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A NOTE ON THE DENSITY THEOREM FOR PROJECTIVE UNITARY REPRESENTATIONS

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ABSTRACT. It is well known that a Gabor representation on $L^2(\mathbb{R}^d)$ admits a frame generator $h \in L^2(\mathbb{R}^d)$ if and only if the associated lattice satisfies the Beurling density condition, which in turn can be characterized as the "trace condition" for the associated von Neumann algebra. It happens that this trace condition is also necessary for any projective unitary representation of a countable group to admit a frame vector. However, it is no longer sufficient for general representations, and in particular not sufficient for Gabor representations when they are restricted to proper time-frequency invariant subspaces. In this short note we show that the condition is also sufficient for a large class of projective unitary representations, which implies that the Gabor density theorem is valid for subspace representations in the case of irrational types of lattices.

A *projective unitary representation* π for a countable group G is a mapping $g \to \pi(g)$ from G into the group $U(\mathcal{H})$ of all the unitary operators on a separable Hilbert space $\mathcal{H}$ such that $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g, h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g,h)$ is then called a *multiplier of* π. In this case we also say that π is a μ-projective unitary representation. It is clear from the definition that we have

(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$,
(ii) $\mu(g, e) = \mu(e, g) = 1$ for all $g \in G$, where e denotes the group unit of G.

Any function $\mu : G \times G \to \mathbb{T}$ satisfying (i) – (ii) above will be called a *multiplier* or 2-*cocycle* of G. It follows from (i) and (ii) that we also have

(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ for all $g \in G$.

Similar to the group unitary representation case, the *left* and *right regular projective representations* with a prescribed multiplier μ for G can be defined by

$$\lambda_g \chi_h = \mu(g,h)\chi_{gh} \quad (h \in G),$$

and

$$\rho_g \chi_h = \mu(h, g^{-1})\chi_{hg^{-1}} \quad (h \in G),$$

where $\{\chi_g : g \in G\}$ is the standard orthonormal basis for $\ell^2(G)$.

One typical example of projective unitary representations is the Gabor representation in time-frequency analysis. In 1946, D. Gabor [8] formulated a fundamental

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approach for signal decomposition in terms of elementary signals: Let $h \in L^2(\mathbb{R})$ be a fixed function and let α, β be fixed positive numbers. The family $(h_{m\alpha,n\beta})$ obtained by translating and modulating h,

$$h_{m\alpha,n\beta}(x) = e^{2\pi i m\alpha x} h(x - n\beta), \quad m, n \in \mathbb{Z},$$

is called a *Weyl-Heisenberg family* or a *Gabor family.* One of the central questions related to the Gabor family is to identify those functions h and α, β such that every function $f \in L^2(\mathbb{R})$ can be decomposed into an infinite linear combination of $h_{m\alpha,n\beta}$. This problem is best studied in the context of time-frequency representations and the induced Gabor frame theory. Let $\mathcal{H} = L^2(\mathbb{R}^d)$ be the Hilbert space of square-integrable functions and $B(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. The time-frequency representation $\pi : \mathbb{R}^d \times \mathbb{R}^d \to B(L^2(\mathbb{R}^d))$ is defined by

$$\pi(\lambda) = E_{\lambda_1} T_{\lambda_2},$$

where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^d \times \mathbb{R}^d$; T_t and E_s are the translation and modulation unitary operators defined by

$$T_t f(x) = f(x - t)$$

and

$$E_s f(x) = e^{2\pi i \langle s, x \rangle} f(x)$$

for all $f \in L^2(\mathbb{R}^d)$.

For the purpose of discrete representations of signals, the Gabor theory requires π to be restricted to a discrete subset of the time-frequency space $\mathbb{R}^d \times \mathbb{R}^d$, and most of the time to be a time-frequency lattice. Let $\mathcal{L} = M\mathbb{Z}^d$ be a full rank lattice in $\mathbb{R}^d$, where M is a non-singular real $d \times d$ matrix. The *volume* of $\mathcal{L}$ is defined to be $\mathrm{v}(\mathcal{L}) = |det(M)|$, which is also the Lebesgue measure of any fundamental domain of $\mathcal{L}$. A *time-frequency lattice* is a full-rank lattice Λ in the time-frequency domain $\mathbb{R}^d \times \mathbb{R}^d$. Note that the restriction $\pi_\Lambda := \pi|_\Lambda$ of π to a time-frequency lattice Λ is a projective unitary representation of the group $\mathbb{Z}^d \times \mathbb{Z}^d$.

