PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 4, April 2017, Pages 1467–1474
http://dx.doi.org/10.1090/proc/13341
Article electronically published on October 26, 2016

CONTACT NILPOTENT LIE ALGEBRAS

M. A. ALVAREZ, M. C. RODRÍGUEZ-VALLARTE, AND G. SALGADO

(Communicated by Kailash C. Misra)

Abstract. In this work we show that for $n \geq 1$, every finite $(2n+3)$-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ can be obtained as a double extension of a contact nilpotent Lie algebra $\mathfrak{h}$ of codimension 2. As a consequence, for $n \geq 1$, every $(2n+3)$-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ can be obtained from the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, by applying a finite number of successive series of double extensions. As a byproduct, we obtain an alternative proof of the fact that a $(2n+1)$-nilpotent Lie algebra $\mathfrak{g}$ is a contact Lie algebra if and only if it is a central extension of a nilpotent symplectic Lie algebra.

1. Introduction

A *contact* manifold is a $(2n+1)$-dimensional smooth manifold M equipped with a 1-form ν on M such that $\nu \wedge (d\nu)^n \neq 0$ pointwise over M. Such a 1-form ν is called a *contact structure* on M. Gromov proved that there exists a contact structure on every odd dimensional connected non-compact Lie group (see [10]). However, such contact structures are non-necessary left invariant under left translations of the Lie group elements. Hence, an interesting and open problem is to determine which Lie groups admit a left invariant contact structure. An approach to solve this problem is to understand contact Lie groups through their Lie algebras. A *contact* Lie algebra $\mathfrak{g}$ is a $(2n+1)$-dimensional Lie algebra endowed with a 1-form $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge (d\alpha)^n \neq 0$. Contact Lie algebras have been studied by several authors (see [7], [9], [15], for instance). In [12], Kutsak provides a classification of all 7-dimensional nilmanifolds admitting a contact structure. This classification is used by Capelletti–Montano et al. in order to provide examples of K-contact manifolds with no Sasakian metric (see [5]). It is important to point out that the classification given by Kutsak is based in the classification of nilpotent Lie algebras of dimension 7 over algebraically closed fields and $\mathbb{R}$, provided by Gong (see [8]).

On the other hand, the notion of *double extension* plays a significant role providing an inductive construction of finite dimensional Lie algebras endowed with a geometric structure. In [11] (see §2, exercise 2.10), Kac points out that a solvable

Received by the editors May 6, 2016 and, in revised form, June 15, 2016.
2010 *Mathematics Subject Classification.* Primary 17B5x, 17B30, 53D10.
The first author was supported in part by Becas Iberoamérica de Jóvenes Profesores e Investigadores, Santander Universidades, and Postdoctoral Fellowship from Centro de Investigación en Matemáticas.
The second author was supported by CONACyT Grants 154340, 222870 and PROMEP Grant UASLP-CA-228.
The third author was supported by CONACyT Grant 222870 and PROMEP Grant UASLP-CA-228.

n-dimensional quadratic Lie algebra $\mathfrak{g}$ is either an orthogonal direct sum of a quadratic solvable ideal of codimension 1 and a 1-dimensional quadratic ideal, or $\mathfrak{g}$ can be constructed as a *double extension* by a *hyperbolic pair* of a quadratic solvable subalgebra of codimension 2. Later, Medina and Revoy generalize this idea to prove that every finite dimensional quadratic indecomposable Lie algebra can be obtained as a double extension of a quadratic subalgebra by an appropriate subspace (see [13]). As a consequence, this fact reduces the study of finite dimensional quadratic Lie algebras to the study of finite dimensional indecomposable quadratic Lie algebras. The same is true for symplectic Lie algebras; in [14] a method of construction of symplectic Lie algebras, called *symplectic double extension*, is described. Similar ideas have been developed in order to give an inductive construction of finite dimensional Lie (super)algebras endowed with other types of geometric structures (see [1], [2], [3] and [4] for more details). However, it is not possible to provide an inductive construction that generates all the finite dimensional Lie algebras endowed with a contact structure: in [15], Rodríguez–Vallarte and Salgado exhibit a family of $(4n+1)$-dimensional indecomposable contact solvable Lie algebras that cannot be obtained either as a suspension of a symplectic Lie algebra of codimension 1 or as a double extension of a Lie algebra of codimension 2; they also show that a suitable double extension of a finite dimensional indecomposable contact Lie algebra is a contact Lie algebra again.

