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EXIT PROBABILITY LEVELS OF DIFFUSION PROCESSES

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ABSTRACT. We are interested in the probability that a diffusion process exits a domain between two curved boundaries through the upper one. In case of given boundaries that problem has closed solutions only in some special cases. We study a modification of the problem in which not only the exit probabilities but also the boundaries are unknown. Introducing the notion of exit probability levels, we show that this new problem can be reduced to a single non-linear second order PDE. In case of some important diffusion processes we find large families of solutions to this equation.

1. INTRODUCTION

Let $U(t)$ and $L(t)$ be two smooth functions, $U(0) > L(0)$, and $T = \inf(t > 0 : U(t) = L(t))$, $\inf \phi = \infty$. We consider a diffusion process, driven by the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{1}$$

where B_t is a standard Brownian motion, and the coefficients $b(t, x)$ and $\sigma(t, x)$ are smooth enough to guarantee existence and uniqueness of its solutions for all initial data. Let us define a stopping time

$$\tau(t, x) = \inf(s > t : X_s = U(s) \text{ or } X_s = L(s)),\ L(t) < x < U(t). \tag{2}$$

In this note we study the function

$$v(t, x) = P(X_{\tau(t,x)} = U(\tau(t, x)),\ t < T,\ L(t) < x < U(t).$$

It is well-known that this function satisfies the boundary-value problem

$$v_t + b(t, x)v_x + \frac{1}{2}\sigma^2(t, x)v_{xx} = 0, \tag{3}$$

$$v(t, U(t)) = 1, \tag{4}$$

$$v(t, L(t)) = 0. \tag{5}$$

A possible approach to that problem consists of straightening the boundaries $U(t)$ and $L(t)$ using the substitution

$$v(t, x) = v_1\left(t, \frac{x - L(t)}{U(t) - L(t)}\right). \tag{6}$$

It has been applied by Donchev [3] in case of a Brownian motion. However, even in this special case the substitution (6) works only for some class of square-root boundaries.

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In this paper we use a different approach. The main idea is to transform the original problem (3)-(5) to a dual problem for a function, which defines exit probability levels of the process X_t through the upper boundary $U(t)$.

Obviously, the solution $v(t,x)$ to the problem (3)-(5) is a function which is strictly increasing in x for all fixed $t,\ 0 \leq t < T$. Therefore, in view of conditions (4) and (5), it is a cumulative distribution function given on the interval $[L(t), U(t)]$. Let $f(t,p)$ be the p-quantile of the distribution $v(t,x)$, that is,

$$v(t, f(t,p)) = p,\ 0 \leq t < T. \tag{7}$$

On the other hand, for a fixed p, the function $f(t,p)$ defines a p-exit level of the process X_t through the upper boundary, i.e. the set of all points such that the process X_t, starting from them, with probability p reaches the boundary $U(t)$ before $L(t)$. Applying the chain rule to equation (7), we find the following representation of the partial derivatives of the function $v(t,x)$:

$$v_x = \frac{1}{f_p},\ v_t = -\frac{f_t}{f_p},\ v_{xx} = -\frac{f_{pp}}{f_p^3}. \tag{8}$$

Thus, in view of (7) and (8), the problem (3)-(5) can be reduced to the following problem for the function $f(t,p)$:

$$-f_t f_p^2 + b(t,f) f_p^2 - \frac{1}{2}\sigma^2(t,f) f_{pp} = 0, \tag{9}$$

$$f(t,1) = U(t), \tag{10}$$

$$f(t,0) = L(t). \tag{11}$$

Now, the original

Problem 1. For given boundaries $L(t)$ and $U(t)$ find a function $v(t,x)$ that satisfies equation (3) with boundary conditions (4) and (5),

is replaced by

Problem 2. Find boundaries $L(t)$ and $U(t)$ and a function $v(t,x)$ that satisfies equation (3) with boundary conditions (4) and (5).

In order to solve Problem 2 it is enough to find solutions to equation (9) which are strictly increasing in p. Then, making use of both conditions (10) and (11) as well as relation (7) we can find the boundaries $L(t),\ U(t)$ and the corresponding exit probabilities $v(t,x)$. Thus, the following theorem holds true.

