

THE SZLENK POWER TYPE AND TENSOR PRODUCTS OF BANACH SPACES

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ABSTRACT. We prove a formula for the Szlenk power type of the injective tensor product of Banach spaces with Szlenk index at most ω . We also show that the Szlenk power type as well as summability of the Szlenk index are separably determined, and we extend some of our recent results concerning direct sums.

1. INTRODUCTION

The notion of Szlenk index was introduced in [21] in order to show that there is no universal space in the class of all separable, reflexive Banach spaces. Since then it has proven to be an extremely useful tool in Banach space theory. The geometry of a given Banach space with Szlenk index ω heavily depends on the so-called Szlenk power type which encodes the rate of cutting out the dual unit ball by iterates of Szlenk derivations; it is strictly connected with the asymptotic moduli of smoothness and convexity (*cf.* [12] and the references therein). In this paper, motivated mainly by the work of Causey [3], we deal with determining the Szlenk power type of injective tensor products of Banach spaces, which in some cases should lead us to getting new information about asymptotic geometry of spaces of compact operators.

For a Banach space X we denote by B_X and S_X the unit ball and the unit sphere of X , respectively. If K is a weak*-compact subset of X^* and $\varepsilon > 0$, then we define the ε -Szlenk derivation of K by

$$\iota_\varepsilon K = \{x^* \in K : \text{diam}(K \cap U) > \varepsilon \text{ for every } w^*\text{-open neighborhood } U \text{ of } x^*\}$$

and its iterates by $\iota_\varepsilon^0 K = K$, $\iota_\varepsilon^{\alpha+1} K = \iota_\varepsilon(\iota_\varepsilon^\alpha K)$ for any ordinal α , and $\iota_\varepsilon^\alpha K = \bigcap_{\beta < \alpha} \iota_\varepsilon^\beta K$ for any limit ordinal α . The ε -Szlenk index of X , $\text{Sz}(X, \varepsilon)$, is defined as the least ordinal α (if any such exists) for which $\iota_\varepsilon^\alpha B_{X^*} = \emptyset$. Finally, the Szlenk index of X is defined as $\text{Sz}(X) = \sup_{\varepsilon > 0} \text{Sz}(X, \varepsilon)$ (we are only concerned with the case where $\text{Sz}(X, \varepsilon)$ is defined for every $\varepsilon > 0$). By compactness $\text{Sz}(X, \varepsilon)$ is always a successor ordinal, and the condition $\text{Sz}(X) \leq \omega$ is equivalent to $\text{Sz}(X, \varepsilon)$ being

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finite for every $\varepsilon > 0$. In such a case, since the function $(0, 1) \ni \varepsilon \mapsto \text{Sz}(X, \varepsilon)$ is submultiplicative (cf. [12, Prop. 4]), there exists a finite limit

$$\mathfrak{p}(X) := \lim_{\varepsilon \rightarrow 0^+} \frac{\log \text{Sz}(X, \varepsilon)}{|\log \varepsilon|},$$

which is called the *Szlenk power type* of X .

For any Banach spaces X and Y we denote by $X \hat{\otimes}_\varepsilon Y$ their injective tensor product (and refer the reader to [20] for any unexplained issues concerning this notion). In [3], Causey proved that the Szlenk index of $X \hat{\otimes}_\varepsilon Y$ behaves generally well; in particular, we have $\text{Sz}(X \hat{\otimes}_\varepsilon Y) \leq \omega$ whenever $\text{Sz}(X) \leq \omega$ and $\text{Sz}(Y) \leq \omega$ and both X and Y are separable. This makes sensible the question of determining the value of $\mathfrak{p}(X \hat{\otimes}_\varepsilon Y)$. Our main result, which concerns not necessarily separable spaces, thus reads as follows.

Main Theorem. *For any nonzero Banach spaces X and Y with $\text{Sz}(X) \leq \omega$ and $\text{Sz}(Y) \leq \omega$ we have*

$$\mathfrak{p}(X \hat{\otimes}_\varepsilon Y) = \max\{\mathfrak{p}(X), \mathfrak{p}(Y)\}.$$

The key motivation for studying the Szlenk power type stems from the fact that it corresponds to the asymptotic moduli of smoothness and convexity introduced by Milman [13]. Knaust, Odell and Schlumprecht [10] showed that a separable Banach space X satisfies $\text{Sz}(X) \leq \omega$ if and only if it can be given an equivalent asymptotically uniformly smooth norm (with a power type modulus) or, equivalently, a norm whose dual norm is weak*-asymptotically uniformly convex (with a power type modulus). This was later extended to the nonseparable case by Raja [19]. Godefroy, Kalton and Lancien [7] gave an exact quantitative result by showing that the Szlenk power type of a separable space corresponds to the optimal power types of both the above-mentioned moduli; for example, $\mathfrak{p}(X)$ is the conjugate of the supremum over all those $q > 1$ for which X can be renormed to have asymptotic modulus of smoothness dominated by Ct^q with some $C > 0$.

Consequently, our main result gives some information on asymptotic geometry of the space $\mathcal{K}(X, Y)$ of compact operators acting between certain Banach spaces X and Y (recall that if either X^* or Y has the approximation property, then $\mathcal{K}(X, Y)$ is isometrically isomorphic to $X^* \hat{\otimes}_\varepsilon Y$; cf. [20, Cor. 4.13]).

Corollary. *If X and Y are nonzero Banach spaces so that either X^* or Y has the approximation property and both $\text{Sz}(X^*)$ and $\text{Sz}(Y)$ are at most ω , then*

$$\mathfrak{p}(\mathcal{K}(X, Y)) = \max\{\mathfrak{p}(X^*), \mathfrak{p}(Y)\}.$$

In particular, for all $1 < p, q < \infty$ we have

$$\mathfrak{p}(\mathcal{K}(\ell_p, \ell_q)) = \max\left\{p, \frac{q}{q-1}\right\},$$

a result which was obtained earlier in [4].

In the next section we recall some terminology and tools concerning block/tree estimates that are essential to proving our main result. In Section 3 we give a proof in the separable case which is based on Proposition 4; the crucial part there is the fact that any separable Banach space X with $\text{Sz}(X) = \omega$ satisfies subsequential ℓ_r -upper tree estimates with r being arbitrarily close to the conjugate exponent of the Szlenk power type of X . We are grateful to the referee for suggesting to us this way of proof, simpler than our original approach. In Section 4 we deal

with the nonseparable case. In Section 5 we extend some of our recent results [5] on summability of the Szlenk index and the Szlenk power type of direct sums. Finally, in the appendix, we prove an ‘ ℓ_1^+ -version’ of Johnson’s lemma [9] which gives subsequential ℓ_r -upper block estimates for a given basic sequence, provided there are no ℓ_1^+ -block sequences of a prescribed length. This approach offers another proof of our main result, more direct than the one presented in Section 3 in the sense that it avoids involving rather deep renorming theorems from [7].