Hence, the aim of this work is to provide an inductive construction of finite dimensional contact nilpotent Lie algebras. To be more precise, we prove the following:

Theorem 4.5. *Let $\mathfrak{g}$ be a $(2n+3)$-dimensional contact nilpotent Lie algebra $(n \geq 1)$ with a contact form $\beta \in \mathfrak{g}^*$. Then, there exist*

- *a $(2n+1)$-dimensional contact nilpotent Lie algebra $\mathfrak{h}$ with a contact form $\alpha \in \mathfrak{h}^*$,*
- *a 2-closed form $\theta \in (\Lambda^2\mathfrak{h})^*$, and*
- *a nilpotent derivation $D \in \mathrm{Der}(\mathfrak{h}_\theta(e))$ of the central extension $\mathfrak{h}_\theta(e)$ of $\mathfrak{h}$ by θ,*

such that $\mathfrak{g}$ is isomorphic to the double extension $\mathfrak{h}(D,\theta)$ of $\mathfrak{h}$ by the pair (D,θ). Moreover, there exists $\lambda \in \mathbb{R}$ such that $\beta = \alpha + \lambda e^*$.

Since every finite dimensional contact nilpotent Lie algebra can be obtained as a double extension of a contact nilpotent Lie algebra of codimension 2, we have as a consequence the following:

Corollary 4.6. *For $n \geq 1$, every $(2n+3)$-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ can be obtained from the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, by applying a finite number of successive series of double extensions by pairs (D,θ), where D is a nilpotent derivation.*

The paper is organized as follows: in section 2, we provide the basic definitions; in section 3, we provide some general facts about double extensions of Lie algebras; finally, in section 4, we prove the main theorem.

2. Basic definitions

Throughout the paper $\mathfrak{g}$ denotes a finite dimensional real Lie algebra and $\mathfrak{g}^*$ denotes its dual space. For the usual definitions about Lie algebra cohomology, we follow [6].

A *contact structure* on a $(2n+1)$-dimensional Lie algebra $\mathfrak{g}$ is a 1-form $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge (d\alpha)^n \neq 0$. A *contact* Lie algebra is a Lie algebra endowed with a contact structure. If $\mathfrak{g}$ is a contact Lie algebra, $\alpha \wedge (d\alpha)^n$ is a volume form for the corresponding Lie group.

A *symplectic structure* on a $2n$-dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega \in (\Lambda^2\mathfrak{g})^*$ such that ω has maximal rank, that is, $d\omega = 0$ and $\omega^n \neq 0$ is a volume form on the corresponding Lie group. A Lie algebra endowed with a symplectic structure is a *symplectic* or *Frobenius* Lie algebra. The symplectic structure is called *exact* if $\omega = d\varphi$ holds for some $\varphi \in \mathfrak{g}^*$.

Let $\mathfrak{g}$ be a real Lie algebra and let $\theta \in (\Lambda^2\mathfrak{g})^*$ be a closed 2-form. Let $\mathrm{Span}_{\mathbb{R}}\{e\} := \langle e\rangle$ be a 1-dimensional trivial Lie algebra. Then, letting

$$
\begin{aligned}
[x,y]_\theta &= [x,y]_{\mathfrak{g}} + \theta(x,y)e, \quad && x,y \in \mathfrak{g},\\
[x,e]_\theta &= 0, && x \in \mathfrak{g},
\end{aligned}
\tag{2.1}
$$

the real vector space $\mathfrak{g} \oplus \langle e\rangle$ is a Lie algebra. This Lie algebra is denoted by $\mathfrak{g}_\theta(e)$ and it is called a *central extension* of $\mathfrak{g}$ by the closed 2-form θ. It is a well-known fact that the elements of $H^2(\mathfrak{g},\mathbb{R})$ are in a one-to-one correspondence with the isomorphism classes of central extensions of a given Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}_\theta(e)$ be a central extension of $\mathfrak{g}$ by a closed 2-form $\theta \in (\Lambda^2\mathfrak{g})^*$, and let $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ be a derivation of $\mathfrak{g}_\theta(e)$. The *double extension* of $\mathfrak{g}$ by the pair (D,θ) is the semidirect product $\mathfrak{g}(D,\theta) := \langle D\rangle \ltimes \mathfrak{g}_\theta(e)$ of the abelian Lie algebra $\langle D\rangle$ with $\mathfrak{g}_\theta(e)$ (see [15] for more details).

