elliptic regularity, $u \in C_0^\alpha(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ for all $\alpha \in (0,1)$. Let $(x_0, y_0) \in \Omega$ be a maximum of u in Ω and consider the function $w = (x - x_0)u_y(x,y) - (y - y_0)u_x(x,y)$. Since $u \in H_0^2(\Omega)$, w belongs to $H_0^1(\Omega)$. We claim that it has zero mean. For, let $\{u_n\} \in C_c^\infty(\Omega)$, $u_n \geq 0$, which tends to u in the H^2-norm. By Rellich's compactness theorem, we may also assume the convergence is uniform in every compact subset of Ω. Let $\Omega_n = \{x \in \Omega : \ u_n(x) > 0\}$. For each fixed n, let $\varepsilon_j \to 0$ be such that $\Omega_n^j = \{x \in \Omega_n : \ u_n(x) > \varepsilon_j\}$ has C^1-boundary. By Green's formula

$$\begin{aligned} & \int_{\Omega_n^j} ((x-x_0)u_{ny} - (y-y_0)u_{nx}) dxdy \\ = & -\int_{\partial\Omega_n^j} (x-x_0)(u_n - \varepsilon_j)dx + (y-y_0)(u_n-\varepsilon_j)dy \\ = & \ 0 \ . \end{aligned}$$

By letting $\varepsilon_j \to 0$, we have

$$\int_{\Omega_n} ((x-x_0)u_{ny} - (y-y_0)u_{nx})\, dxdy = 0 \ .$$

Using the fact that $\{u_n\}$ tends to u uniformly on every compact set in Ω, we see that $\lim_{n\to\infty} |\Omega \setminus \Omega_n| = 0$. Therefore,

$$\begin{aligned} \left| \int_{\Omega\setminus\Omega_n} ((x-x_0)u_{ny} - (y-y_0)u_{nx})\, dxdy \right| & \leq |\Omega \setminus \Omega_n|^{1/2} \left(\|u\|_{H^1(\Omega)} + 1 \right) \\ & \to 0 \ , \quad \text{as } n \to \infty \ . \end{aligned}$$

It follows that

$$\begin{aligned} & \int_\Omega w dxdy \\ = & \lim_{n\to\infty} \int_\Omega ((x-x_0)u_{ny} - (y-y_0)u_{nx})\, dxdy \\ = & \lim_{n\to\infty} \int_{\Omega\setminus\Omega_n} ((x-x_0)u_{ny} - (y-y_0)u_{nx})\, dxdy \\ & + \lim_{n\to\infty} \int_{\Omega_n} ((x-x_0)u_{ny} - (y-y_0)u_{nx})\, dxdy \\ = & \ 0. \end{aligned}$$

On the other hand, by a direct computation, $\triangle w + f'(u)w = 0$ in Ω. Multiplying this equation with w and then integrating over Ω yields

$$\int_\Omega |\nabla w|^2 = \int_\Omega f'(u)w^2. \tag{3}$$

Assume that w is non-zero. We have $w = \nabla w = 0$ at (x_0, y_0), that is, (x_0, y_0) is a degenerate critical point of w. By a general result on the nodal set of the solution of an elliptic equation (see, for instance, [4]), there exists a homogeneous harmonic polynomial p_N of degree $N, N \geq 2$, such that

$$w(x,y) = p_N(x,y) + O((x^2+y^2)^{(2+\varepsilon)/2}), \quad \varepsilon \in (0,1) \ ,$$

where O depends on N. Every p_N is a linear combination of the real and imaginary parts of z^N. It can be expressed in polar coordinates as a non-zero multiple of $r^N \cos(N\theta + \theta_0)$. After a rotation of axes we may assume $\theta_0 = 0$. The nodal set of p_N divides the plane into $2N$ many connected components. For simplicity, in the following we take $N = 2$. The proof of the other cases is completely analogous. When N is equal to two, we have $p(x, y) = ar^2 \cos 2\theta = a(x^2 - y^2)$ for some non-zero a. Let us take $a = 1$. We fix a small $r_0 > 0$ such that

$$\left|w(x, y) - (x^2 - y^2)\right| \leq \frac{1}{2}(x^2 + y^2), \quad x^2 + y^2 \leq r_0^2 \ .$$