2. TOOLS

Recall that a sequence $E = (E_n)$ of finite-dimensional subspaces of a Banach space X is called a *finite-dimensional decomposition* (FDD for short) if every $x \in X$ has a unique representation $x = \sum_{n=1}^\infty x_n$ with $x_n \in E_n$ for every $n \in \mathbb{N}$. In such a case we denote by P_n^E the n th canonical projection $X \rightarrow E_n$, and for every $z \in c_{00}(\bigoplus_{n=1}^\infty E_n)$ we set $\text{supp}_E z = \{n \in \mathbb{N} : P_n^E z \neq 0\}$. A (finite or infinite) sequence (z_n) in X is called a *block sequence* (with respect to E) if for all suitable n ’s we have

$$\max \text{supp}_E z_n < \min \text{supp}_E z_{n+1}.$$

If $E = (E_n)$ is an FDD for X , then we have natural (not necessarily isometric) embeddings of E_n^* ’s in X^* . Under this identification E is called *shrinking* provided that X^* coincides with the norm closure of $c_{00}(\bigoplus_{n=1}^\infty E_n^*)$, that is, the set that consists of all functionals $(x_n^*)_{n=1}^\infty \in \prod_{n=1}^\infty E_n^*$ with $x_n^* \neq 0$ for finitely many n ’s. If V is a Banach space with a normalized, 1-unconditional basis (v_n) , then we say that E *satisfies subsequential C - V -upper block estimates*, with some $C \geq 1$, provided that for every normalized block sequence $(z_n) \subset X$ (with respect to E) and any finitely supported sequence of scalars (a_n) we have

$$\left\| \sum_{n=1}^\infty a_n z_n \right\| \leq C \left\| \sum_{n=1}^\infty a_n v_{m_n} \right\|, \quad \text{where } m_n = \min \text{supp}_E z_n.$$

We analogously define *subsequential C - V -lower block estimates*.

We shall now recall some terminology concerning trees in Banach spaces. First, define

$$T_l = \{(n_1, \dots, n_l) : n_1 < \dots < n_l \text{ are in } \mathbb{N}\} \quad \text{for } l \in \mathbb{N}.$$

We consider the trees $S_l = \bigcup_{j=1}^l T_j$ for $l \in \mathbb{N} \cup \{\infty\}$ ordered by the initial segment relation; that is, for $\alpha = (m_1, \dots, m_k)$ and $\beta = (n_1, \dots, n_l)$ we write $\alpha \leq \beta$ iff $k \leq l$ and $m_i = n_i$ for each $1 \leq i \leq k$. For any $\alpha = (m_1, \dots, m_k)$ we set $|\alpha| = k$ and call it the *length* of α . For each $l \in \mathbb{N} \cup \{\infty\}$, we say that S_l is *of order l* ; in other words, the order of S_l is the largest possible length of a *node* in S_l . We say that β is a *successor* of α if $|\beta| = |\alpha| + 1$ and $\alpha \leq \beta$, so $\beta = \alpha \frown k$ for some $k \in \mathbb{N}$, $k > \max \alpha$, where \frown stands for concatenation.

Let σ be any set and let $\sigma^{<\omega}$ be the collection of all finite sequences in σ . A family $\mathcal{F} \subset \sigma^{<\omega}$, ordered by the initial segment relation, is called a *tree on σ* if it is tree-isomorphic to one of the S_l ’s ($l \in \mathbb{N} \cup \{\infty\}$) and is closed under taking initial segments. The order of \mathcal{F} is, by definition, the same as the order of the corresponding tree S_l , and we denote it by $\text{ord}(\mathcal{F})$. It is sometimes convenient to write a tree on σ in the form $(x_\alpha)_{\alpha \in S_l}$, where each $x_\alpha \in \sigma$; this is then identified with

$$\mathcal{F} = \{(x_{(m_1)}, x_{(m_1, m_2)}, \dots, x_{(m_1, \dots, m_k)}) : \alpha = (m_1, \dots, m_k) \in S_l\}.$$

By a *branch* of \mathcal{F} we mean any maximal linearly ordered subset of \mathcal{F} , which we identify with a (finite or infinite) set of the form $\{x_{(m_1)}, x_{(m_1, m_2)}, \dots\}$.

If $(\beta_i)_{i=1}^\infty$ is the sequence of all successors of some α with $0 \leq |\alpha| < l$, then (under the above convention) the sequence $(x_{\beta_i})_{i=1}^\infty$ is called an *s-subsequence* of \mathcal{F} . If σ is a subset of a vector space equipped with some topology τ , then we say that \mathcal{F} is τ -null provided every *s*-subsequence of \mathcal{F} is τ -null. We shall be mainly concerned with weakly null trees on Banach spaces and weak*-null trees on dual Banach spaces.

Definition (cf. [2]). Let X be a Banach space. We say that a normalized sequence $(x_j)_{j=1}^n$ is an ℓ_1^+ - ρ -sequence, for some $\rho \in (0, 1]$, if

$$\left\| \sum_{j=1}^n a_j x_j \right\| \geq \rho \sum_{j=1}^n a_j \quad \text{for every } (a_j)_{j=1}^n \subset [0, \infty).$$

If \mathcal{F} is a tree on X , then we say that it is an ℓ_1^+ - ρ -weakly null tree provided it is weakly null and its every node is an ℓ_1^+ - ρ -sequence.

According to results by Alspach, Judd and Odell [2], the behavior of Szlenk derivations of B_{X^*} can conveniently be described in terms of the quantities

$$I_{w,\rho}^+(X) := \sup \{ \text{ord}(\mathcal{F}) : \mathcal{F} \text{ is an } \ell_1^+ \text{-} \rho \text{-weakly null tree on } S_X \} \quad (0 < \rho < 1).$$

Originally, they considered derivations defined by

$$P_\varepsilon(K) = \left\{ x^* \in K : \exists (x_n^*) \subset K, \quad x_n^* \xrightarrow{w^*} x^* \text{ and } \liminf_n \|x_n^* - x^*\| \geq \varepsilon \right\}.$$

However, it is easily seen that for every weak*-compact set $K \subset X^*$ and $\varepsilon \in (0, 1)$ we have $\iota_\varepsilon K \subseteq P_{\varepsilon/2}(K)$ and $P_\varepsilon(K) \subseteq \iota_{\varepsilon'} K$ for each $0 < \varepsilon' < \varepsilon$, and hence we can rephrase their results in the following form.

