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INSTABILITY OF EQUATORIAL EDGE WAVES IN THE BACKGROUND FLOW

LILI FAN AND HONGJUN GAO

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Abstract. In this paper, we first present an explicit exact solution to the edge wave problem in stratified geophysical flows with an underlying longshore current. Then we analyze the short-wavelength perturbation approach for barotropic incompressible fluids. Finally, we prove, by applying this method to geophysical equatorial edge waves in the background flow, that these waves are unstable when their steepness exceeds a specific threshold.

1. Introduction

In this paper, we consider geophysical equatorial edge waves in stratified water which admit an underlying longshore current. Edge waves were first described by Stokes [45]. They are a type of wave which propagates in the longshore direction, their crests point offshore, and their amplitude is maximal at the shoreline and decays asymptotically to zero towards the sea. These waves play a significant role in nearshore hydrodynamics, and one can refer [24] for a detailed discussion.

In water of constant density, Matioc [40] derived an explicit solution for geophysical edge waves, which is a Gerstner-type solution of the governing equations in the f-plane approximation in the Lagrangian description. Finding an exact solution is important as it can describe the nonlinear dynamic of complex fluid flows in detail. The explicit exact solution of the governing equations for periodic two-dimensional travelling gravity water waves was first discovered by Gerstner [15] and was rediscovered in [13, 42, 43], while an explicit form of the edge-wave solutions was provided by Constantin [3], where he gave a detailed analysis of the edge-wave dynamics as well as examples of the surface profiles and of the run-up patterns. We refer the reader to [9, 23, 36, 48] to see other important contributions and interesting studies on the theory of edge waves. Recently, many other authors have derived Gerstner-type solutions and adapted them to model a number of different physical and geophysical scenarios with explicit exact solutions in a Lagrangian description; cf. [4–6, 18, 19, 25, 26, 34, 40, 47], etc.

In water of arbitrary stratification, it is still possible for the Gerstner solution and the related edge-wave solution to propagate along a sloping beach [8, 41, 46, 49]. This remarkable fact is due to the special barotropic nature of these travelling waves, namely, that in a frame of reference moving with the waves, the lines of constant pressure are identical with lines of constant density and with the streamlines.

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Thus, the stratification of the fluid makes the density different from streamline to streamline but constant on the same streamline and does not alter the main structure of the dynamical equations. In fact, Godin recently managed an extension of sorts to "compressible" flow, though again the motion is an "incompressible" motion [16, 17].

Considering flows with underlying longshore currents, Howd et al. described the linear edge wave in [24], and Hsu made a further step to extend the explicit exact solution in Constantin [3] to include an underlying uniform current. For extensions to equatorial surface geophysical flows and geophysical internal waves with underlying currents see Henry [20] and Hsu [27]. In Section 3, we first extend the explicit exact solution in Matioc [40] to include an underlying uniform current and then show that it is possible for the obtained equatorial geophysical edge-wave solution to exist in heterogeneous liquids mainly by finding a suitable pressure and showing that the structure of the particle trajectories is such that lines of constant pressure are identical with lines of constant density.

Once an exact solution is available, the stability issue becomes important. An efficient way to study the hydrodynamic stability/instability theory of the general non-steady three-dimensional fluids is the short-wavelength method. This method was developed independently in [1, 7, 10–12, 39]. It is noted here that for barotropic incompressible fluids, Ionescu-Kruse present the short-wavelength instability method in [31, 32] by employing the WKB (Wentzel-Kramers-Brillouin)-form wave packet proposed by Lifschitz and Hameiri [39]. In this paper, we follow the idea presented by Constantin and Germain [7] to get the same results as Ionescu-Kruse's surveys [31, 32]. The obtained results are that, at leading order, wave phase and the wave amplitude of the velocity satisfy the same system of equations as in the constant density case, but the component of the pressure has a different expression.

The short-wavelength method is applicable for certain solutions which have an explicit Lagrangian form: see [38] for Gerstner's waves and recently for geophysical flows (with and without currents or stratification) in [7, 14, 21, 31], for edge waves (with and without stratification) in [32, 33], for internal equatorial waves (with and without currents) in [22] and for geophysical edge waves with currents in [28]. Since the transport equation for the amplitude vector and the eikonal equation for the wave phase are the same as in the constant density case, we can transfer the analysis of short-wavelength instability of equatorial edge waves by [28] to the stratified case and obtain that in stratified water the equatorial edge waves with steepness parameter higher than $\frac{(4\Omega\sin\alpha-kc_0)(8\Omega\sin\alpha-7kc_0)}{(4\Omega\sin\alpha-3kc_0)^2}\times\frac{\sin\alpha}{2}$, with α being the sloping angle of the beach, are unstable. And we observe that the obtained threshold is slightly smaller or slightly bigger than the threshold of Gerstner's unstable threshold $\frac{7}{18}\sin\alpha$ for different directions of wave propagation.