3. Some general facts about double extensions of Lie algebras

For the sake of completeness, in this section we collect some results that will be helpful in order to understand the structure of contact nilpotent Lie algebras.

A direct calculation shows the following

Lemma 3.1. *Let $\mathfrak{g}_\theta(e) = \mathfrak{g} \oplus \langle e\rangle$ be a central extension of a real Lie algebra $\mathfrak{g}$ by a closed 2-form θ. Then the center of $\mathfrak{g}_\theta(e)$ is given by*

$$Z(\mathfrak{g}_\theta(e)) = (Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)) \oplus \langle e\rangle,$$

where $\mathrm{Rad}(\theta) = \{z \in \mathfrak{g} \mid \theta(z,x) = 0,\ \forall x \in \mathfrak{g}\}$.

It is straightforward to prove that a non-perfect Lie algebra $\mathfrak{g}$ always admits an ideal $\mathfrak{h}$ of codimension 1.

Lemma 3.2. *Let $\mathfrak{g} = \mathfrak{h} \oplus \langle x\rangle$ be a non-perfect Lie algebra having an ideal $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1, and such that $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{h}$. Then there exists a derivation $D_x \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g})$ such that the Lie algebras $\mathfrak{g}$ and $\langle D_x\rangle \ltimes \mathfrak{h}$ are isomorphic.*

Proof. It is enough to take $D_x = \mathrm{ad}(x)|_{\mathfrak{h}}$. □

A straightforward calculation proves the following result:

Proposition 3.3. *Let $\mathfrak{g}$ be a real Lie algebra and suppose that there exists a non-trivial element $e \in Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$. Then $\mathfrak{g} = \mathfrak{h}_\theta(e)$ where $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra and $\theta \in (\Lambda^2\mathfrak{h})^*$.*

Observe that if $\mathfrak{g}$ is a $(2n+1)$-dimensional Lie algebra satisfying the conditions stated in Proposition 3.3 above, we can determine the conditions for which $\mathfrak{g}$ is a contact Lie algebra. In the next result, the hat $\widehat{\ }$ over e_1 means that this element is omitted.

Corollary 3.4. *Let $\mathfrak{g} = \langle e_1, e_2, \dots, e_{2n+1}\rangle$ be a real Lie algebra. Suppose that $e_1 \in Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{h} = \langle \widehat{e_1}, e_2, \dots, e_{2n+1}\rangle$ be a Lie subalgebra of codimension 1, as in Proposition 3.3 above. If $\theta \in (\Lambda^2\mathfrak{h})^*$ is a 2-form of maximal rank, then $(\mathfrak{h}, \theta)$ is a symplectic Lie algebra. Moreover, $\mathfrak{g}$ is a contact Lie algebra with a contact form $\alpha \in \mathfrak{g}^*$ given by e_1^*.*

Proof. It is enough to see that $\langle e_2, \dots, e_{2n+1}\rangle$ is a symplectic basis for $\mathfrak{h}$. Therefore $e_1^* \wedge (de_1^*)^n \neq 0$. □

Now, from Lemma 3.2 and Proposition 3.3 it follows that every nilpotent Lie algebra $\mathfrak{g}$ can be described as a double extension of a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ of codimension 2 by a pair (D, θ). More specifically,

Theorem 3.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then there exist*

(1) *a nilpotent Lie subalgebra $\mathfrak{m} \subsetneq \mathfrak{g}$ of codimension 2, say $\mathfrak{g} = \mathfrak{m} \oplus \langle x, e\rangle$ for some $x, e \in \mathfrak{g}$,*
(2) *a closed 2-form $\theta \in (\Lambda^2\mathfrak{m})^*$, and*
(3) *a derivation $D_x \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{m}_\theta(e))$,*

such that $\mathfrak{g}$ is isomorphic to a double extension of $\mathfrak{m}$ by (D_x, θ).