Theorem 1 (cf. [2, Props. 4.3, 4.10]). *If X is a separable Banach space with $\text{Sz}(X) \leq \omega$, then for all $n \in \mathbb{N}$ and $\varepsilon, \rho \in (0, 1)$ we have:*

- if $\iota_\varepsilon^n B_{X^*} \neq \emptyset$, then there is an ℓ_1^+ - $\frac{1}{16}\varepsilon$ -weakly null tree on S_X of order n ;
- if there is an ℓ_1^+ - ρ -weakly null tree on S_X of order n , then $\iota_\delta^n B_{X^*} \neq \emptyset$ for every $0 < \delta < \rho$.

Remark. The above assertion holds true for X^* being separable and without assuming that $\text{Sz}(X) \leq \omega$, provided that one considers weakly null trees of higher orders being countable ordinals (cf. [2, §3]). Then the ℓ_1^+ -weak index of X defined by the formula

$$I_w^+(X) = \sup_{0 < \rho < 1} I_{w,\rho}^+(X)$$

happens to be exactly equal to $\text{Sz}(X)$ (cf. [2, Thm. 4.2]). We shall not go into these details here, as we are exclusively concerned with the case where $\text{Sz}(X) \leq \omega$.

Lemma 2 (cf. [7, Prop. 3.4]). *Assume that X is a separable Banach space and $\varepsilon_1, \dots, \varepsilon_n > 0$. In order that $\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset$ it is necessary that there exist a weak*-null tree $(x_\alpha^*)_{\alpha \in S_n}$ on X^* of order n such that $\|x_\alpha^*\| \geq \frac{1}{4}\varepsilon_{|\alpha|}$ for each $\alpha \in S_n$ and $\| \sum_{\alpha \in \Gamma} x_\alpha^* \| \leq 1$ for every branch $\Gamma \subset S_n$, and it is sufficient that there exists a weak*-null tree $(x_\alpha^*)_{\alpha \in S_n}$ on X^* of order n such that $\|x_\alpha^*\| \geq \varepsilon_{|\alpha|}$ for each $\alpha \in S_n$ and $\| \sum_{\alpha \in \Gamma} x_\alpha^* \| \leq 1$ for every branch $\Gamma \subset S_n$.*

3. THE SEPARABLE CASE

The following result is a ‘power type’ analogue to [3, Cor. 4.5].

Proposition 3. *Let V be a Banach space with a normalized 1-unconditional basis (v_n) and assume $\text{Sz}(V) \leq \omega$. If X is a Banach space with a shrinking FDD satisfying subsequential V -upper block estimates with respect to (v_n) , then $\mathfrak{p}(X) \leq \mathfrak{p}(V)$.*

Proof. Suppose $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$ are such that $v_\varepsilon^n B_{X^*} \neq \emptyset$. By Theorem 1, there exists an $\ell_1^+ - \frac{1}{16}\varepsilon$ -weakly null tree $\mathcal{F} = (x_\alpha)_{\alpha \in S_n}$ on S_X of order n . By slightly decreasing the value of $\frac{1}{16}\varepsilon$ and using an easy pruning procedure we may assume that every s -subsequence of \mathcal{F} and every branch in \mathcal{F} forms a block sequence with respect to the given FDD \mathbf{E} of X .

Now, we define a new tree $\mathcal{V} = (w_\alpha)_{\alpha \in S_n}$ on S_V by setting

$$w_\alpha = v_{N(\alpha)}, \quad \text{where } N(\alpha) := \min \text{supp}_{\mathbf{E}} x_\alpha \quad (\alpha \in S_n).$$

If $C \geq 1$ is such that \mathbf{E} satisfies subsequential C - V -upper block estimates, then every node of \mathcal{V} is an $\ell_1^+ - C^{-1}\varrho$ -sequence in V , where ϱ can be any prescribed positive number smaller than $\frac{1}{16}\varepsilon$. For each α such that $0 \leq |\alpha| < n$ we obviously have $N(\alpha \frown k) \rightarrow \infty$ as $k \rightarrow \infty$, and since the basis (v_n) is shrinking, we infer that every s -subsequence of \mathcal{V} is weakly null. Therefore, \mathcal{V} is an $\ell_1^+ - C^{-1}\varrho$ -weakly null tree in S_V , and by appealing to Theorem 1 once again, we obtain $v_\delta^n B_{V^*} \neq \emptyset$ for every $0 < \delta < C^{-1}\varrho$. Hence, $\text{Sz}(X, \varepsilon) \leq \text{Sz}(V, \delta)$ for every $0 < \delta < \varepsilon/(16C)$, which completes the proof. \square

For any $p \in [1, \infty)$ we denote by p' the conjugate exponent, i.e. $p' = p/(p - 1)$ if $p > 1$ and $p' = \infty$ if $p = 1$. We say that X satisfies *subsequential ℓ_q -upper tree estimates* if there exists a constant $C > 0$ so that every weakly null tree on S_X contains a branch (x_n) which for every finitely supported sequence of scalars (a_n) satisfies $\|\sum_n a_n x_n\| \leq C(\sum_n |a_n|^q)^{1/q}$.

Proposition 4. *Assume X is a separable Banach space with $\text{Sz}(X) = \omega$ and let $1 < p < \infty$. The following conditions are equivalent:*

- (i) $\mathfrak{p}(X)' \geq p$.
- (ii) For every $1 < r < p$, X admits an equivalent asymptotically uniformly smooth norm with power type r .
- (iii) For every $1 < r < p$, X admits an equivalent norm such that X^* is weak*-asymptotically uniformly convex with power type r' .
- (iv) For every $1 < r < p$, X satisfies subsequential ℓ_r -upper tree estimates.
- (v) For every $1 < r < p$, X embeds in a Banach space Z with a shrinking FDD satisfying subsequential C - ℓ_r -upper block estimates with some $C \geq 1$.
- (vi) For every $1 < r < p$, X embeds in a Banach space Z with a bimonotone, shrinking FDD satisfying subsequential 1 - ℓ_r -upper block estimates.

Proof. The equivalence of (i), (ii) and (iii) follows from the paper by Godefroy, Kalton and Lancien (cf. [7, Thm. 4.8]).