Theorem 1. *Let $f(t,p),\ 0 \leq t \leq T,\ 0 \leq p \leq 1$, be a solution to equation* (9) *which is strongly increasing in p. Then the function $v(t,x)$ determined by* (7) *is a solution to Problem* 2*, corresponding to the boundaries $L(t) = f(t,0),\ U(t) = f(t,1)$ and $T = \inf(t > 0 : U(t) = L(t)),\ \inf \phi = \infty$.*

The main difficulty for applying Theorem 1 is to find a solution to equation (9) itself. Usually, it is a non-linear second order PDE. Here, we are looking for solutions to this equation in the form

$$f(t,p) = y(h(p)r(t)), \tag{12}$$

where all three functions $y,\ h$ and r are unknown. We obtain ODEs for them that can be solved in a closed form.

There are many important applications of the theory discussed here in the sequential analysis Anderson [1] and Lerche [7] and the theory of American and barrier options, Peskir and Shiryaev [10] and Frisgling et al. [9]. However, all existing results relate to Problem 1 and its generalization about the exit densities. Unfortunately, this problem admits closed solutions only in some special cases. For linear boundaries, we refer to Anderson [1], Durbin [4], and Salminen and Yor [13], for square-root boundaries to Shepp [14], Novikov [8], and Yor [13] and for parabolic boundaries to Salminen [12]. Besides the method of PDEs used here, an alternative approach has been developed by Buonocore et al. [2]. The authors have shown that the first crossing densities satisfy a system of second kind Volterra integral equations. For the method of images, see Lerche [7].

Our goal is to show that Problem 2, which is a weaker version of Problem 1, leads to wide family of boundaries for which the corresponding exit probabilities can be found in a closed form. In the rest of the paper we find solutions to Problem 2 in case of some important diffusion processes.

2. A Brownian motion

In case of a Brownian motion, equations (3) and (9) take the form

$$v_t + \frac{1}{2}v_{xx} = 0, \tag{13}$$

$$f_{pp} = -2f_t f_p^2. \tag{14}$$

Let us apply the substitution (12) to equation (14). We shall try to determine both functions $h(p)$ and $r(t)$ in such a way that the function $y = y(z)$ satisfies some ODE. Applying the chain rule to the right-hand side of (12) and substituting it in (14), we obtain an equation for $y(z)$ with coefficients depending on $r(t), h(p), r'(t), h'(p)$ and $h''(p)$. That equation will reduce to an ODE for $y(z)$ if and only if these coefficients depend only on $h(p)r(t)$. Thus, we get the following equations for $h(p)$ and $r(t)$:

$$r' = \frac{\beta}{2}r \tag{15}$$

and

$$\frac{h''}{(h')^2} = \frac{\alpha}{h}, \tag{16}$$

for some constants α and β. We are not interested in the case $\beta = 0$, since then the boundaries are lines parallel to the t-axis. If (15) and (16) hold, then the equation for $y(z)$ becomes

$$y'' = -\beta z(y')^3 - \frac{\alpha}{z}y'. \tag{17}$$

Solving equations (15) and (16) we get

$$r(t) = Ce^{\beta t/2}, \tag{18}$$

$$h(p) = \begin{cases} (Ap + B)^{\frac{1}{1-\alpha}} & \text{if } \alpha \neq 1, \\ Be^{Ap} & \text{if } \alpha = 1, \end{cases} \tag{19}$$

where A, B and C are integration constants. Since y depends on the product $h(p)r(t)$, we may assume that $C = 1$. The equation (17) is Bernoulli's equation for the function $y_1 = y'$. The substitution $y_1 = 1/\sqrt{u}$ reduces it to the following linear ODE for the unknown function u:

$$u' = \frac{2\alpha}{z}u + 2\beta z.$$

Its solution is

$$u(z) = \begin{cases} Cz^{2\alpha} + \frac{\beta}{1-\alpha} z^2 & \text{if } \alpha \neq 1, \\ Cz^2 + 2\beta z^2 \log z & \text{if } \alpha = 1, \end{cases}$$

where C is an integration constant.