Proof. By assumption, $\mathfrak{g}$ is a non-perfect Lie algebra. From Lemma 3.2 it follows that there exist an ideal $\mathfrak{h}$ of codimension 1 such that $\mathfrak{g} = \mathfrak{h} \oplus \langle x\rangle$ for some $x \in \mathfrak{g}$, and a derivation $D_x = \mathrm{ad}(x) \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{h})$ such that the Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\langle D_x\rangle \ltimes \mathfrak{h}$. Let $D = D_x$ be as above. Since $\mathfrak{h}$ is a nilpotent ideal of $\mathfrak{g}$, there exists $k \in \mathbb{Z}^+$ such that $\mathfrak{h}^k \neq 0$ and $\mathfrak{h}^{k+1} = 0$. Then $\mathfrak{h}^k \subseteq Z(\mathfrak{h}) \subseteq [\mathfrak{h}, \mathfrak{h}]_{\mathfrak{h}}$. Now from Proposition 3.3 there exists a Lie subalgebra $\mathfrak{m} \subsetneq \mathfrak{h}$ of codimension 1 such that $\mathfrak{h} = \mathfrak{m} \oplus \langle e\rangle$ for some $e \in \mathfrak{h}$, and a closed 2-form $\theta \in (\Lambda^2\mathfrak{m})^*$ such that $\mathfrak{h}$ is a central extension of $\mathfrak{m}$ by θ, that is, $\mathfrak{h} = \mathfrak{m}_\theta(e)$. Therefore, the Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{m}(D, \theta) = \langle D\rangle \ltimes \mathfrak{m}_\theta(e)$, which proves the theorem. □

For the next result we shall use the following notation: d_θ (respectively, d) stands for the Chevalley–Eilenberg differential operator associated with the trivial representation on a double extension $\mathfrak{g}(D, \theta)$ of a Lie algebra $\mathfrak{g}$ by a pair (D, θ) (respectively, on $\mathfrak{g}$).

Lemma 3.6. *Let $\mathfrak{g}(D, \theta)$ be a double extension of a Lie algebra $\mathfrak{g}$ by a pair (D, θ), where $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ and $\theta \in (\Lambda^2\mathfrak{g})^*$. Suppose that $\alpha \in \mathfrak{g}^*$ is such that $d\alpha = 0$ and $\beta = \alpha + \lambda\, e^* \in \mathfrak{g}(D, \theta)^*$ is a contact form on $\mathfrak{g}(D, \theta)$. Then the closed 2-form $\theta \in (\Lambda^2\mathfrak{g})^*$ has maximal rank.*

Proof. Given a derivation $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ and a closed 2-form $\theta \in (\Lambda^2\mathfrak{g})^*$, consider a double extension of $\mathfrak{g}$ by (D, θ), say $\mathfrak{g}(D, \theta)$. Let $\{e_1, \dots, e_{2n+1}, e, D\}$ be a basis for $\mathfrak{g}(D, \theta)$ and $\{e^1, \dots, e^{2n+1}, e^*, D^*\}$ be its dual basis. Then $\alpha = \sum_{i=1}^{2n+1} \alpha_i e^i$ for $\alpha_i \in \mathbb{R}$, and $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ has a matrix representation given by

$$[D]_{\mathfrak{g}_\theta(e)} = \begin{pmatrix} [D|_{\mathfrak{g}}] & v \\ u^t & a \end{pmatrix},$$

where $v, u \in \mathbb{R}^n, a \in \mathbb{R}$ and $[D|_{\mathfrak{g}}]$ is a matrix representation for $D|_{\mathfrak{g}} \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$, with $([D|_{\mathfrak{g}}])_{ij} = D_{ij} \in \mathbb{R}$.