The remainder of this paper is organized as follows. In Section 2, we present the governing equations for the equatorial geophysical edge-wave problem. In Section 3, we provide the exact solutions describing the equatorial geophysical edge waves in stratified flows with underlying currents. In Section 4, we present the short-wavelength instability approach for the barotropic geophysical equatorial flows. In Section 5, we employ the obtained short-wavelength instability approach to prove that if wave steepness exceeds a sharp threshold, the equatorial edge waves in the background flow are unstable.

2. Governing equations for geophysical equatorial edge waves

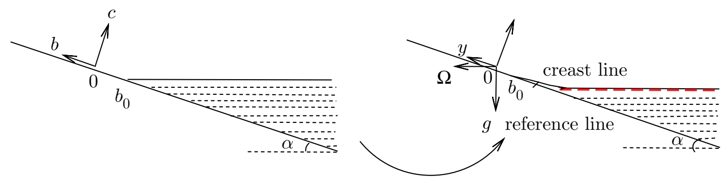

FIGURE 1. Cross section of the parameter space (left) and the corresponding cross section of the edge-wave solution (right).

In a reference frame with the origin located at the point O on earth's surface and rotating with the earth, we consider a sloping bed with an angle $\alpha \in (0, \pi/2)$, the xy-plane is taken to be parallel to the sloping beach and the z-axis is normal to it. Moreover, the x-axis is parallel to the shoreline and it is tangent to the equator, pointing eastwards while the y-axis and the earth's rotation vector $\overrightarrow{\Omega}$ form an angle equal to α. The coordinate system we adopt is shown in Figure 1, with the still sea in the region

$$R = \{(x, y, z), x \in \mathbb{R}, y \le b_0 \le 0, 0 \le z \le (b_0 - y) \tan \alpha\}.$$

The governing equations for the geophysical equatorial water waves in the f-plane approximation are given by [25–27, 30]

$$\begin{cases} \frac{Du}{Dt} + 2\Omega(w \cos \alpha + v \sin \alpha) & = -\frac{1}{\rho} P_x, \\ \frac{Dv}{Dt} - 2\Omega u \sin \alpha & = -\frac{1}{\rho} P_y - g \sin \alpha, \\ \frac{Dw}{Dt} - 2\Omega u \cos \alpha & = -\frac{1}{\rho} P_z - g \cos \alpha, \end{cases} \tag{2.1}$$

the incompressibility condition

$$\nabla \cdot \mathbf{U} = 0, \tag{2.2}$$

and the equation of mass conservation

$$\frac{D\rho}{Dt} = 0. \tag{2.3}$$

In the above equations, t is the time, $\mathbf{U} = (u, v, w)$ is the fluid velocity, $\Omega = 7.3 \times 10^{-5}$ rad/s is the rotational speed of the earth, $g = 9.81$ m/s is the gravitational acceleration at the earth's surface, $\rho(t, x)$ is an arbitrary density of the fluid, P is the pressure of the fluid, and $D/Dt = \frac{\partial}{\partial t} + (\mathbf{U} \cdot \nabla)$ is the material derivative. In this paper we consider the barotropic fluid whose density $\rho(t, x)$ is a function only of the pressure P:

$$\rho = f(P), \quad f'(P) \neq 0. \tag{2.4}$$

The kinematic boundary conditions for the fluid are [2, 35]:

$$\begin{cases} w = \eta_t + u\eta_x + v\eta_y, & \text{on } z = \eta(x, y, t), \\ w = 0, & \text{on } z = 0. \end{cases} \tag{2.5}$$

The kinematic boundary conditions express the fact that the same particles always form the free water surface $\eta(t, x, y)$ and the sloping bed is impermeable. The dynamic boundary condition is [2, 35]

$$P = P_0, \qquad \text{on } z = \eta(x, y, t). \tag{2.6}$$

3. A geophysical equatorial edge wave in the background flow

In the Lagrangian description, trajectories (x, y, z) of fluid particles are considered as functions of time t and some labels (a, b, c) which uniquely specify the fluid particles. Let a, b, c be parameters which fix the position of a particular water particle before the passage of a wave. The equatorial geophysical edge-wave solution representing waves travelling in the longshore direction at a constant speed of propagation c_0, in the presence of a constant current of strength $s_0 < \frac{g}{2\Omega}$, is given by

$$\begin{cases} x(t,a,b,c) = a + s_0 t - \frac{1}{k}e^{k(b-c)}\sin[k(a+c_0 t)], \\ y(t,a,b,c) = b - c + \frac{1}{k}e^{k(b-c)}\cos[k(a+c_0 t)] + \frac{2\Omega c_0 \cot\alpha}{g-2\Omega s_0} z(c), \\ z(c) = \frac{g-2\Omega s_0}{g-2\Omega s_0+2\Omega c_0}[c + c\tan\alpha - \frac{\tan\alpha}{2k}e^{2kb_0}(1-e^{-2kc(1+\cot\alpha)})], \end{cases} \tag{3.1}$$

with

$$c_{0+/-} = \frac{\Omega\sin\alpha \pm \sqrt{\Omega^2\sin^2\alpha + (g-2\Omega s_0)k\sin\alpha}}{k} \tag{3.2}$$