The equivalence of (iv) and (v) follows from the Freeman–Odell–Schlumprecht–Zsák theorem [6, Thm. 1.1]. For the equivalence of (v) and (vi) observe that if Z has a shrinking FDD \mathbf{E} , then the formula

$$|z^*| = \sup \left\{ \left(\sum_{i=1}^{\infty} \|P_{[k_i, k_{i+1})}^{\mathbf{E}^*} z^*\|^{r'} \right)^{1/r'} : 1 \leq k_1 < k_2 < \dots \right\}$$

yields an equivalent dual norm on Z^* so that the dual FDD E^* satisfies subsequential $1-\ell_{r'}$ -lower block estimates (cf. [18, Lemma 3.3]). Under the predual norm E is then bimonotone and satisfies $1-\ell_r$ -upper block estimates.

The implication (ii) \implies (iv) is a part of [17, Prop. 5].

The implication (v) \implies (i) follows from Proposition 3. □

We can now prove our main result in the separable case by repeating the general scheme of the proof of Causey’s theorem mentioned in the introduction.

Proof of Main Theorem (the separable case). Set $p = \max\{p(X), p(Y)\}$. Proposition 4 guarantees that for every $1 < q < p'$ both X and Y can be embedded in some Banach spaces, say W and Z , having shrinking bimonotone FDD’s which satisfy subsequential $1-\ell_q$ -upper block estimates. In view of Causey’s result [3, Lemma 6.6], the space $W \hat{\otimes}_\varepsilon Z$ has a shrinking FDD which satisfies subsequential $2-\ell_q$ -upper block estimates, and hence Proposition 3 implies that

$$p(W \hat{\otimes}_\varepsilon Z) \leq p(\ell_q) = q'.$$

Since q can be taken arbitrarily close to p' and $X \hat{\otimes}_\varepsilon Y \hookrightarrow W \hat{\otimes}_\varepsilon Z$, the assertion follows. □

4. THE NONSEPARABLE CASE

We start with a simple fact which guarantees that summability of the Szlenk index and its power type are inherited by subspaces in a ‘uniform’ way.

Lemma 5. *Let X a Banach space, Y be a subspace of X and $\varepsilon_1, \dots, \varepsilon_n > 0$. Then we have*

$$\iota_{\varepsilon_1/2} \dots \iota_{\varepsilon_n/2} B_{X^*} \neq \emptyset \quad \text{whenever} \quad \iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{Y^*} \neq \emptyset.$$

Proof. In the proof of [8, Lemma 2.39] it was shown that if $K \subseteq B_{X^*}$ and $L \subseteq B_{Y^*}$ are weak*-compact sets such that $L \subseteq j^*(K)$, then $\iota_\varepsilon L \subseteq j^*(\iota_{\varepsilon/2} K)$ for every $\varepsilon > 0$, where j^* stands for the adjoint of the inclusion operator $j: Y \rightarrow X$. Inductively, putting $K = B_{X^*}$ and $L = B_{Y^*}$, and then $K = \iota_{\varepsilon_i/2} \dots \iota_{\varepsilon_n/2} B_{X^*}$ and $L = \iota_{\varepsilon_i} \dots \iota_{\varepsilon_n} B_{Y^*}$, we easily obtain

$$\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{Y^*} \subseteq j^*(\iota_{\varepsilon_1/2} \dots \iota_{\varepsilon_n/2} B_{X^*}),$$

which gives the assertion. □

The following lemma is, in a sense, a quantitative version of [11, Lemma 3.4].

Lemma 6. *Let X be a Banach space and $\varepsilon_1, \dots, \varepsilon_n > 0$. Then there is a separable space $Y \subseteq X$ such that*

$$\iota_{\varepsilon_1/4} \dots \iota_{\varepsilon_n/4} B_{Y^*} \neq \emptyset \quad \text{whenever} \quad \iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset.$$

Proof. Given $n \in \mathbb{N}$, let $(\alpha_m)_{m=1}^\infty$ be an enumeration of all elements of the tree S_n so that $\max \alpha_k \leq \max \alpha_l$ whenever $k < l$. Define $\varphi: \mathbb{N} \rightarrow S_n$ by $\varphi(m) = \alpha_m$; note that φ is surjective and $\max \varphi(m) \leq m$ for each $m \in \mathbb{N}$.

Assume $\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset$. By induction on $\max \alpha$, we shall construct a tree $(x_\alpha^*)_{\alpha \in S_n}$ on X^* of order n and a family $\{x_\alpha : \alpha \in S_n\} \subset B_X$ such that the following conditions are satisfied:

- (i) $x_\alpha^*(x_\alpha) > \frac{1}{4}\varepsilon_{|\alpha|}$ for each $\alpha \in S_n$;
- (ii) $\begin{cases} \sum_{\beta \leq \alpha} x_\beta^* \in \iota_{\varepsilon_{|\alpha|+1}/2} \dots \iota_{\varepsilon_n/2} B_{X^*} & \text{for } |\alpha| \leq n-1, \\ \sum_{\beta \leq \alpha} x_\beta^* \in B_{X^*} & \text{for } |\alpha| = n; \end{cases}$
- (iii) $|x_{\alpha \frown m}^*(x_{\varphi(k)})| \leq 2^{-m}$ for $\alpha \frown m \in S_n$ and $1 \leq k < m$.

Since $\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset$, we have

$$0 \in \frac{1}{2}\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} + \frac{1}{2}B_{X^*} \subseteq \iota_{\varepsilon_1/2} \dots \iota_{\varepsilon_n/2} B_{X^*}.$$

Given $m \in \mathbb{N} \cup \{0\}$, suppose that all the elements of the form $x_{\alpha \frown k}$ and $x_{\alpha \frown k}^*$, for $1 \leq k \leq m$ and $\alpha \frown k \in S_n$, have been constructed in such a way that they satisfy conditions (i)–(iii). Take $\alpha \in S_n \cup \{\emptyset\}$ with $|\alpha| < n$ and $\max \alpha < m+1$ (we set $\max \emptyset = 0$); we are to define $x_{\alpha \frown m+1}$ and $x_{\alpha \frown m+1}^*$. Observe that condition (ii) implies that

$$\text{diam}\left(V \cap \iota_{\varepsilon_{|\alpha|+2}/2} \dots \iota_{\varepsilon_n/2} B_{X^*}\right) > \frac{\varepsilon_{|\alpha|+1}}{2}$$

for each weak*-neighborhood V of $\sum_{\beta \leq \alpha} x_\beta^*$. In particular, this shows that there is $x^* \in \iota_{\varepsilon_{|\alpha|+2}/2} \dots \iota_{\varepsilon_n/2} B_{X^*}$ such that

$$\left\|x^* - \sum_{\beta \leq \alpha} x_\beta^*\right\| > \frac{\varepsilon_{|\alpha|+1}}{4} \quad \text{and} \quad \left| \left(x^* - \sum_{\beta \leq \alpha} x_\beta^*\right)(x_{\varphi(k)}) \right| \leq \frac{1}{2^{m+1}} \quad \text{for } 1 \leq k \leq m.$$

Define $x_{\alpha \frown m+1}^* = x^* - \sum_{\beta \leq \alpha} x_\beta^*$. Plainly, conditions (ii) and (iii) are satisfied. To finish the construction pick any $x_{\alpha \frown m+1} \in B_X$ with $x_{\alpha \frown m+1}^*(x_{\alpha \frown m+1}) > \frac{1}{4}\varepsilon_{|\alpha|+1}$.