Take $\beta = \alpha + \lambda\, e^* \in \mathfrak{g}(D, \theta)^*$. Clearly, $d_\theta\beta = d_\theta\alpha + \lambda d_\theta e^*$ where

$$d_\theta\alpha = d\alpha + \frac{1}{2}\sum_{i=1}^{2n+1}\left(\sum_{k=1}^{2n+1} \alpha_k D_{ki}\right) e^i \wedge D^* + \frac{1}{2}\left(\sum_{i=1}^{2n+1} \alpha_i v_i\right) e^* \wedge D^*,$$

and

$$d_\theta e^* = -\frac{\lambda}{2}\left(\sum_{i<j=1}^{2n+1}\theta(e_i,e_j)e^i\wedge e^j - \sum_{i=1}^{2n+1}u_ie^i\wedge D^* - ae^*\wedge D^*\right).$$

Letting

$$\bar{u}_i(\lambda) = \lambda u_i + \sum_{k=1}^{2n+1}\alpha_k D_{ki}, \qquad \bar{a}(\lambda) = \lambda a + \sum_{i=1}^{2n+1}\alpha_i v_i,$$

a straightforward calculation shows that

$$\beta\wedge(d_\theta\beta)^{n+1} = (n+1)\left(\frac{1}{2}\sum_{i=1}^{2n+1}(\alpha_i\bar{a}(\lambda) - \lambda\bar{u}_i(\lambda))\ e^i\right)\wedge(d\alpha - \frac{\lambda}{2}\theta)^n\wedge e^*\wedge D^*. \tag{3.1}$$

Since $\beta = \alpha + \lambda\ e^* \in \mathfrak{g}(D,\theta)^*$ is a contact form on $\mathfrak{g}(D,\theta)$ such that $d\alpha = 0$, from equation (3.1) it is clear that θ has maximal rank on $\mathfrak{g}$. □

4. Contact nilpotent Lie algebras as double extensions

In this section we use the results obtained in section 3 in order to provide a description of an arbitrary contact nilpotent Lie algebra as a double extension of a contact nilpotent Lie algebra of codimension 2.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of nilindex $k+1$. Then, it is clear that $0 \neq \mathfrak{g}^k \subseteq Z(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]$. Moreover, if such $\mathfrak{g}$ is also a contact Lie algebra it is well known that $\dim Z(\mathfrak{g}) \leq 1$, and hence

$$\dim\mathfrak{g}^k = \dim Z(\mathfrak{g}) = 1.$$

Now, from [9] we recall the following result:

Lemma 4.1. *Let $\mathfrak{g}$ be a contact Lie algebra with contact form $\alpha \in \mathfrak{g}^*$. Then there exists a basis $\{\alpha_1,\dots,\alpha_{2n+1}\}$ of $\mathfrak{g}^*$ such that $\alpha_{2n+1} = \alpha$ and $d\alpha_{2n+1} = -\alpha_1\wedge\alpha_2 - \dots - \alpha_{2n-1}\wedge\alpha_{2n}$. Moreover, if $\{x_1,\dots,x_{2n+1}\}$ is the dual basis of $\mathfrak{g}^*$, it follows that*

$$[x_{2k-1},x_{2k}] = x_{2n+1} + \sum_{s=1}^{2n}C^s_{2k-1,2k}x_s, \quad k=1,\dots,n,$$

$$[x_i,x_j] = \sum_{s=1}^{2n}C^s_{i,j}x_s, \quad 1\leq i<j\leq 2n+1,\ (i,j)\neq(2k-1,2k),$$

where $C^s_{i,j}\in\mathbb{R}$ for all $i,j,s=1,\dots,2n$.

Suppose now that $\mathfrak{g}$ is a $(2n+1)$-dimensional contact Lie algebra and let us fix a basis as in Lemma 4.1 above. Supposing that $C^s_{2k-1,2k} = C^s_{i,j} = 0$ for all $s=1,\dots,2n, k=1,\dots,n$ and $1\leq i<j\leq 2n+1$ with $(i,j)\neq(2k-1,2k)$, it is clear that $\mathfrak{g}$ is the $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. On the other hand, supposing that either $C^s_{2k-1,2k}\neq 0$ or $C^s_{i,j}\neq 0$ for some $s=1,\dots,2n, k=1,\dots,n$ and $1\leq i<j\leq 2n+1$ with $(i,j)\neq(2k-1,2k)$, we can prove that:

Lemma 4.2. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional contact Lie algebra with a contact form $\alpha\in\mathfrak{g}^*$. Consider the basis $\{\alpha_1,\dots,\alpha_{2n+1}\}$ of $\mathfrak{g}^*$ such that $\alpha_{2n+1} = \alpha$ and $d\alpha_{2n+1} = -\alpha_1\wedge\alpha_2 - \dots - \alpha_{2n-1}\wedge\alpha_{2n}$. If $\mathfrak{g}$ is nilpotent, then $Z(\mathfrak{g}) = \langle x_{2n+1}\rangle$.*

In [9] it is proved that a contact Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}$ is a central extension of a symplectic nilpotent Lie algebra (see [9, Theorem 20]). We can use Lemma 4.2 above to provide another proof for this result:

Theorem 4.3. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional nilpotent Lie algebra. Then $\mathfrak{g}$ is a contact Lie algebra if and only if it is a central extension of a nilpotent symplectic Lie algebra.*

Proof. Suppose that $\mathfrak{g}$ is a contact nilpotent Lie algebra of nilindex $k+1$. From Lemma 4.2 above it follows that there exists a basis $\{x_1, \dots, x_{2n+1}\}$ of $\mathfrak{g}$ such that $Z(\mathfrak{g}) = \langle x_{2n+1} \rangle$ and $\alpha = x_{2n+1}^*$ is a contact form. For this basis, one has the linear maps $\pi^i : \mathfrak{g} \to \mathbb{R}$ given by $\pi^i(x) = a_i$, where $x = \sum_{i=1}^n a_i x_i,\ a_i \in \mathbb{R}$. Then, for each $i = 1, \dots, 2n+1$, we can define $\theta_i \in (\Lambda^2\mathfrak{g})^*$ by $\theta_i(x,y) = \pi^i([x,y])$ for all $x, y \in \mathfrak{g}$. Letting $\mathfrak{m} = \langle x_1, \dots, x_{2n}, \widehat{x}_{2n+1} \rangle$ and $\theta = \theta_{2n+1}|_{\mathfrak{m}\times\mathfrak{m}}$, it is easy to check that $\theta \in (\Lambda^2\mathfrak{m})^*$ is a 2-closed form of maximal rank. Therefore, from Proposition 3.3 it follows that $\mathfrak{g}$ is a central extension of $\mathfrak{m}$ by θ. $\square$

Now, one can state the following:

Theorem 4.4. *Let $\mathfrak{g}$ be a $(2n+3)$-dimensional contact nilpotent Lie algebra ($n \geq 1$). Then $\mathfrak{g}$ is a double extension of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ by a pair $(D, 0)$ where $D \in \mathrm{Der}((\mathfrak{h}_{2n+1})_0(e))$, if and only if, $\mathfrak{g}$ is a double extension of the abelian Lie algebra $\mathbb{R}^{2n+1}$ by a pair (T, θ) where $T \in \mathrm{Der}((\mathbb{R}^{2n+1})_\theta(z))$ and $\theta \in \Lambda^2(\mathbb{R}^{2n+1})^*$ is a closed 2-form of maximal rank.*

Proof. Suppose that $\mathfrak{g}$ is a $(2n+3)$-dimensional contact nilpotent Lie algebra such that it can be expressed as a double extension of an abelian Lie algebra by a pair (θ, D), that is, $\mathfrak{g} = \langle D \rangle \ltimes (\mathbb{R}^{2n+1})_\theta(z)$ for some 2-closed form $\theta \in \Lambda^2(\mathbb{R}^{2n+1})^*$ and some derivation $D \in \mathrm{Der}((\mathbb{R}^{2n+1})_\theta(z))$. With no loss of generality, we can suppose that we have chosen a basis $\{e_1, \dots, e_{2n+3}\}$ in $\mathfrak{g}$ such that the contact form $\alpha \in \mathfrak{g}^*$ is given by $\alpha = e_{2n+3}^*$. From this it follows that e_{2n+3} can be written as $e_{2n+3} = x + \lambda z$, where $x \in Z(\mathbb{R}^{2n+1}) \cap \mathrm{Rad}(\theta)$ and $\lambda \in \mathbb{R}$. Denoting by d_θ (respectively, d) the exterior differential in $\mathbb{R}^{2n+1}(\theta, D)$ (respectively, in $\mathbb{R}^{2n+1}$), the condition $e_{2n+3}^* \wedge (d_\theta e_{2n+3}^*)^{n+1} \neq 0$ together with the fact that $dx^* = 0$, imply that $\theta \in \Lambda^2(\mathbb{R}^{2n+1})^*$ has maximal rank in $\mathbb{R}^{2n+1}$ (see Lemma 3.6 of section 3). Since having a central extension of $\mathbb{R}^{2n+1}$ by a 2-closed form of maximal rank is equivalent to having a trivial central extension of $\mathfrak{h}_{2n+1}$, one can conclude that the Lie algebra $\mathfrak{g} = \langle D \rangle \ltimes (\mathbb{R}^{2n+1})_\theta(z)$ is isomorphic to the Lie algebra $\langle T \rangle \ltimes (\mathfrak{h}_{2n+1})_0(e)$, where $T \in \mathrm{Der}((\mathfrak{h}_{2n+1})_0(e))$. $\square$