Then

$$w(x, 0) \geq \frac{1}{2}x^2, \quad x \in (0, r_0] \cup [-r_0, 0) \ .$$

Similarly, we have

$$w(0, y) \leq -\frac{1}{2}y^2, \quad y \in (0, r_0] \cup [-r_0, 0) \ .$$

In particular, w is positive at the points $(0, r_0), (-r_0, 0)$ and negative at $(0, r_0)$, $(0, -r_0)$. If these four points belong to different components, the non-zero set of w has at least four components and we are done. Let us assume that two of these points, $(r_0, 0)$ and $(-r_0, 0)$, say, belong to the same component C. By path-connectedness, $(r_0, 0)$ to $(-r_0, 0)$ can be connected by a simple curve γ_1 inside the open set C. We can make it into a simple closed curve γ by adjoining the line segment connecting the points $(r_0, 0)$ and $(-r_0, 0)$ to it. The curve γ now divides the plane into two components, and it is clear that the two points $(0, r_0)$ and $(0, -r_0)$ are lying on each one of these components. As a result, $(0, r_0)$ and $(0, -r_0)$ cannot belong to the same component of the non-zero set of w. We conclude that there are at least three components for the non-zero set of w.

Taking $V = f'(u)$ in Lemma 3, as we have just shown that there are at least three connected components in the non-zero set of w, w cannot be a minimizer of (1). In view of (2) and (3), one must have $\mu_0 < 1$. As a result, there exists some $\varphi_1 \in C_c^\infty(\Omega)$ with zero mean such that

$$\int |\nabla \varphi_1|^2 - \int f'(u)\varphi_1^2 < 0.$$

We have shown that u is not stable if w is non-zero. It follows that w must vanish identically, i.e., Ω is a disk with center (x_0, y_0) and u is radially symmetric on this disk. It is also radially decreasing by the general result in [16]. The proof of Theorem 1 is completed.

3. Droplets of the thin film equation

Let $f \in C(\mathbb{R})$. The fourth order parabolic equation

$$u_t + \operatorname{div}\left[m(u)\nabla\left(\Delta u + f(u)\right)\right] = 0 \tag{4}$$

has been studied extensively in recent years. When $m(u)$ is a positive constant, this equation is the Cahn-Hilliard equation which describes the spinodal decomposition of binary alloys ([13]). Let Ω_0 be a bounded domain. Under suitable conditions on f and some Neumann-type boundary conditions, a global solution of the Cahn-Hilliard equation defines a flow in the subset of the Sobolev space $H^1(\Omega_0)$ given by

$$\mathcal{S} = \left\{ u \in H^1(\Omega_0) : \int_{\Omega_0} u(x,t)dx = \int_{\Omega_0} u(x,0)dx \right\} ,$$

which decreases the energy defined in Section 1. Its steady states are solutions to (1) in Ω_0.

In physical models such as the lubrication approximation to thin films ([23], [22]), or Hele-Show flows, people are interested in non-negative solutions. Unlike the Cahn-Hilliard equation, now the diffusion coefficient $m(z)$ degenerates. Typically it is a positive function on $(0,\infty)$ which is equal to 0 at $z = 0$. A common choice in the thin film equation is $m(z) = z^3$. As a result, even starting with a positive initial data, the solution may touch down at 0 in finite time. Nevertheless, non-negative, weak global solutions have been constructed in many cases for the one dimensional thin film equation ([3], [2], [5], [6]). Although these solutions still constitute flows in $\mathcal{S}$, due to the possibility of touch-down, the steady state is defined as a non-negative function in $H^1(\Omega_0)$ which satisfies (1) in each connected component of $\{u > 0\}$ where the constant c may vary on the component. Steady states not only describe the ultimate pattern of (4) but also are important in the study of the dynamical behavior of the flow. As seen from linearization, the constant states will lose their stability in the so-called long wave situation. The classification and stability of the steady states for the 1-D thin film equation have been studied in [18]–[21], [9], [11]. The notion of energy stability has turned out to be useful in these studies.