Frame theory originally introduced by Duffin and Schaeffer [4] belongs to the area of applied harmonic analysis, but its underpinnings involve large areas of functional analysis including operator theory and deep connections with the theory of operator algebras on Hilbert spaces. Roughly speaking, frames are generalizations of bases but still provide robust, basis-like (but generally non-unique) representations of vectors in a Hilbert space $\mathcal{H}$. The potential redundancy of frames often allows them to be more easily constructible than bases and to possess better properties than are achievable using bases, and therefore today they have applications in a wide range of areas in engineering, natural and even social sciences.

Given a projective unitary representation π of a countable group G on a Hilbert space $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is called a frame vector (or generator) for π if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{g \in G} |\langle x, \pi(g)\xi \rangle|^2 \leq B\|x\|^2 \quad (x \in \mathcal{H}).$$

The optimal constants are known as the upper and lower *frame bounds.* The vector ξ is called a *Parseval frame vector* if $A = B = 1$. If we only require the upper bound condition in the frame inequality, then ξ is called a *Bessel vector.* The set of all Bessel vectors for π will be denoted by $\mathcal{B}_\pi$. A projective unitary representation π is called a *frame representation* if it admits a frame generator for $\mathcal{H}$, and a *cyclic representation* if it admits a cyclic vector.

A natural question in Gabor analysis is to characterize the lattice Λ that admits a single function generated frame $\{\pi(\lambda)g : g \in \Lambda\}$ for $L^2(\mathbb{R}^d)$. Searching for the answer to this question led to one of the fundamental theorems in time-frequency analysis: *The Density Theorem in Gabor analysis.*

Theorem 1. *Let Λ be a full rank lattice in $\mathbb{R}^d \times \mathbb{R}^d$. Then the Gabor representation π_Λ admits a frame generator for $L^2(\mathbb{R}^d)$ if and only if $\mathrm{v}(\Lambda) \leq 1$.*

Roughly speaking, existence of a Gabor frame (even a complete Gabor family) requires that the time-frequency lattice Λ cannot be spread out "too sparsely" throughout the time-frequency plane $\mathbb{R}^{2d}$. This can be exactly measured by the *Beurling density* $\frac{1}{\mathrm{v}(\Lambda)}$ for the lattice Λ. The density theorem has a long history of evolution for which we refer to two excellent survey papers by C. Heil [13, 14]. In particular we point out that much of the credit goes to the early contributions to the necessary part of the theorem from Baggett [1], Daubechies [3] and Rieffel [16]. The proof for the sufficient part is trivial for the one-dimensional separable lattice (i.e., $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$) case. However, the proof is difficult for the higher dimensional lattice case even if Λ is separable. The proof was settled in [2, 11], where complicated tools from either operator algebras or common lattice tilings were used in the proofs.

Considering the fact that π_Λ is a projective unitary representation of the group $\mathbb{Z}^d \times \mathbb{Z}^d$ and the deep connections between time-frequency analysis and operator algebras [9, 10, 15, 16], it is not surprising that this theorem can be put into a more general context in terms of general projective group representations. For this purpose we introduce a useful trace formula for the commutant of the von Neumann algebra generated by a projective group representation.

Lemma 2 ([6])**.** *Let π be a projective representation of a countable group G on a Hilbert space $\mathcal{H}$ such that $\mathcal{B}_\pi$ is dense in $\mathcal{H}$. If there exist vectors ξ_i ($i \in \mathbb{J}$) such that $\bigcup_{i\in\mathbb{J}}\{\pi(g)\xi_i : g \in G\}$ is a Parseval frame for $\mathcal{H}$ and $\sum_{i\in\mathbb{J}} ||\xi_i||^2 < \infty$, then*

$$Tr_{\pi(G)'}(A) = \sum_{i\in\mathbb{J}} \langle A\xi_i, \xi_i\rangle$$

defines a faithful trace on $\pi(G)'$, the commutant of $\pi(G)$. Moreover, $Tr_{\pi(G)'}$ is independent of the choices of the Parseval frame generators $\{\xi_i\}$.