We can prove now the main result of this section: every contact nilpotent Lie algebra can be obtained as a double extension of a contact nilpotent Lie algebra of codimension 2.

Theorem 4.5. *Let $\mathfrak{g}$ be a $(2n+3)$-dimensional contact nilpotent Lie algebra ($n \geq 1$) with a contact form $\beta \in \mathfrak{g}^*$. Then, there exist*

- *a $(2n+1)$-dimensional contact nilpotent Lie algebra $\mathfrak{h}$ with a contact form $\alpha \in \mathfrak{h}^*$,*
- *a 2-closed form $\theta \in (\Lambda^2\mathfrak{h})^*$, and*
- *a nilpotent derivation $D \in \mathrm{Der}(\mathfrak{h}_\theta(e))$ of the central extension $\mathfrak{h}_\theta(e)$ of $\mathfrak{h}$ by θ,*

such that $\mathfrak{g}$ *is isomorphic to the double extension* $\mathfrak{h}(D,\theta)$ *of* $\mathfrak{h}$ *by the pair* (D,θ). *Moreover, there exists* $\lambda \in \mathbb{R}$ *such that* $\beta = \alpha + \lambda e^*$.

Proof. Since $\mathfrak{g}$ is a contact nilpotent Lie algebra, from Theorem 4.3 it follows that $\mathfrak{g}$ is a central extension of a symplectic nilpotent Lie algebra $\mathfrak{m}$ by a 2-closed form $\omega \in (\Lambda^2\mathfrak{m})^*$, namely $\mathfrak{g} = \mathfrak{m}_\omega(e)$, where e^* is a contact form on $\mathfrak{g}$.

Now, since $\mathfrak{m}$ is also a nilpotent Lie algebra, there exists a non-trivial element $e_1 \in \mathfrak{m}$ such that $e_1 \in Z(\mathfrak{m}) \cap [\mathfrak{m},\mathfrak{m}]$. Since the codimension of $\mathfrak{m}$ is one, there exists $e_2 \in \mathfrak{m}$ such that $e_2 \notin [\mathfrak{m},\mathfrak{m}]$ and $\omega(e_1,e_2) = 1$. We can extend $\{e_1,e_2\}$ to a symplectic basis of $\mathfrak{m}$, say $\{e_1,e_2,e_3,e_4,\dots,e_{2n+2}\}$ such that $\omega(e_{2k-1},e_{2k}) = 1$ if $k = 1,\dots,n+1$, and $\omega(e_i,e_j) = 0$ if $1 \le i < j \le 2n+2$ and $(i,j) \neq (2k-1,2k)$.

Letting $\mathfrak{h} = \langle e_3,e_4,\dots,e_{2n+2},e\rangle$, it is easy to check that $\mathfrak{h}$ is a $(2n+1)$-dimensional contact nilpotent Lie algebra with contact form given by $\alpha = e^*|_{\{e_1,e_2\}^\perp}$. Moreover, since there exists a 2-closed form $\theta \in (\Lambda^2\mathfrak{h})^*$, we can consider the central extension $\mathfrak{h}_\theta(e_1)$ of $\mathfrak{h}$ by θ. Clearly, $D = \mathrm{ad}(e_2) \in \mathrm{Der}(\mathfrak{h}_\theta(e_1))$ is a nilpotent derivation of $\mathfrak{h}_\theta(e_1)$. Hence, $\mathfrak{g}$ is isomorphic to the double extension $\mathfrak{h}(D,\theta)$ of a $(2n+1)$-dimensional nilpotent Lie algebra $\mathfrak{h}$ by the pair (D,θ), which proves the theorem. $\square$