Let us recall the definition of energy stability. A critical point u in $\mathcal{A}$ is called energy stable if for every $\varphi \in C_c^1(\Omega)$ with zero mean, there is some $\varepsilon_0 > 0$ such that

$$\mathcal{E}(u) \le \mathcal{E}(u + \varepsilon\varphi) , \quad \forall \varepsilon \in [0, \varepsilon_0] .$$

Note that when u is positive and continuous, $u + \varepsilon\varphi$ belongs to $\mathcal{A}$ for all sufficiently small ε. It is evident that a local minimizer of the energy in $\mathcal{A}$ (regarded as a subset of $H_0^1(\Omega)$) is energy stable. However, as ε_0 depends on φ, an energy stable critical point may not be a local minimizer. On the other hand, by Taylor's expansion one sees that every positive, continuous, energy stable critical point is linearly stable as described in Section 1.

A steady state is called a droplet if its non-zero set is a subdomain of Ω_0, that is, its non-zero set has exactly one component. In the one dimensional case a droplet is a positive solution to $u_{xx} + f(u) = c$ on some (a,b) . It is easy to see that u_x exists at the endpoints a and b. When $u_x(a) = u_x(b) = 0$, u is called a droplet with zero contact angle. In [9] it is shown that in many situations droplets with zero contact angle are energy stable but other droplets are not energy stable.

The two dimensional thin film equation has also been studied recently. Non-negative global weak solutions were constructed in [12], [7]. However, there are very few results concerning its steady states. As an analog of the one dimensional droplet, we define a droplet in this case to be a non-negative function $u \in H^2(\Omega_0)$ such that $\Omega = \{x \in \Omega_0 : \ u(x) > 0\}$ is a subdomain compactly contained in Ω_0 on which (1) is satisfied. Note that u is continuous by Sobolev embedding. The following result shows that the extra regularity assumption requiring u not only in $H^1(\Omega_0)$ but also in $H^2(\Omega_0)$ is not too artificial.

Proposition. *Let $u(.\ ,t)$ be a classical, positive solution of* (4) *under the boundary conditions*

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\left(m(u)\nabla(\Delta u + f(u))\right) = 0 \qquad \text{on } \partial\Omega_0\ ,$$

which is uniformly bounded in $H^1(\Omega_0)$. There exists $\{t_j\}$, $t_j \to \infty$, such that $\{u(.\ ,t_j)\}$ converges weakly to some steady state in $H^2(\Omega_0)$.

A proof of this proposition in the one dimensional case, based on an entropy estimate ([3]), could be found in [9]. It works in the two dimensional case after some minor changes.

When $f \in C^2((0,\infty))$, the droplet u defined above belongs to $C^3(\Omega)$. By observing the fact that $\min\{u-\delta, 0\} \in H_0^1(\Omega)$ (δ is a regular value of u) and then letting $\delta \to 0$, one sees that every droplet belongs to $H_0^1(\Omega)$. Our definition of a droplet with zero contact angle is natural as it requires not only u but also ∇u vanish at the boundary of Ω in a certain sense. Put in this context, Theorem 1 asserts that for any non-radially symmetric droplet with zero contact angle, there must be some admissible perturbations decreasing its energy. Therefore, only a radially symmetric droplet with zero contact angle could be energy stable. Examples of energy stable droplets with zero contact angle can be found in [10]. In fact, in the model studied in this work, these droplets are global minimizers of the energy.

Acknowledgements

The first author was partially supported by the HKIEd Internal Research Grants RG60/2010-11 and MIT/SRG02/15-16. The second author was partially supported by an Earmarked Grant for Research, Hong Kong: CUHK 14301014. We are grateful to the referee for suggestions and the careful reading of the manuscript.

References

[1] F. Brock, *Symmetry for a general class of overdetermined elliptic problems*, arviv.org /pdf/1512.05126.pdf.