Set $Y = \overline{\text{span}}\{x_\alpha : \alpha \in S_n\}$ and $y_\alpha^* = x_\alpha^* \upharpoonright Y$ for every $\alpha \in S_n$. Condition (iii) guarantees that $(y_\alpha^*)_{\alpha \in S_n}$ is a weak*-null tree on Y^* of order n . Moreover, by conditions (i) and (ii), we have

- (i') $\|y_\alpha^*\| > \frac{1}{4}\varepsilon_{|\alpha|}$ for each $\alpha \in S_n$;
- (ii') $\|\sum_{\alpha \in \Gamma} y_\alpha^*\| \leq 1$ for every branch $\Gamma \subset S_n$.

Therefore, Lemma 2 yields $\iota_{\varepsilon_1/4} \dots \iota_{\varepsilon_n/4} B_{Y^*} \neq \emptyset$. □

We are ready to show that summability of the Szlenk index and the Szlenk power type are separably determined. First, recall that a Banach space X is said to have *summable Szlenk index* if there is a constant M such that for all positive $\varepsilon_1, \dots, \varepsilon_n$ we have $\sum_{i=1}^n \varepsilon_i \leq M$ whenever $\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset$. Then we also say that X has *summable Szlenk index with constant M* . Given any family of Banach spaces, we shall say that they have *uniformly summable Szlenk index* provided that all of them have summable Szlenk index with the same constant.

Proposition 7. *A Banach space has summable Szlenk index if every one of its separable subspaces does.*

Proof. Given a Banach space X and positive numbers $\varepsilon_1, \dots, \varepsilon_n$, let $Y(\varepsilon_1, \dots, \varepsilon_n)$ be a separable subspace of X constructed according to Lemma 6. Denote by \mathcal{E} the

collection of all finite sequences of positive rational numbers and set

$$Y = \overline{\text{span}} \bigcup_{(\delta_1, \dots, \delta_n) \in \mathcal{E}} Y(\delta_1, \dots, \delta_n).$$

As Y is a separable subspace of X , it is enough to show that Y has nonsummable Szlenk index provided that X does too.

Suppose, towards a contradiction, that Y has summable Szlenk index with constant M and consider any $\varepsilon_1, \dots, \varepsilon_n > 0$ with $\iota_{\varepsilon_1} \dots \iota_{\varepsilon_n} B_{X^*} \neq \emptyset$. For each k pick a rational number $\delta_k \in (\varepsilon_k/2, \varepsilon_k)$; of course, $\iota_{\delta_1} \dots \iota_{\delta_n} B_{X^*} \neq \emptyset$. Therefore, $\iota_{\delta_1/4} \dots \iota_{\delta_n/4} B_{Y(\delta_1, \dots, \delta_n)^*} \neq \emptyset$. Hence, by Lemma 5 we have $\iota_{\delta_1/8} \dots \iota_{\delta_n/8} B_{Y^*} \neq \emptyset$, which implies that $\delta_1 + \dots + \delta_n \leq 8M$. Thus $\varepsilon_1 + \dots + \varepsilon_n \leq 16M$, which proves that X has summable Szlenk index. \square

Proposition 8. *If X is a Banach space with $\text{Sz}(X) = \omega$, then there is a separable space $Y \subseteq X$ with $\mathfrak{p}(Y) = \mathfrak{p}(X)$.*

Proof. For any $\varepsilon > 0$, $n \in \mathbb{N}$ let $Y(n, \varepsilon)$ be a separable subspace of X constructed according to Lemma 6 applied to $\varepsilon_1 = \dots = \varepsilon_n = \varepsilon$. Define

$$Y = \overline{\text{span}} \bigcup_{n=1}^{\infty} \bigcup_{\delta \in \mathbb{Q}_+} Y(n, \delta).$$

Clearly, Y is a separable subspace of X . We shall show that $\mathfrak{p}(Y) = \mathfrak{p}(X)$.

Consider $\varepsilon > 0$ and $n \in \mathbb{N}$ with $\iota_{\varepsilon}^n B_{X^*} \neq \emptyset$. Pick a rational number $\delta \in (\varepsilon/2, \varepsilon)$; of course, we have $\iota_{\delta}^n B_{X^*} \neq \emptyset$ and hence $\iota_{\delta/4}^n B_{Y(n, \delta)^*} \neq \emptyset$. By Lemma 5, we have $\iota_{\delta/8}^n B_{Y^*} \neq \emptyset$, whence $\iota_{\varepsilon/16}^n B_{Y^*} \neq \emptyset$. Consequently, we have shown that $\text{Sz}(Y, \varepsilon/16) \geq \text{Sz}(X, \varepsilon)$, which yields $\mathfrak{p}(Y) \geq \mathfrak{p}(X)$. \square

Proof of Main Theorem (contd.) Clearly, $\mathfrak{p}(X \hat{\otimes}_{\varepsilon} Y) \geq \max\{\mathfrak{p}(X), \mathfrak{p}(Y)\}$. Therefore, in view of Proposition 8, it suffices to show that for any separable subspace Z of $X \hat{\otimes}_{\varepsilon} Y$ we have $\mathfrak{p}(Z) \leq \max\{\mathfrak{p}(X), \mathfrak{p}(Y)\}$. Let X_0 and Y_0 be separable subspaces of X and Y , respectively, such that Z can be embedded in $X_0 \hat{\otimes}_{\varepsilon} Y_0$. Therefore, by the ‘separable part’, we have

$$\mathfrak{p}(Z) \leq \mathfrak{p}(X_0 \hat{\otimes}_{\varepsilon} Y_0) = \max\{\mathfrak{p}(X_0), \mathfrak{p}(Y_0)\} \leq \max\{\mathfrak{p}(X), \mathfrak{p}(Y)\},$$

and the assertion follows. \square

Question. Suppose X and Y are (separable) Banach spaces with summable Szlenk index. Does $X \hat{\otimes}_{\varepsilon} Y$ necessarily have summable Szlenk index as well?