Therefore, the solution to equation (17) is

$$y(z) = \begin{cases} \int \frac{dz}{\sqrt{Cz^{2\alpha} + \frac{\beta}{1-\alpha} z^2}} & \text{if } \alpha \neq 1, \\ \int \frac{dz}{z\sqrt{C + 2\beta \log z}} & \text{if } \alpha = 1. \end{cases} \tag{20}$$

Both integrals in the right-hand side of formula (20) admit closed representations. Obviously, if $\alpha = 1$ we have

$$y(z) = \frac{1}{\beta}\sqrt{C + 2\beta \log z}. \tag{21}$$

The case $\alpha \neq 1$ is more complicated. The substitution $s = z^{2\alpha-2}$ reduces the first integral in (20) to

$$\frac{1}{2\alpha - 2} \int \frac{ds}{s\sqrt{Cs + \frac{\beta}{1-\alpha}}}.$$

We neglect possible changes in integration constants and the sign, because they do not affect the solution, and replace the last integral with

$$\frac{1}{2} \int \frac{ds}{s\sqrt{Cs + k}}, \quad k = \beta(1-\alpha).$$

This integral can be found in Brychkov et al. [11, Formula 1.2.19.6] and is equal to

$$\frac{1}{2\sqrt{k}} \log \frac{\sqrt{Cs+k} - \sqrt{k}}{\sqrt{Cs+k} + \sqrt{k}}, \quad \text{if } k > 0,$$

and

$$\frac{1}{-k} \arctan \frac{\sqrt{Cs+k}}{\sqrt{-k}}, \quad \text{if } k < 0.$$

Returning to z and making use of (18) and the first formula in (19) we see that

$$s = z^{2\alpha-2} = e^{-k}(Ap+B)^{-2}.$$

Thus, we obtain the following expression for the function $f(t,p) = y(h(p)r(t))$:

$$f(t,p) = \begin{cases} \frac{1}{2\sqrt{k}} \log \frac{\sqrt{Ce^{-kt} + k(Ap+B)^2} - \sqrt{k}(Ap+B)}{\sqrt{Ce^{-kt} + k(Ap+B)^2} + \sqrt{k}(Ap+B)} & \text{if } k > 0, \\ \frac{1}{\sqrt{-k}} \arctan \frac{\sqrt{Ce^{-kt} + k(Ap+B)^2}}{(Ap+B)\sqrt{-k}} & \text{if } k < 0. \end{cases}$$

The last formula simplifies if we pass to arcsinh and arcsin functions. Then, we finally get

$$f(t,p) = \begin{cases} \frac{1}{\sqrt{k}} \operatorname{arcsinh}((Ap+B)e^{kt/2}) & \text{if } k > 0, \\ \frac{1}{\sqrt{-k}} \arcsin((Ap+B)e^{kt/2}) & \text{if } k < 0. \end{cases} \tag{22}$$

If $\alpha = 1$, formulas (18), (19) and (21) imply that

$$f(t,p) = \sqrt{Ap + B + t}. \tag{23}$$

Now, we are in a position to apply Theorem 1. In order to guarantee that $f(t,p)$ is increasing in p we should take $A > 0$ in (22) and (23). Moreover, obviously

$B > -A$ in the second formula of (22) and (23) in order that both arcsin and the square root are well defined. Setting $f(t,p) = x$ in (22) and (23) and solving the corresponding equations w.r.t. p, we find the exit probabilities $v(t,x)$ that are solutions to Problem 2 in all three cases.