[2] Elena Beretta, Michiel Bertsch, and Roberta Dal Passo, *Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation*, Arch. Rational Mech. Anal. **129** (1995), no. 2, 175–200, DOI 10.1007/BF00379920. MR1328475

[3] Francisco Bernis and Avner Friedman, *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations **83** (1990), no. 1, 179–206, DOI 10.1016/0022-0396(90)90074-Y. MR1031383

[4] Lipman Bers, *Local behavior of solutions of general linear elliptic equations*, Comm. Pure Appl. Math. **8** (1955), 473–496. MR0075416

[5] A. L. Bertozzi and M. Pugh, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Comm. Pure Appl. Math. **49** (1996), no. 2, 85–123, DOI 10.1002/(SICI)1097-0312(199602)49:2⟨85::AID-CPA1⟩3.3.CO;2-V. MR1371925

[6] A. L. Bertozzi and M. C. Pugh, *Long-wave instabilities and saturation in thin film equations*, Comm. Pure Appl. Math. **51** (1998), no. 6, 625–661, DOI 10.1002/(SICI)1097-0312(199806)51:6⟨625::AID-CPA3⟩3.3.CO;2-2. MR1611136

[7] Michiel Bertsch, Roberta Dal Passo, Harald Garcke, and Günther Grün, *The thin viscous flow equation in higher space dimensions*, Adv. Differential Equations **3** (1998), no. 3, 417–440. MR1751951

[8] T. Carleman, *Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes* (French), Ark. Mat., Astr. Fys. **26** (1939), no. 17, 9. MR0000334

[9] Ka-Luen Cheung and Kai-Seng Chou, *On the stability of single and multiple droplets for equations of thin film type*, Nonlinearity **23** (2010), no. 12, 3003–3028, DOI 10.1088/0951-7715/23/12/002. MR2734502

[10] K. L. Cheung and K. S. Chou, *Energy stability of droplets and dry spots in a thin film model of hanging drops*, preprint 2016.

[11] Kai-Seng Chou and Zhenyu Zhang, *A mountain pass scenario and heteroclinic orbits for thin-film type equations*, Nonlinearity **25** (2012), no. 12, 3343–3388, DOI 10.1088/0951-7715/25/12/3343. MR2997698

[12] Roberta Dal Passo, Harald Garcke, and Günther Grün, *On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions*, SIAM J. Math. Anal. **29** (1998), no. 2, 321–342 (electronic), DOI 10.1137/S0036141096306170. MR1616558

[13] Charles M. Elliott and Zheng Songmu, *On the Cahn-Hilliard equation*, Arch. Rational Mech. Anal. **96** (1986), no. 4, 339–357, DOI 10.1007/BF00251803. MR855754

[14] L. E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge Tracts in Mathematics, vol. 128, Cambridge University Press, Cambridge, 2000. MR1751289

[15] M. Fermigier, L. Limat, J. E. Wesfreid, P. Boudinet, and C. Quilliet, *Two-dimensional patterns in Rayleigh-Taylor instability of a thin layer*, J. Fluid Mech. **236** (1992), 349-383.

[16] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR544879

[17] Antoine Henrot and Gérard A. Philippin, *Some overdetermined boundary value problems with elliptical free boundaries*, SIAM J. Math. Anal. **29** (1998), no. 2, 309–320 (electronic), DOI 10.1137/S0036141096307217. MR1616562

[18] R. S. Laugesen and M. C. Pugh, *Properties of steady states for thin film equations*, European J. Appl. Math. **11** (2000), no. 3, 293–351, DOI 10.1017/S0956792599003794. MR1844589

[19] R. S. Laugesen and M. C. Pugh, *Linear stability of steady states for thin film and Cahn-Hilliard type equations*, Arch. Ration. Mech. Anal. **154** (2000), no. 1, 3–51, DOI 10.1007/PL00004234. MR1778120

[20] R. S. Laugesen and M. C. Pugh, *Energy levels of steady states for thin-film-type equations*, J. Differential Equations **182** (2002), no. 2, 377–415, DOI 10.1006/jdeq.2001.4108. MR1900328

[21] Richard S. Laugesen and Mary C. Pugh, *Heteroclinic orbits, mobility parameters and stability for thin film type equations*, Electron. J. Differential Equations (2002), No. 95, 29. MR1938391

[22] T. G. Myers, *Thin films with high surface tension*, SIAM Rev. **40** (1998), no. 3, 441–462, DOI 10.1137/S003614459529284X. MR1642807

[23] A. Oron, S. H. Davis, and S. G. Bankoff, *Long–scale evolution of thin liquid films*, Rev. Mod. Phys. **69** (1997), 931–980.

[24] James Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318. MR0333220

Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong
E-mail address: `kaluen@.eduhk.hk`

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `kschou@math.cuhk.edu.hk`