being the solutions of the quadratic equation

$$kc_0^2 - 2\Omega\sin\alpha c_0 - (g-2\Omega s_0)\sin\alpha = 0. \tag{3.3}$$

If $c_0 = c_{0+} > 0$, then the wave propagates along the equator from west to east, and if $c_0 = c_{0-} < 0$, it travels from east to west. Defining

$$\chi = k(b-c), \qquad \theta = k(a+c_0 t), \tag{3.4}$$

the Jacobi matrix of the transformation (3.1) is

$$F := \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix} = \begin{pmatrix} 1-e^{\chi}\cos\theta & -e^{\chi}\sin\theta & e^{\chi}\sin\theta \\ -e^{\chi}\sin\theta & 1+e^{\chi}\cos\theta & -1-e^{\chi}\cos\theta+\mu z'(c) \\ 0 & 0 & z'(c) \end{pmatrix}, \tag{3.5}$$

where

$$z'(c) = \frac{g-2\Omega s_0}{g-2\Omega s_0+2\Omega c_0}(1+\tan\alpha)(1-e^{2kb_0}e^{-2kc(1+\cot\alpha)}) \tag{3.6}$$

and

$$\mu = \frac{2\Omega c_0\cot\alpha}{g-2\Omega s_0}. \tag{3.7}$$

The fluid domain is bounded below by the image of the place $c = 0$ under the map (3.1), that is, the impermeable bed $z = 0$, and it is bounded above by the image of the still water surface $c = (b_0 - b)\tan\alpha$ under the map (3.1), that is, the free surface parameterized by

$$\begin{cases} x = a + s_0 t - \frac{1}{k}e^{kb(1+\tan\alpha)-kb_0\tan\alpha}\sin\theta, \\ y = b(1+\tan\alpha) - kb_0\tan\alpha + \frac{1}{k}e^{kb(1+\tan\alpha)-kb_0\tan\alpha}\cos\theta + \frac{2\Omega c_0\cot\alpha}{g-2\Omega s_0}z((b_0-b)\tan\alpha), \\ z = \frac{g-2\Omega s_0}{g-2\Omega s_0+2\Omega c_0}[(b_0-b)(1+\tan\alpha)\tan\alpha - \frac{\tan\alpha}{2k}e^{2kb_0}(1-e^{2k(b_0-b)(1+\tan\alpha)})], \end{cases} \tag{3.8}$$

with $a \in \mathbb{R}, b \le b_0 \le 0$ and $t \ge 0$.

By (3.3) and the fact that $0 \le c \le (b_0 - b)\tan\alpha$, we have $z'(c) > 0$. Then the Jacobian is $J = \det F = (1 - e^{2\chi})z'(c) > 0$, which ensures that $(x(t), y(t), z(t))$ is a local diffeomorphism. The inverse of F has the expression

$$G := \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1+e^{\chi}\cos\theta}{1-e^{2\chi}} & \frac{e^{\chi}\sin\theta}{1-e^{2\chi}} & -\frac{\mu e^{\chi}\sin\theta}{1-e^{2\chi}} \\ \frac{e^{\chi}\sin\theta}{1-e^{2\chi}} & \frac{1-e^{\chi}\cos\theta}{1-e^{2\chi}} & \frac{1}{z'(c)} - \frac{\mu(1-e^{\chi}\cos\theta)}{1-e^{2\chi}} \\ 0 & 0 & \frac{1}{z'(c)} \end{pmatrix}. \tag{3.9}$$

The particle velocity can be computed from (3.1) as

$$\begin{cases} u = s_0 - c_0 e^{\chi}\cos\theta, \\ v = -c_0 e^{\chi}\sin\theta, \\ w = 0. \end{cases} \tag{3.10}$$

Using (3.4) and (3.9), we can get the velocity gradient tensor

$$\begin{aligned} L = \nabla \mathbf{U} &= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{dF}{dt}G \\ &= \frac{1}{1-e^{2\chi}} \begin{pmatrix} kc_0 e^{\chi}\sin\theta & -kc_0 e^{\chi}(\cos\theta - e^{\chi}) & kc_0\mu e^{\chi}(\cos\theta - e^{\chi}) \\ -kc_0 e^{\chi}(\cos\theta + e^{\chi}) & -kc_0 e^{\chi}\sin\theta & kc_0\mu e^{\chi}\sin\theta \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.11}$$

By (3.11) the vorticity of the water flow (2.1) is given by

$$\begin{aligned} \gamma :&= (\gamma_1, \gamma_2, \gamma_3) = (w_y - v_z, u_z - w_x, v_x - u_y) \\ &= \Big(-\frac{c_0 k\mu e^{\chi}\sin\theta}{1-e^{2\chi}}, \frac{-c_0 k\mu e^{2\chi} + c_0 k\mu e^{\chi}\cos\theta}{1-e^{2\chi}}, \frac{-2c_0 k e^{2\chi}}{1-e^{2\chi}}\Big). \end{aligned} \tag{3.12}$$