Corollary 4.6. *For* $n \ge 1$, *every* $(2n+3)$*-dimensional contact nilpotent Lie algebra* $\mathfrak{g}$ *can be obtained from the* 3*-dimensional Heisenberg Lie algebra* $\mathfrak{h}_3$, *by applying a finite number of successive series of double extensions by pairs* (D,θ), *where* D *is a nilpotent derivation.*

References

[1] Helena Albuquerque, Elisabete Barreiro, and Saïd Benayadi, *Odd-quadratic Lie superalgebras*, J. Geom. Phys. **60** (2010), no. 2, 230–250, DOI 10.1016/j.geomphys.2009.09.013. MR2587391

[2] Ignacio Bajo, Saïd Benayadi, and Alberto Medina, *Symplectic structures on quadratic Lie algebras*, J. Algebra **316** (2007), no. 1, 174–188, DOI 10.1016/j.jalgebra.2007.06.001. MR2354858

[3] Hedi Benamor and Saïd Benayadi, *Double extension of quadratic Lie superalgebras*, Comm. Algebra **27** (1999), no. 1, 67–88, DOI 10.1080/00927879908826421. MR1668212

[4] Saïd Benayadi, *Quadratic Lie superalgebras with the completely reducible action of the even part on the odd part*, J. Algebra **223** (2000), no. 1, 344–366, DOI 10.1006/jabr.1999.8067. MR1738266

[5] Beniamino Cappelletti-Montano, Antonio De Nicola, Juan Carlos Marrero, and Ivan Yudin, *Examples of compact K-contact manifolds with no Sasakian metric*, Int. J. Geom. Methods Mod. Phys. **11** (2014), no. 9, 1460028, 10, DOI 10.1142/S0219887814600287. MR3270291

[6] Claude Chevalley and Samuel Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124. MR0024908

[7] André Diatta, *Left invariant contact structures on Lie groups*, Differential Geom. Appl. **26** (2008), no. 5, 544–552, DOI 10.1016/j.difgeo.2008.04.001. MR2458280

[8] Ming-Peng Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and R)*, ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–University of Waterloo (Canada). MR2698220

[9] Michel Goze and Elisabeth Remm, *Contact and Frobeniusian forms on Lie groups*, Differential Geom. Appl. **35** (2014), 74–94, DOI 10.1016/j.difgeo.2014.05.008. MR3231748

[10] M. L. Gromov, *Stable mappings of foliations into manifolds* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 707–734. MR0263103

[11] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[12] Sergii Kutsak, *Invariant contact structures on 7-dimensional nilmanifolds*, Geom. Dedicata **172** (2014), 351–361, DOI 10.1007/s10711-013-9922-6. MR3253785

[13] Alberto Medina and Philippe Revoy, *Algèbres de Lie et produit scalaire invariant* (French), Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 553–561. MR826103

[14] Alberto Medina and Philippe Revoy, *Groupes de Lie à structure symplectique invariante* (French), Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 20, Springer, New York, 1991, pp. 247–266, DOI 10.1007/978-1-4613-9719-9_17. MR1104932

[15] M. C. Rodríguez-Vallarte and G. Salgado, *5-dimensional indecomposable contact Lie algebras as double extensions*, J. Geom. Phys. **100** (2016), 20–32, DOI 10.1016/j.geomphys.2015.10.014. MR3435759

Departamento de Matemáticas, Universidad de Antofagasta, Antofagasta, Chile
E-mail address: maria.alvarez@uantof.cl

Facultad de Ciencias, UASLP, Av. Salvador Nava s/n, Zona Universitaria, CP 78290, San Luis Potosí, S.L.P., México
E-mail address: mcvallarte@fc.uaslp.mx

Facultad de Ciencias, UASLP, Av. Salvador Nava s/n, Zona Universitaria, CP 78290, San Luis Potosí, S.L.P., México
E-mail address: gsalgado@fciencias.uaslp.mx
E-mail address: gil.salgado@gmail.com