For our convenience we still adopt the notation $Tr_{\pi(G)'}(I) = \sum_{i\in\mathbb{J}} ||\xi_i||^2$ even for the case when $\sum_{i\in\mathbb{J}} ||\xi_i||^2 = \infty$, where I is the identity operator on $\mathcal{H}$. For Gabor representation π_Λ, it is known that $Tr(I) = \mathrm{v}(\Lambda)$ [6, 11]. Therefore the following clearly generalizes one direction of the density theorem for Gabor representation:

Theorem 3 ([6])**.** *Let π be a projective representation of a countable group G on a Hilbert space $\mathcal{H}$ such that $\mathcal{B}_\pi$ is dense in $\mathcal{H}$. If there exists $x \in \mathcal{H}$ such that $\{\pi(g)x : g \in G\}$ is a complete sequence in $\mathcal{H}$, then $Tr_{\pi(G)'}(I) \leq 1$.*

Remark 4. (i) Suppose that π is a frame representation with a frame vector ξ. Then $\mathcal{B}_\pi$ is dense in $\mathcal{H}$ since $A\xi \in \mathcal{B}_\pi$ for every $A \in \pi(G)''$ (cf. [10]). Therefore the density of $\mathcal{B}_\pi$ is a necessary condition in order to study frame representations. In fact, it can be proved that $\mathcal{B}_\pi$ is dense in $\mathcal{H}$ if and only if π is a direct sum of frame representations [6].

(ii) While the condition $Tr_{\pi(G)'}(I) \leq 1$ is also sufficient for a Gabor representation π_Λ to admit a complete frame vector for $L^2(\mathbb{R}^d)$, this condition is not sufficient in general for π to be a frame representation of a group G. In fact this condition is not even sufficient for the Gabor representation when it is restricted to proper π_Λ-invariant subspaces [5,7]. Another simple example is the following: let $G = \mathbb{Z}$ and $\mathcal{H} = L^2[0,1/2] \oplus L^2[0,1/2]$. Define $\pi(n) = diag(E_n, E_n)$ with $E_n f(x) = e^{2\pi inx} f(x)$ for $f \in L^2[0,1/2]$. Then $Tr_{\pi(G)'}(I) = 1$. But π does not admit a cyclic vector for $\mathcal{H}$.

The purpose of this note is to present a class of groups G and 2-cocycles μ such that a μ-projective unitary representation π is a frame representation if and only if $Tr_{\pi(G)'}(I) \leq 1$.

Theorem 5. *Let G be a countable group and μ be a 2-cocycle of G. Assume that the von Neumann algebra generated by the μ-projective left regular representation λ is a factor von Neumann algebra. Then for any μ-projective unitary representation π of G on a Hilbert space $\mathcal{H}$ such that $\mathcal{B}_\pi$ is dense in $\mathcal{H}$, the following are equivalent:*

(i) π is a cyclic representation;

(ii) π is a frame representation;

(iii) $Tr_{\pi(G)'}(I) \leq 1$.

Proof. The equivalence of (i) and (ii) has been proved in [6], and the implication "$(ii) \Rightarrow (iii)$" follows from Theorem 3. So it suffices to prove "$(iii) \Rightarrow (ii)$".

Select vectors $x_i \in \mathcal{H}$ $(i \in \mathbb{J})$ such that $[\pi(G)x_i] \perp [\pi(G)x_j]$ for $i \neq j$ and

$$\mathcal{H} = \bigoplus_{i \in \mathbb{J}} \mathcal{H}_i$$

where $\mathcal{H}_i = [\pi(G)x_i] = \overline{span}\{\pi(g)x_i : g \in G\}$. Note that since each $\mathcal{H}_i$ is π-invariant, we get that $\pi_i = \pi|_{\mathcal{H}_i}$ is also a μ-projective unitary representation of G. Moreover, $\mathcal{B}_{\pi_i} \supseteq P_i(\mathcal{B}_\pi)$, where P_i is the orthogonal projection on $\mathcal{H}_i$. Thus $\mathcal{B}_{\pi_i}$ is dense in $\mathcal{H}_i$. By the implication "$(i) \Rightarrow (ii)$" we obtain that there exists $\eta_i \in \mathcal{H}_i$ such that $\{\pi_i(g)\eta_i\}_{g\in G}$ is a frame for $\mathcal{H}_i$. Let Θ_i be the analysis operator for $\{\pi_i(g)\eta_i\}_{g\in G}$ and $S_i = \Theta_i^*\Theta_i$. Then S is an invertible positive operator on $\mathcal{H}_i$ and it commutes with $\pi(g)$ for every $g \in G$. Write $\xi_i = S^{-1/2}\eta_i$. Then we get that