Moreover, by (3.5), (3.10) and (2.1), we get that the first-order partial derivatives of P in Lagrangian coordinates are

$$\begin{aligned} \frac{1}{\rho}\begin{pmatrix} P_a \\ P_b \\ P_c \end{pmatrix} &= \frac{1}{\rho}F^T \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (g - 2\Omega s_0)\sin\alpha(e^{2\chi} - 1) \\ -(g - 2\Omega s_0)\left(\sin\alpha e^{2\chi} + \cos\alpha - (\sin\alpha + \cos\alpha)e^{2k(b_0 - c(1+\cot\alpha))}\right) \end{pmatrix}; \end{aligned} \tag{3.13}$$

here F^T denotes the transpose of the Jacobi matrix F. Then the pressure is given by

$$\begin{aligned} \tilde{P} &= P/\rho \\ &= C_0 + (g - 2\Omega s_0)\left(\frac{\sin\alpha}{2k}e^{2\chi} - (c\cos\alpha + (b - b_0)\sin\alpha) - \frac{\sin\alpha}{2k}e^{2k(b_0 - c(1+\cot\alpha))}\right), \end{aligned} \tag{3.14}$$

in which C_0 is a constant. Now, if ρ is variable, the equations in (3.13) are satisfied if

$$\frac{1}{\rho}\frac{\partial P}{\partial a} = \frac{\partial \tilde{P}}{\partial a}, \quad \frac{1}{\rho}\frac{\partial P}{\partial b} = \frac{\partial \tilde{P}}{\partial b}, \quad \frac{1}{\rho}\frac{\partial P}{\partial c} = \frac{\partial \tilde{P}}{\partial c}, \tag{3.15}$$

which can be satisfied for

$$P = r(\tilde{P}), \quad \rho = r'(\tilde{P}) \tag{3.16}$$

for some function g satisfying the boundary condition $r(C_0) = P_0$. Obviously, the pressure $\tilde{P}$ is independent of a. This means that the velocity field obtained by (3.1) is dynamically possible if ρ is independent of a. In fact, since on any plane of constant z the equatorial edge waves are travelling waves, one may shift to a moving coordinate system by making the change of variables $(x, y, z) \to (\zeta = x + c_0 t - s_0 t, y, z)$. Then we get from (2.3) that

$$\frac{D\rho}{Dt} = \rho_t + \rho_x u + \rho_y v + \rho_z w = \rho_\zeta(u + c_0 - s_0) + \rho_y v = 0. \tag{3.17}$$

On the other hand, we have from (3.1) that

$$\frac{\partial \zeta}{\partial a} = \frac{\partial(x + c_0 t - s_0 t)}{\partial a} = 1 + \frac{u - s_0}{c_0} = \frac{u + c_0 - s_0}{c_0} \tag{3.18}$$

and

$$\frac{\partial y}{\partial a} = \frac{v}{c_0}, \quad \frac{\partial z}{\partial a} = 0. \tag{3.19}$$

Hence

$$\frac{\partial \rho}{\partial a} = \frac{\partial \rho}{\partial \zeta}\frac{\partial \zeta}{\partial a} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial a} = \rho_\zeta \frac{u + c_0 - s_0}{c_0} + \rho_y \frac{v}{c_0} = 0, \tag{3.20}$$

where we've used the equality (3.17), and we obtain that ρ is independent of a. Obviously, the barotropic fluid (2.4) we consider in this paper satisfies the condition (3.16).

Finally, we point out that the wave steepness of the edge-wave profile, defined to be half the amplitude multiplied by the wavenumber, is given by

$$\tau = kd = \frac{\sin\alpha}{2}(e^{2kb(1+\tan\alpha) - 2kb_0 \tan\alpha} + 2e^{kb(1+\tan\alpha) - kb_0\tan\alpha}). \tag{3.21}$$

To get the amplitude of the edge-wave solution (3.1), we first need to compute the reference half-plane by letting $c \to \infty$ in (3.1),

$$\begin{cases} y(t, a, b, c) = b - c + \frac{2\Omega c_0 \cot\alpha}{g - 2\Omega s_0} z(c), \\ z(c) = \frac{g - 2\Omega s_0}{g - 2\Omega s_0 + 2\Omega c_0}(c + c\tan\alpha - \frac{\tan\alpha}{2k} e^{2kb_0}), \end{cases} \tag{3.22}$$

and considering the free surface $c = (b_0 - b)\tan\alpha$ in (3.22),

$$\begin{cases} y(t, a, b, c)\tan\alpha = b(1 + \tan\alpha)\tan\alpha - b_0 \tan^2\alpha + \frac{2\Omega c_0}{g - 2\Omega s_0} z(c), \\ z(c) = \frac{g - 2\Omega s_0}{g - 2\Omega s_0 + 2\Omega c_0}(b_0(1 + \tan\alpha)\tan\alpha - b(1 + \tan\alpha)\tan\alpha - \frac{\tan\alpha}{2k} e^{2kb_0}). \end{cases} \tag{3.23}$$

Then we get from (3.23) that the edge-wave solution (3.1) approaches the reference half-plane

$$z = -\frac{\tan\alpha}{2k} e^{2kb_0} + (b_0 - y)\tan\alpha. \tag{3.24}$$

Next, we rotate the coordinate system Oyz counterclockwise with the angle α and get the new coordinate system OYZ. In these new coordinates, we have the edge-wave solution