5. DIRECT SUMS

In this section, we extend some of our theorems recently obtained in [5]. First, observe that using results of Section 4 one can easily prove nonseparable analogues to [5, Thms. 3.2 and 5.9].

Theorem 9. *For any family $\{X_{\gamma} : \gamma \in \Gamma\}$ of Banach spaces with uniformly summable Szlenk index the space $X = (\bigoplus_{\gamma \in \Gamma} X_{\gamma})_{c_0(\Gamma)}$ also has summable Szlenk index.*

Proof. If Γ is finite, then [5, Lemma 3.1] applies; henceforth we assume that Γ is infinite. We shall show that any separable subspace Y of X has summable Szlenk index. Indeed, observe that Y is contained in $(\bigoplus_{\gamma \in \Gamma_0} Y_{\gamma})_{c_0(\Gamma_0)}$ with a countable set $\Gamma_0 \subseteq \Gamma$ and Y_{γ} being a separable subspace of X_{γ} for $\gamma \in \Gamma_0$. By [5, Thm. 3.2], the space $(\bigoplus_{\gamma \in \Gamma_0} Y_{\gamma})_{c_0(\Gamma_0)}$ has summable Szlenk index, which ends the proof. \square

In a very similar way one can derive the next result, which is a nonseparable version of [5, Thm. 5.9].

Theorem 10. *Let E be a Banach space with a normalized, shrinking, 1-unconditional basis (e_n) such that for some $p \in [1, \infty)$ its dual E^* is asymptotic ℓ_p with respect to (e_n^*) . Then for every sequence $(X_n)_{n=1}^\infty$ of nonzero Banach spaces such that the number*

$$\mathfrak{p}(X_n)_{n=1}^\infty := \inf \{p \in [1, \infty) : \text{there exists a constant } C > 0 \text{ such that} \\ \text{Sz}(X_n, \varepsilon) \leq C\varepsilon^{-p} \text{ for all } n \in \mathbb{N} \text{ and } \varepsilon \in (0, 1)\}$$

is finite, we have

$$\mathfrak{p}\left(\left(\bigoplus_{n=1}^\infty X_n\right)_E\right) = \max\{p, \mathfrak{p}(X_n)_{n=1}^\infty\}.$$

As was shown in [5, Exs. 5.12 and 5.13], the assumption that E^* is asymptotic ℓ_p with respect to the dual basis is generally essential. However, in the case where all X_n 's are finite-dimensional one can reduce the assumptions on E to a minimum.

Theorem 11. *Let E be a Banach space with a normalized 1-unconditional basis $\{e_\gamma : \gamma \in \Gamma\}$ and assume $\text{Sz}(E) = \omega$. Then for any family $\{F_\gamma : \gamma \in \Gamma\}$ of nonzero finite-dimensional Banach spaces we have*

$$\mathfrak{p}\left(\left(\bigoplus_{\gamma \in \Gamma} F_\gamma\right)_E\right) = \mathfrak{p}(E).$$

Proof. Since the Szlenk power type of $(\bigoplus_{\gamma \in \Gamma} F_\gamma)_E$ is separably determined, we may pass to a subspace of the form $(\bigoplus_{\gamma \in \Gamma_0} F_\gamma)_{E_0}$ with a countable set $\Gamma_0 \subseteq \Gamma$ and $E_0 = \overline{\text{span}}\{e_\gamma : \gamma \in \Gamma_0\}$. Therefore, we may assume that $\Gamma = \mathbb{N}$.

Define $\mu : X := (\bigoplus_{n=1}^\infty F_n)_E \rightarrow E$ by $\mu((x_n)) = \sum_{n=1}^\infty \|x_n\|e_n$. Obviously, μ is norm-preserving and sends weakly null sequences in X to weakly null sequences in E . Moreover, if $(\mathbf{x}_n) \subset S_X$ is a block sequence with respect to the FDD (F_n) , then $(\mu(\mathbf{x}_n))$ is a block sequence in E isometrically equivalent to (\mathbf{x}_n) .

Choose $1 < r < \mathfrak{p}(E)'$. In order to prove that $\mathfrak{p}(X) = \mathfrak{p}(E)$ we shall show that X satisfies subsequential ℓ_r -upper tree estimates. Consider a weakly null tree $(\mathbf{x}_\alpha)_{\alpha \in S_\infty}$ on S_X . Using standard pruning and perturbation arguments we may assume that every branch yields a block sequence with respect to (F_n) . By the properties of μ , $(\mu(\mathbf{x}_\alpha))_{\alpha \in S_\infty}$ is a weakly null tree on S_E each branch of which is a block sequence isometrically equivalent to the corresponding branch in $(\mathbf{x}_\alpha)_{\alpha \in S_\infty}$. By Proposition 4, E satisfies subsequential ℓ_r -upper tree estimates. Therefore there is a branch in $(\mu(\mathbf{x}_\alpha))_{\alpha \in S_\infty}$ which is dominated by the canonical basis of ℓ_r , and so is the corresponding branch in $(\mathbf{x}_\alpha)_{\alpha \in S_\infty}$. \square

6. APPENDIX: ℓ_1^+ -METHODS

In this section, we prove a slightly more delicate quantitative version of Johnson's result [9, Lemma III.1], which originally says the following: For any unconditionally monotone basic sequence (e_i) and every $n \in \mathbb{N}$ there exists $p > 1$ (namely, any p with $2n^{1/p} < 3$) such that if (e_i) does not admit any normalized block subsequence 10-equivalent to the unit vector basis of ℓ_1^n , then it satisfies subsequential $3\text{-}\ell_q$ -upper block estimates with $q = p'$. Our version offers an alternative method of proving the main result (more precisely, the implication (i) \implies (iv) of Proposition 4),

but also seems to be interesting in its own right. It is optimal in the sense that it quite automatically gives the best possible exponent for subsequential ℓ_r -upper tree estimates. Let us first explain how such a result is related to the Szlenk power type; this will require recalling some terminology.

For a separable Banach space X we denote by $\text{cof}(X)$ the family of all finite codimensional subspaces of X and consider the following game between two players:

Player I chooses $Y_1 \in \text{cof}(X)$

Player II chooses $y_1 \in S_{Y_1}$

Player I chooses $Y_2 \in \text{cof}(X)$

Player II chooses $y_2 \in S_{Y_2}$

...