$$\{\pi(g)\xi_i : g \in G\} = \{S^{-1/2}\pi(g)\eta_i : g \in G\}$$

is a Parseval frame for $\mathcal{H}_i$ and thus $\{\pi(g)\xi_i : g \in G, i \in \mathbb{J}\}$ is a Parseval frame for $\mathcal{H}$. Therefore we have

$$Tr_{\pi(G)'}(I) = \sum_{i \in \mathbb{J}} ||\xi_i||^2.$$

For each $i \in \mathbb{J}$, define the "analysis operator" $\Theta_i : \mathcal{H}_i \to \ell^2(G)$ by

$$\Theta_i(x) = \sum_{g \in G} \langle x, \pi(g)\xi_i \rangle \chi_g \quad (x \in \mathcal{H}_i).$$

Then the range space of Θ_i is invariant under the μ-projective left regular representation λ of G. Moreover, $\Theta_i : \mathcal{H}_i \to \Theta_i(\mathcal{H}_i)$ is unitary and induces a unitary equivalence between π_i and $\lambda|_{Q_i}$, where Q_i is the orthogonal projection of $\ell^2(G)$ onto $\Theta_i(\mathcal{H}_i)$. Additionally, we also have $||\xi_i||^2 = ||Q_i\chi_e||^2$, where e is the group unit of G.

Recall that $\Phi(A) = \langle A\chi_e, \chi_e \rangle$ is a faithful normalized trace on the von Neumann algebra $\lambda(G)'$. Since $\sum_{i\in\mathbb{J}} \Phi(Q_i) = \sum_{i\in\mathbb{J}} ||\xi_i||^2 = Tr_{\pi(G)'}(I) \leq 1$ and $\lambda(G)'$ is a factor, there exists a sequence of mutually orthogonal projections $R_i \in \lambda(G)'$ such that Q_i and R_i are equivalent; i.e., there exist partial isometries $V_i \in \lambda(G)'$

satisfying $V_i^*V_i = Q_i$ and $V_iV_i^* = R_i$. This implies that $\lambda|_{Q_i}$ (and hence π_i) is unitarily equivalent to $\lambda|_{R_i}$.

Let $R = \sum_{i\in\mathbb{J}} R_i$. Then R is a projection in $\lambda(G)'$ and $\lambda|_R = \bigoplus_{i\in\mathbb{J}} \lambda|_{R_i}$. This implies that π is unitarily equivalent to the subrepresentation $\lambda|_R$ of the μ-projective left regular representation λ of G, and therefore π is a frame representation. $\square$

Following from the above theorem and Proposition 1.2 in [6] we immediately have the following:

Corollary 6. *Let μ be a 2-cocycle for a countable group G. Assume that for each $e \neq g \in G$, either $\{hgh^{-1} : h \in G\}$ or $\{\mu(hgh^{-1}, h)\overline{\mu(h,g)} : \ h \in G\}$ is an infinite set. Then a μ-projective unitary representation π on a Hilbert space $\mathcal{H}$ is a frame representation if and only if $\mathcal{B}_\pi$ is dense in $\mathcal{H}$ and $Tr_{\pi(G)'}(I) \leq 1$.*

Example 7 (A trace formula for the restrictions of time-frequency representations)**.** Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be the associated time-frequency lattice with A and B being $d\times d$ non-singular real matrices. Let $\{\Omega_1, \dots, \Omega_k\}$ be a partition of $A[0,1]^d$ such that $\{\Omega_i + (B^t)^{-1}n : n \in \mathbb{Z}^d\}$ is a disjoint (up to measure zero) family for $i = 1, \dots, k$. Define $f_i = \sqrt{|detB|}\chi_{\Omega_i}$. Then $\{\pi_\Lambda(\lambda)f_i : \lambda \in \Lambda, 1 \leq i \leq k\}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. Now let M be a π_Λ-invariant subspace of $L^2(\mathbb{R}^d)$ and let P be the orthogonal projection onto M. Then $\{Pf_1, \dots, Pf_k\}$ generates a Parseval frame for the restriction σ of π_Λ to M. Thus we get

$$Tr_{\sigma(G)'} = \sum_{i=1}^{k} ||Pf_i||^2,$$

where $G = \mathbb{Z}^d \times \mathbb{Z}^d$.