(3.25)
$$\begin{cases} Y = \left(b - c + \frac{1}{k}e^{k(b-c)}\cos[k(a+c_0t)] + \frac{2\Omega c_0 \cot\alpha}{g-2\Omega s_0} z(c)\right)\cos\alpha - z(c)\sin\alpha, \\ Z = \left(b - c + \frac{1}{k}e^{k(b-c)}\cos[k(a+c_0t)] + \frac{2\Omega c_0 \cot\alpha}{g-2\Omega s_0} z(c)\right)\sin\alpha + z(c)\cos\alpha \end{cases}$$

and the reference half-plane

$$Z = \left(-\frac{\tan\alpha}{2k}e^{2kb_0} + (b_0 - y)\tan\alpha\right)\cos\alpha + y\sin\alpha = -\frac{\sin\alpha}{2k}e^{2kb_0} + b_0\sin\alpha. \tag{3.26}$$

Setting $c = (b_0 - b)\tan\alpha$ in the second equation of (3.25), we obtain that the elevation with respect to the plane (3.26) is

$$\frac{\sin\alpha}{2k}\left(e^{2kb(1+\tan\alpha)-2kb_0\tan\alpha} + 2e^{kb(1+\tan\alpha)-kb_0\tan\alpha}\cos[k(a+c_0t)]\right), \tag{3.27}$$

which enables us to obtain the wave steepness (3.21).

4. Short-wavelength instability approach for the barotropic geophysical equatorial flows

Inspired by [7], we will present the short-wavelength instability approach for the barotropic geophysical equatorial flows. We suppose that a geophysical equatorial flow $(\mathbf{U}(t,\mathbf{X}), P(t,\mathbf{X}))$ which satisfies the system (2.1)-(2.3), with a density ρ of the form (2.4), is disturbed by a small perturbation of the form

$$\mathbf{u}(\mathbf{X},t) \approx \varepsilon\mathbf{b}(\mathbf{X},\xi_0,\mathbf{b}_0,t)e^{i\Phi(\mathbf{X},\xi_0,\mathbf{b}_0,t)/\delta} \tag{4.1}$$

and

$$\mathbf{p}(\mathbf{X},t) \approx \varepsilon\delta d(\mathbf{X},\xi_0,\mathbf{b}_0,t)e^{i\Phi(\mathbf{X},\xi_0,\mathbf{b}_0,t)/\delta}, \tag{4.2}$$

with the initial conditions

$$\Phi(\mathbf{X},\xi_0,\mathbf{b}_0,0) = \mathbf{X}\cdot\xi_0, \qquad \mathbf{b}(\mathbf{X},\xi_0,\mathbf{b}_0,0) = \mathbf{b}_0(\mathbf{X},\xi_0), \tag{4.3}$$

where $\mathbf{X} = (x,y,z)$, Φ is a scalar function, $\mathbf{b}_0$ represents the normalized amplitude, ξ_0 is the normalized wave vector subject to the transversality condition $\xi_0\cdot\mathbf{b}_0 = 0$, the scalar function d measures the amplitude of the pressure perturbation $\mathbf{p}$, and the small parameters ε and δ ensure that the small disturbance oscillates rapidly in space.

By the incompressibility (2.2) and the relation (2.4), equation (2.3) becomes

$$P_t + \mathbf{U}\cdot\nabla P = 0. \tag{4.4}$$

Substituting $\mathbf{U}+\mathbf{u}$ and $P+\mathbf{p}$ into the equations (2.1), (2.2) and (4.4) leads to

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{U} + (\mathbf{U}+\mathbf{u})\cdot\nabla\mathbf{u} + L_\Omega\mathbf{u} = -\nabla(\frac{\mathbf{p}}{f(P)}) + \frac{1}{2}\frac{f'(P)}{f^2(P)}\nabla(\mathbf{p}^2) + o(\mathbf{p}^2), \tag{4.5}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{4.6}$$

and

$$\mathbf{p}_t + \mathbf{u}\cdot\nabla P + (\mathbf{U}+\mathbf{u})\cdot\nabla\mathbf{p} = 0, \tag{4.7}$$

where we developed

$$\frac{1}{f(P+\mathbf{p})} = \frac{1}{f(P)} - \frac{f'(P)}{f^2(P)}\mathbf{p} + o(\mathbf{p}^2), \tag{4.8}$$

and we denoted

$$L_\Omega = \begin{pmatrix} 0 & 2\Omega\sin\alpha & 2\Omega\cos\alpha \\ -2\Omega\sin\alpha & 0 & 0 \\ -2\Omega\cos\alpha & 0 & 0 \end{pmatrix}. \tag{4.9}$$

Due to (4.1), we get at highest order in the expansion of (4.6) in powers of δ that

$$\mathbf{b}\cdot\nabla\Phi = 0, \tag{4.10}$$

while (4.5) leads to

$$\Phi_t + (\mathbf{U}+\mathbf{u})\cdot\nabla\Phi = 0. \tag{4.11}$$

As (4.10) is equivalent to $\mathbf{u}\cdot\nabla\Phi = 0$, (4.11) yields to the eikonal equation

$$\Phi_t + \mathbf{U}\cdot\nabla\Phi = 0. \tag{4.12}$$

Taking the gradient of (4.12) gives the evolution equation

$$\xi_t + (\mathbf{U}\cdot\nabla)\xi + (\nabla\mathbf{U})^T\xi = 0 \tag{4.13}$$

for the field of wave vectors $\xi = \nabla\Phi$, where $(\nabla\mathbf{U})^T$ is the transpose of the basic velocity gradient tensor L.