Given $n \in \mathbb{N}$ and $\mathcal{A} \subseteq S_X^n$ we say that *Player II has a winning strategy in the \mathcal{A} -game* if he can always end up with $(y_j)_{j=1}^n \in \mathcal{A}$ after n steps, no matter what subspaces Y_j 's were picked by Player I. Let \mathcal{M}_n be the collection of all normalized basic sequences of length n with basis constant at most 2, where we identify all sequences which are 1-equivalent. Then $(\mathcal{M}_n, \log d_b)$ is a compact metric space, where d_b stands for the equivalence constant between basic sequences, that is, $d_b((e_i)_{i=1}^n, (f_i)_{i=1}^n) = \|I\| \|I^{-1}\|$, where $I: \text{span}\{e_i\} \rightarrow \text{span}\{f_i\}$ is the isomorphism given by $I(e_i) = f_i$ for $1 \leq i \leq n$.

Definition (cf. [14]). Let X be a Banach space and $n \in \mathbb{N}$. We say that a sequence $(e_j)_{j=1}^n \in \mathcal{M}_n$ is an *element of the n th asymptotic structure of X* , and then we write $(e_j)_{j=1}^n \in \{X\}_n$, provided that

$$\forall \varepsilon > 0 \forall Y_1 \in \text{cof}(X) \exists y_1 \in S_{Y_1} \dots \forall Y_n \in \text{cof}(X) \exists y_n \in S_{Y_n}, \\ d_b((y_j)_{j=1}^n, (e_j)_{j=1}^n) < 1 + \varepsilon.$$

In other words, $(e_j)_{j=1}^n \in \{X\}_n$ if and only if for every $\delta > 0$ Player II has a winning strategy in the \mathcal{A}_δ -game, where \mathcal{A}_δ is the ball in \mathcal{M}_n with center $(e_j)_{j=1}^n$ and radius δ .

In the case where X^* is separable this property can be restated in terms of trees (cf. [16, Cor. 5.2]). Namely, $\{X\}_n$ is the minimal closed subset of \mathcal{M}_n such that for any $\varepsilon > 0$ every weakly null tree on S_X of order n has a node $(y_j)_{j=1}^n$ with $d_b((y_j)_{j=1}^n, \{X\}_n) < 1 + \varepsilon$. Therefore, Theorem 1 and a simple pruning argument guarantee that for every separable Banach space X with $\text{Sz}(X) \leq \omega$ and any $\varrho \in (0, 1)$ there is some uniform bound on the lengths of ℓ_1^+ - ϱ -sequences lying in an asymptotic structure of X .

The following assertion is a part of a theorem due to Odell and Schlumprecht.

Theorem 12 (cf. [17, Thm. 3]). *Let X be a Banach space with X^* separable. Then the following assertions are equivalent:*

- (i) $\text{Sz}(X) \leq \omega$.
- (ii) *There exist $q > 1$ and $K < \infty$ so that for all $n \in \mathbb{N}$, $(e_i)_{i=1}^n \in \{X\}_n$ and $(a_i)_{i=1}^n \subset \mathbb{R}$ we have*

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leq K \left(\sum_{i=1}^n |a_i|^q \right)^{1/q}.$$

- (iii) *There exists $\bar{q} > 1$ so that X satisfies subsequential $\ell_{\bar{q}}$ -upper tree estimates. In fact, one can take any $\bar{q} \in (1, q)$ with q satisfying assertion (ii) above.*

Our goal is thus to show that condition (ii) above holds true with any $q < \mathfrak{p}(X)'$.

To formalize the announced version of Johnson’s lemma it is convenient to introduce the following terminology. Let a be a parameter running through some set A and let $\Phi_a: \mathbb{N} \times (c_a, \infty) \rightarrow [0, \infty)$, where $c_a \geq 1$, and $\varrho_a: (c_a, \infty) \rightarrow (0, 1]$ ($a \in A$). We say that $\{(\Phi_a, \varrho_a)\}_{a \in A}$ is an ℓ_1^+ -method provided that for all $a \in A$, $n \in \mathbb{N}$, $C > c_a$, any Banach space X and every normalized monotone basic sequence (e_i) in X the following condition is satisfied:

- (J) if (e_i) does not admit any block $\ell_1^+ - \varrho_a(C)$ -sequence of length n , then for every exponent $p > 1$ so that $p \geq \Phi_a(n, C)$ it satisfies subsequential $C - \ell_q$ -upper block estimates with $q = p'$.

Lemma 13. *Suppose there exists an ℓ_1^+ -method $\{(\Phi_a, \varrho_a)\}_{a \in A}$ such that:*

- (a) *for all $\eta > 0, p \geq 1$ there exist $\delta > 0, a \in A$ such that $\Phi_a(\lfloor \varrho_a(C)^{-p-\delta} \rfloor, C) \leq p + \eta$ whenever C is sufficiently large;*
- (b) *$\lim_{C \rightarrow \infty} \varrho_a(C) = 0$ for each $a \in A$.*

Then for every separable Banach space X with $\text{Sz}(X) \leq \omega$ and every $q < \mathfrak{p}(X)'$ there exists $K_q < \infty$ so that for all $n \in \mathbb{N}, (e_i)_{i=1}^n \in \{X\}_n$ and $(a_i)_{i=1}^n \subset \mathbb{R}$ we have

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leq K_q \left(\sum_{i=1}^n |a_i|^q \right)^{1/q}.$$

Proof. By the remarks following the definition of asymptotic structures, there exists a function $N: (0, 1) \rightarrow \mathbb{N}$ so that if $(e_i)_{i=1}^n \in \{X\}_n$ is an $\ell_1^+ - \varrho$ -sequence, then $n < N(\varrho)$. Moreover, if N stands for the pointwise smallest function with this property, then Theorem 1 yields that $N(\varrho) \leq \text{Sz}(X, \varepsilon)$ for every $0 < \varepsilon < \varrho$. Fix any $1 < q < \mathfrak{p}(X)'$ and put $\eta = q' - \mathfrak{p}(X) > 0$. In view of condition (a), there exist $\delta > 0$ and $a \in A$ so that for sufficiently large C we have

$$(1) \quad \Phi_a(\lfloor \varrho_a(C)^{-\mathfrak{p}(X)-\delta} \rfloor, C) \leq \mathfrak{p}(X) + \eta = q'.$$

Plainly, we have $\text{Sz}(X, \varepsilon) \leq \varepsilon^{-\mathfrak{p}(X)-\delta}$ if ε is sufficiently small (just by the definition of $\mathfrak{p}(X)$). Hence, with the aid of condition (b) and taking C sufficiently large, we may guarantee that the last inequality holds true for every $\varepsilon \leq \varrho_a(C)$; we may also assume that (1) is valid for our choice of C . Consequently,

$$N(\varrho_a(C)) \leq \varrho_a(C)^{-\mathfrak{p}(X)-\delta},$$

which means that there are no $\ell_1^+ - \varrho_a(C)$ -sequences of length $\lfloor \varrho_a(C)^{-\mathfrak{p}(X)-\delta} \rfloor$ in the corresponding asymptotic structure of X . Therefore, condition (J) implies that all members of any asymptotic structure of X satisfy subsequential $C - \ell_q$ -upper block estimates. □

In order to show that a suitable ℓ_1^+ -method exists, we shall need an ‘ ℓ_1^+ -version’ of the well-known James blocking argument used in the proof of his ℓ_1 -distortion theorem (see, e.g., [15, Prop. 2]). The original argument applies *mutatis mutandis* to our situation, so we omit the proof.