Consider the Gabor representation π_Λ for some full rank time-frequency lattice $\Lambda = M(\mathbb{Z}^d \times \mathbb{Z}^d)$. Then the associated 2-cocycle μ for $G = \mathbb{Z}^d \times \mathbb{Z}^d$ is given by

$$\mu(\vec{n}, \vec{m}) = e^{-2\pi i\langle\lambda_2, u_1\rangle},$$

where $M\vec{n} = (\lambda_1, \lambda_2)^t$ and $M\vec{m} = (u_1, u_2)^t$ for $\vec{n}, \vec{m} \in \mathbb{Z}^d \times \mathbb{Z}^d$. Thus the von Neumann algebra generated by the μ-projective left regular representation of $\mathbb{Z}^d \times \mathbb{Z}^d$ is a factor von Neumann algebra if the set

$$\{\langle\lambda_2, u_1\rangle - \langle\mu_2, \lambda_1\rangle : (u_1, u_2)^t = M\vec{m}, \vec{m} \in \mathbb{Z}^d \times \mathbb{Z}^d\}$$

modulo $\mathbb{Z}$ is an infinite set for each $(\lambda_1, \lambda_2)^t = M\vec{n} \neq 0$. In particular, this is true for the separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ case when the group generated by $A\mathbb{Z}^d \cup (B^t)^{-1}\mathbb{Z}^d$ is $\mathbb{R}^d$. So, as a consequence of Corollary 6 we immediately have the first part of the following result:

Corollary 8. *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a separable time-frequency lattice, and V be a π_Λ-invariant subspace of $L^2(\mathbb{R}^d)$. Suppose that the group generated by $A\mathbb{Z}^d \cup (B^t)^{-1}\mathbb{Z}^d$ is $\mathbb{R}^d$. Then there is a function $h \in V$ such that $\{e^{2\pi i\langle t, Bn\rangle}h(t - Am) : n, m \in \mathbb{Z}^d\}$ is a frame for V if and only if $\sum_{i=1}^{k} ||Pf_i||^2 \leq 1$. In the case that $V = L^2(S)$, there is a function $h \in L^2(S)$ such that $\{e^{2\pi i\langle t, Bn\rangle}h(t - Am) : n, m \in \mathbb{Z}^d\}$ is a frame for $L^2(S)$ if and only if $|det(B)| \cdot |S \cap A[0,1]^d| \leq 1$, where S is an $A\mathbb{Z}^d$-translation invariant measurable subset of $\mathbb{R}^d$ and $|\cdot|$ denotes the Lebesgue measure.*

Proof. We only need to prove the second part. Let σ be the restriction of π_Λ to $V = L^2(S)$ and $G = \mathbb{Z}^d \times \mathbb{Z}^d$. Let f_i and Ω_i be as in Example 7. Then $||Pf_i||^2 = |det(B)| \cdot |\Omega_i|$, and thus

$$Tr_{\sigma(G)'}(I) = \sum_{i=1}^{k} ||Pf_i||^2 = |det(B)| \sum_{i=1}^{k} |S \cap \Omega_i| = |det(B)| \cdot |S \cap A[0,1]^d|. \qquad \square$$

Remark 9. The one-dimensional case is simple and clean: Let $\alpha, \beta > 0$ be such that $\alpha\beta$ is irrational and let $S \subset \mathbb{R}$ be an $\alpha\mathbb{Z}$-translation invariant. Then there is $h \in L^2(S)$ such that $\{e^{2\pi i \beta n t} h(t - \alpha n) : m, n \in \mathbb{Z}\}$ is a frame for $L^2(S)$ if and only if $\beta|S \cap [0, \alpha]| \leq 1$. But this condition is not sufficient if $\alpha\beta$ is rational (see Example 2.5 in [5]).

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