On the other hand, at the highest order in the expansion of (4.5) in powers of ε, we obtain

$$\mathbf{b}_t + \mathbf{b}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{b} + L_\Omega\mathbf{b} = -\frac{id}{f(P)}\xi. \tag{4.14}$$

Taking the vector product of (4.14) with the vector ξ, we have

$$d = if(P)\frac{\xi\cdot[\mathbf{b}_t + \mathbf{b}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{b} + L_\Omega\mathbf{b}]}{|\xi|^2}. \tag{4.15}$$

The total time derivative of (4.10) yields

$$\xi\cdot[\mathbf{b}_t + (\mathbf{U}\cdot\nabla)\mathbf{b}] = -\mathbf{b}\cdot[\xi_t + (\mathbf{U}\cdot\nabla)\xi]. \tag{4.16}$$

Taking advantage of (4.13) and (4.16), the expression of d (4.15) can be simplified to

$$d = if(P)\frac{\xi\cdot[2\mathbf{b}\cdot\nabla\mathbf{U} + L_\Omega\mathbf{b}]}{|\xi|^2}. \tag{4.17}$$

Using the above equation, (4.14) can be reexpressed as

$$\mathbf{b}_t + \mathbf{U}\cdot\nabla\mathbf{b} = -L_\Omega\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{U} + \frac{\xi}{|\xi|^2}(\xi\cdot(2\mathbf{b}\cdot\nabla\mathbf{U} + L_\Omega\mathbf{b})). \tag{4.18}$$

Finally, we get at highest order in the expansion of (4.7) in powers of ε

$$\mathbf{b}\cdot\nabla P = 0 \tag{4.19}$$

if we recall (4.2).

Summing up, for barotropic incompressible geophysical equatorial flows, the evolution in time of $\mathbf{X}$, of the amplitude $\mathbf{b}$ and of the wave vector $\xi = \nabla\Phi$ is governed at leading order in expansion in powers of ε and δ by the system of ODEs

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{U}(\mathbf{X}, t), \\ \dot{\xi} = -L^T \xi, \\ \dot{\mathbf{b}} = -L_\Omega \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{U} + \frac{\xi}{|\xi|^2}(\xi \cdot (2\mathbf{b} \cdot \nabla \mathbf{U} + L_\Omega \mathbf{b})), \end{cases} \tag{4.20}$$

with the initial condition

$$\mathbf{X}(0) = \mathbf{X}_0, \quad \xi(0) = \xi_0, \quad \mathbf{b}(0) = \mathbf{b}_0, \quad \xi_0 \cdot \mathbf{b}_0 = 0. \tag{4.21}$$

Remark 4.1. In the barotropic case considered here, the component d which appears in the pressure is different from the constant density case [7], as it is multiplied by $f(P)$ (4.15). This result as well as the orthogonality condition (4.19) coincides with the results in [31]. The system of ODEs (4.20) describing the evolution of a rapidly varying perturbation were independently derived in [1, 7, 11, 12, 31, 39, 44].

The associated instability criterion for Lagrangian flows for which $\mathbf{X}(0) = \mathbf{X}_0$ is determined by the exponent

$$\Lambda(\mathbf{X}_0) = \limsup_{t\to\infty} \frac{1}{t} \left(\ln \sup_{|\xi_0| = |\mathbf{b}_0| = 1, (\xi_0 \cdot \mathbf{b}_0) = 0} \{ |\mathbf{b}(\mathbf{X}_0, \xi_0, \mathbf{b}_0, t)| \} \right). \tag{4.22}$$

If $\Lambda(\mathbf{X}_0) > 0$ for a given fluid trajectory, then particles become separated at an exponential rate, and accordingly the flow is unstable [11]. Hence, establishing (4.22) provides us with a criterion for instability of a flow.

5. Instability of the equatorial edge wave in the background flows

The first equation in system (4.20) is already solved as the particle trajectories are already known explicitly (3.1). Let us now solve the second equation of system (4.20).