Lemma 14. *Let $N, k \in \mathbb{N}, \varrho > 0$ and let $(x_i)_{i=1}^{N^k}$ be a normalized $\ell_1^+ - \varrho$ -sequence in some Banach space. Then $(x_i)_{i=1}^{N^k}$ admits a normalized block subsequence of length N which forms an $\ell_1^+ - \varrho^{1/k}$ -sequence.*

Lemma 15. *There exists an ℓ_1^+ -method $\{(\Phi_a, \varrho_a)\}_{a>1}$ satisfying conditions (a) and (b).*

Proof. Fix $a > 1$ and pick any sequence $(\omega_i)_{i=0}^\infty$ of natural numbers such that $1 < \omega_i/\omega_{i-1} < a$ for $i \in \mathbb{N}$. Pick $\alpha \in (0, 1)$ and define

$$\Phi_a(n, C) = \frac{\omega_0 \log n}{\log(C - \sigma)} \quad \text{for } C - 1 > \sigma := \sum_{i=0}^\infty \alpha^{\omega_i}.$$

Define also $\varrho_a(C) = \alpha(C - \sigma)^{-a/\omega_0}$.

Fix $n \in \mathbb{N}$, $C > \sigma + 1$ and consider any exponent $p > 1$ with $p \geq \Phi_a(n, C)$. Let (e_i) be a normalized monotone basic sequence and assume that there is a (finite) block subsequence (y_j) of (e_i) such that

$$\left\| \sum_j y_j \right\| > C \left(\sum_j \|y_j\|^q \right)^{1/q}, \quad \text{where } q := p'.$$

We shall produce a block $\ell_1^+ - \varrho_a(C)$ -subsequence of (y_j) of length n . Hence, it is enough to find a norm one functional f so that $f(z_j) \geq \varrho_a(C)$ for each $1 \leq j \leq n$ and some $(z_j)_{j=1}^n$ being a normalized block subsequence of (y_j) . (By the geometric Hahn–Banach theorem, it is actually equivalent to the existence of the sequence $(z_j)_{j=1}^n$.)

Take a norm one functional f so that $f(\sum_j y_j) = \|\sum_j y_j\|$. Set $\gamma = \alpha n^{-a/p}$ and define

$$E_0 = \{j : \gamma^{\omega_0} \|y_j\| < f(y_j) \leq \|y_j\|\}$$

and

$$E_i = \{j : \gamma^{\omega_i} \|y_j\| < f(y_j) \leq \gamma^{\omega_{i-1}} \|y_j\|\} \quad \text{for } i \geq 1.$$

We claim that the cardinality $|E_i| \geq n^{\omega_i}$ for at least one $i \geq 0$. If this is not true, then by applying Hölder’s inequality we obtain

$$\begin{aligned} \left\| \sum_j y_j \right\| &= f\left(\sum_j y_j\right) \leq \sum_{i=0}^\infty \sum_{j \in E_i} f(y_j) \\ &\leq \sum_{j \in E_0} \|y_j\| + \sum_{i=1}^\infty \gamma^{\omega_{i-1}} \sum_{j \in E_i} \|y_j\| \\ &\leq |E_0|^{1/p} \left(\sum_{j \in E_0} \|y_j\|^q\right)^{1/q} + \sum_{i=1}^\infty \gamma^{\omega_{i-1}} |E_i|^{1/p} \left(\sum_{j \in E_i} \|y_j\|^q\right)^{1/q} \\ &< \left(n^{\omega_0/p} + \sum_{i=1}^\infty \alpha^{\omega_{i-1}} \cdot n^{(\omega_i - a\omega_{i-1})/p}\right) \left(\sum_j \|y_j\|^q\right)^{1/q} \\ &< (n^{\omega_0/p} + \sigma) \left(\sum_j \|y_j\|^q\right)^{1/q} \leq C \left(\sum_j \|y_j\|^q\right)^{1/q} \end{aligned}$$

because $p \geq \Phi_a(n, C)$, and hence we arrive at a contradiction.

Pick an index i with $|E_i| \geq n^{\omega_i}$. By normalizing the vectors from $\{y_j : j \in E_i\}$ we obtain a sequence (z_j) of length at least n^{ω_i} which consists of unit block vectors and satisfies $f(z_j) \geq \gamma^{\omega_i}$ for each j . This means that (z_j) forms an $\ell_1^+ - \gamma^{\omega_i}$ -sequence,

and an appeal to Lemma 14 produces an ℓ_1^+ - γ -sequence of length n . Notice that since $n^{\omega_0/p} \leq C - \sigma$, we have

$$\gamma = \alpha n^{-a/p} \geq \alpha(C - \sigma)^{-a/\omega_0} = \varrho_a(C),$$

so the resulting sequence is an ℓ_1^+ - $\varrho_a(C)$ -sequence. This shows that $\{(\Phi_a, \varrho_a)\}_{a>1}$ yields an ℓ_1^+ -method.

It remains to verify conditions (a) and (b). For arbitrarily fixed $\eta > 0$ and $p \geq 1$ note that

$$\Phi_a(\lfloor \varrho_a(C)^{-p-\delta} \rfloor, C) \leq p + \eta \quad \text{whenever} \quad (C - \sigma)^{(a - \frac{p+\eta}{p+\delta})/\omega_0} \leq \alpha.$$

This can be easily guaranteed once we take $0 < \delta < \eta$, $1 < a < \frac{p+\eta}{p+\delta}$ and C sufficiently large. Hence, condition (a) holds true. Condition (b) is obvious by the very definition. \square

Finally, notice that combining Lemmas 13, 15 and Theorem 12 we obtain the implication (i) \implies (iv) of Proposition 4 and therefore another proof of the main result.

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