Differentiating with respect to t the Jacobian matrix (3.5) of the transformation (3.1) gives

$$\frac{dF}{dt} = \frac{d}{dt}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{a}}\right) = \frac{\partial \mathbf{U}}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{a}} = LF, \tag{5.1}$$

with L the matrix defined in (3.11). On the other hand, differentiating with respect to t the relation

$$F^T G^T = Id, \tag{5.2}$$

we get

$$\frac{dG^T}{dt} = -L^T G^T, \tag{5.3}$$

which combined with the initial condition (4.21) yields the solution of the second equation of the system (4.20) as

$$\xi(t) = G^T(t) F^T(0) \xi_0. \tag{5.4}$$

Recalling the expressions of the matrices F (3.5) and G (3.9), we have the solution

$$\xi(t) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \xi_0, \tag{5.5}$$

where $*$ denotes different functions depending on ka, χ, θ, μ and $z'(c)$. To demonstrate the instability of the geophysical equatorial edge flow (3.1), it is not necessary to investigate the associated system (4.20) for all initial data. Our aim is to make a choice for the initial disturbance that results in an exponentially growing amplitude $\mathbf{b}$. So we choose the vertical wave vector $\xi_0 = (0\ 0\ 1)^T$ and we get from (5.5) the solution of the second equation of system (4.20),

$$\xi(t) = (0\ 0\ 1)^T, \quad \text{for all } t \geq 0. \tag{5.6}$$

Substituting the solution (5.6) into the third equation of the system (4.20) leads to the evolution of $\mathbf{b} = (b_1, b_2, b_3)$:

$$\frac{d\mathbf{b}}{dt} = -L\mathbf{b} - \begin{pmatrix} 0 & 2\Omega\sin\alpha & 2\Omega\cos\alpha \\ -2\Omega\sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{b}. \tag{5.7}$$

Equation (5.7) and the expression (3.11) of the matrix L ensure that $b_3(t) = 0$ for all $t \geq 0$ with the choice of $b_3(0) = 0$. Then we rewrite (5.7) in the form

$$\frac{d\mathbf{b}}{dt} = [\frac{kc_0 e^{\chi}}{1-e^{2\chi}} M(t) + (2\Omega\sin\alpha + \frac{kc_0 e^{2\chi}}{1-e^{2\chi}}) N(t)]\mathbf{b}, \tag{5.8}$$

where

$$M(t) = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

By rotating the canonical basis with the angle $\frac{kc_0 t}{2}$ around the axis $(0\ 0\ 1)^T$, the rotating matrix becomes

$$R(t) = \begin{pmatrix} \cos(\frac{kc_0 t}{2}) & -\sin(\frac{kc_0 t}{2}) & 0 \\ \sin(\frac{kc_0 t}{2}) & \cos(\frac{kc_0 t}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

The components $(\tilde{b}_1(t), \tilde{b}_2(t), \tilde{b}_3(t))$ of the vector $\widetilde{\mathbf{b}}$ in the new basis are related to the components in the canonical basis by

$$\mathbf{b}(t) = R(t)\widetilde{\mathbf{b}}(t) \tag{5.11}$$

and

$$\widetilde{\mathbf{b}}(t) = R^{-1}(t)\mathbf{b}(t). \tag{5.12}$$

Differentiating with respect to t the relation (5.12), we obtain

$$\frac{d\widetilde{\mathbf{b}}}{dt} = \frac{d}{dt}[R^{-1}(t)]\mathbf{b}(t) + R^{-1}(t)\frac{d\mathbf{b}}{dt}. \tag{5.13}$$

Taking (5.11) and (5.8) into account, we get for the system (5.13):

$$\frac{d\widetilde{\mathbf{b}}}{dt} = [\frac{d}{dt}[R^{-1}(t)]R(t) + \frac{kc_0 e^{\chi}}{1-e^{2\chi}}R^{-1}(t)M(t)R(t) + (2\Omega\sin\alpha + \frac{kc_0 e^{2\chi}}{1-e^{2\chi}})R^{-1}(t)N(t)R(t)]\widetilde{\mathbf{b}}. \tag{5.14}$$

Easy computations lead to

$$\frac{d}{dt}[R^{-1}(t)]R(t) = -\frac{kc_0}{2}N(t), \quad R^{-1}(t)N(t)R(t) = N(t),$$
$$R^{-1}(t)M(t)R(t) = \begin{pmatrix} -\sin(ka) & \cos(ka) & 0 \\ \cos(ka) & \sin(ka) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

Then (5.14) becomes autonomous in the rotating basis, which is of the form

$$\frac{d\widetilde{\mathbf{b}}}{dt} = D\widetilde{\mathbf{b}}, \tag{5.16}$$

where

(5.17)
$$D=\begin{pmatrix} -\frac{kc_0e^{\chi}}{1-e^{2\chi}}\sin(ka) & \frac{kc_0e^{\chi}}{1-e^{2\chi}}\cos(ka)-2\Omega\sin\alpha-\frac{kc_0e^{2\chi}}{1-e^{2\chi}}+\frac{kc_0}{2} & 0 \\ \frac{kc_0e^{\chi}}{1-e^{2\chi}}\cos(ka)+2\Omega\sin\alpha+\frac{kc_0e^{2\chi}}{1-e^{2\chi}}-\frac{kc_0}{2} & \frac{kc_0e^{\chi}}{1-e^{2\chi}}\sin(ka) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{b} = R\widetilde{\mathbf{b}}$ and $R(t)$ is time periodic, we deduce that the short-wavelength rapidly varying perturbation $\mathbf{u}$ defined by (4.1) grows exponentially with time if the matrix D has a positive eigenvalue. We find that these eigenvalues of the matrix D have to satisfy the following equation:

(5.18)
$$\lambda^2 = \frac{10k^2c_0^2e^{2\chi} + 32\Omega\sin\alpha(\Omega\sin\alpha - kc_0)e^{2\chi} - (kc_0-4\Omega\sin\alpha)^2 - (3kc_0-4\Omega\sin\alpha)^2e^{4\chi}}{4(1-e^{2\chi})^2}.$$

Thus, instability of the system occurs if

(5.19)
$$e^{\chi} > \frac{4\Omega\sin\alpha - kc_0}{4\Omega\sin\alpha - 3kc_0} = \frac{3\Omega\sin\alpha \mp \sqrt{\Omega^2\sin^2\alpha + (g-2\Omega s_0)k\sin\alpha}}{\Omega\sin\alpha \mp 3\sqrt{\Omega^2\sin^2\alpha + (g-2\Omega s_0)k\sin\alpha}} = \frac{3\tilde{\varepsilon}\mp 1}{\tilde{\varepsilon}\mp 3},$$

with

$$\tilde{\varepsilon} = \frac{\Omega\sin\alpha}{\sqrt{\Omega^2\sin^2\alpha + (g-2\Omega s_0)k\sin\alpha}}, \tag{5.20}$$

where we have take dispersion relation (3.2) into account. This shows that the geophysical equatorial edge wave (3.1) with wavenumber k is linearly unstable when the wave steepness (3.21)

$$\tau > \frac{(4\Omega\sin\alpha - kc_0)(8\Omega\sin\alpha - 7kc_0)}{(4\Omega\sin\alpha - 3kc_0)^2} \times \frac{\sin\alpha}{2} \tag{5.21}$$

or the vorticity (3.12) in the z-direction satisfies

$$|\gamma_3| > \frac{|-2kc_0|(4\Omega\sin\alpha - kc_0)^2}{(4\Omega\sin\alpha - 3kc_0)^2 - (4\Omega\sin\alpha - kc_0)^2}. \tag{5.22}$$

And the exponential growth rate of the perturbation is given by the positive value of the root λ. Therefore, we have proved

Proposition 5.1. *In a stratified fluid with an underlying current* $s_0 < \frac{g}{2\Omega}$, *the geophysical equatorial edge waves* (3.1), *travelling along a sloping beach with an angle* $\alpha \in (0, \frac{\pi}{2})$, *are unstable under short-wavelength perturbation if the vorticity* (3.12) *in z-direction satisfies*

$$|\gamma_3| > \frac{|-2kc_0|(4\Omega\sin\alpha - kc_0)^2}{(4\Omega\sin\alpha - 3kc_0)^2 - (4\Omega\sin\alpha - kc_0)^2} \tag{5.23}$$

or if the wave steepness τ *is higher than* $\frac{(4\Omega\sin\alpha - kc_0)(8\Omega\sin\alpha - 7kc_0)}{(4\Omega\sin\alpha - 3kc_0)^2} \times \frac{\sin\alpha}{2}$. *The exponential growth rate of instabilities is*

$$\lambda = \frac{\sqrt{10k^2c_0^2e^{2\chi} + 32\Omega\sin\alpha(\Omega\sin\alpha - kc_0)e^{2\chi} - (kc_0 - 4\Omega\sin\alpha)^2 - (3kc_0 - 4\Omega\sin\alpha)^2 e^{4\chi}}}{2(1 - e^{2\chi})}, \tag{5.24}$$

with c_0 *defined by* (3.2).

Remark 5.1. If we ignore the Coriolis effects of the earth's rotation and uniform current by setting $\Omega = 0$ and $s_0 = 0$, then we recover the instability criterion for Gerstner's edge wave presented in [31]. We observe that for

(i) $c_0 > 0$, i.e. the equatorial edge waves propagating from west to east, for the right-hand side of (5.19), we have $\frac{3\tilde{\varepsilon}-1}{\tilde{\varepsilon}-3} \lessapprox \frac{1}{3}$. Hence we see the threshold of the wave steepness (5.22) is slightly smaller than the threshold of Gerstner's edge wave steepness $\frac{7}{18}\sin\alpha$.

(ii) $c_0 < 0$, i.e. the equatorial edge waves propagating from east to west, for the right-hand side of (5.19), we have $\frac{3\tilde{\varepsilon}+1}{\tilde{\varepsilon}+3} \gtrapprox \frac{1}{3}$. Hence we see the threshold of the wave steepness (5.22) is slightly bigger than the threshold of Gerstner's edge wave steepness $\frac{7}{18}\sin\alpha$.

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School of Mathematical Sciences, Institute of Mathematics, Nanjing Normal University, Nanjing 210023, People's Republic of China — and — College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, People's Republic of China

E-mail address: fanlily89@126.com

School of Mathematical Sciences, Institute of Mathematics and Jiangsu Center for Collaborative Innovation in Geographical Information Resource Development and Application, Nanjing Normal University, Nanjing 210023, People's Republic of China — and — Institute of Mathematics, Jilin University, Changchun 130012, People's Republic of China, corresponding author

E-mail address: gaohj@njnu.edu.cn