Theorem 2. *The following functions are solutions to Problem* 2:

(i) $v(t,x) = (x^2 - t - B)/A$ *for the boundaries*

$$\begin{aligned} L(t) &= \sqrt{B+t}, \\ U(t) &= \sqrt{A+B+t}. \end{aligned}$$

(ii) $v(t,x) = (\sin(\sqrt{-k}x)e^{-kt/2} - B)/A,\ k < 0,$ *for the boundaries*

$$\begin{aligned} L(t) &= \frac{1}{\sqrt{-k}}\arcsin(Be^{kt/2}), \\ U(t) &= \frac{1}{\sqrt{-k}}\arcsin((A+B)e^{kt/2}). \end{aligned}$$

(iii) $v(t,x) = (\sinh(\sqrt{k}x)e^{-kt/2} - B)/A,\ k > 0,$ *for the boundaries*

$$\begin{aligned} L(t) &= \frac{1}{\sqrt{k}}\operatorname{arcsinh}(Be^{kt/2}), \\ U(t) &= \frac{1}{\sqrt{k}}\operatorname{arcsinh}((A+B)e^{kt/2}). \end{aligned}$$

In all three cases the probabilities of reaching the boundary $L(t)$ *before* $U(t)$ *are equal to* $1 - v(t,x)$.

Remarks. The boundaries $L(t)$ and $U(t)$ from point (i) do not coincide with any known cases (Novikov [8], Donchev [3]) of square-root boundaries. For all two-sided square-root boundaries studied so far, it holds that $\lim_{t\to\infty}(U(t) - L(t)) = \infty$. The same limit for the square-root boundaries in Theorem 2 is zero.

For large t, the boundaries from point (iii) behave like two parallel straight lines. In this case, the exit probabilities $v(t,x)$ are good approximations of the complicated series expansion formulas obtained by Anderson [1], and Salminen and Yor [13].

3. Square-root processes

In this section we consider square-root processes driven by the equation

$$dX_t = (b - X_t)dt + \sqrt{X_t}dB_t. \tag{24}$$

It is well-known (see Ikeda and Watanabe [5, Part 4, Section 8] that the boundary $x = 0$ is non-singular if $0 < b < 1/2$ and unattainable if $b \geq 1/2$. We are interested in the last case and take $b = 1$ in (24). Then, equations (3) and (9) reduce to

$$v_t + (1-x)v_x + \frac{1}{2}xv_{xx} = 0, \tag{25}$$

$$-f_t + 1 - f + \frac{1}{2}\frac{ff_{pp}}{f_p^2} = 0. \tag{26}$$

We are going to simplify equation (25) by means of two substitutions. First, we set $f = 1/f_1$ and get the following equation for f_1:

$$\frac{1}{2}\frac{f_1(f_1)_{pp}}{(f_1)_p^2} = \frac{1}{f_1} - \frac{(f_1)_t}{f_1^2}.$$

We use the substitution $f_1 = \log f_2$, which is well defined since all exit probability levels of the process are strictly positive. Then, the function f_2 satisfies the equation

$$\frac{1}{2}\left(\frac{(f_2)_{pp}}{(f_2)_p^2}+1\right) = e^{-f_2}(1-(f_2)_t). \tag{27}$$

Equation (27) admits separation of variables if we set $f_2(t,p) = r(t) + h(p)$. Then, for the unknown functions $r(t)$ and $h(p)$ we get

$$\frac{1}{2}\left(\frac{h''}{(h')^2}+1\right)e^h = (1-r')e^{-r} = \lambda = const. \tag{28}$$

The equation for $r(t)$ is a first order ODE. The equation for $h(p)$,

$$\frac{h''}{(h')^2} - 2\lambda e^{-h} + 1 = 0, \tag{29}$$

is of a second order which permits reduction to a first-order ODE, since it does not depend on p. The standard substitution to solve it is $g = h',\ g = g(h)$. Then $h'' = gg'$, and we obtain the following equation for g:

$$\frac{g'}{g} - 2\lambda e^{-h} + 1 = 0.$$

The solution to this equation is

$$g(h) = C_1 \exp(h + 2\lambda e^{-h}),$$

and the unknown function $h(p)$ can be found from the relation

$$p = \int \frac{dh}{g(h)}.$$

Thus, we get

$$p = C_1 \exp(-2\lambda e^{-h}) + C_2,$$

C_1 and C_2 being integration constants. Solving the last equation w.r.t. h we find a solution to (29),

$$h(p) = -\log\log(ap+b)^{-1/2\lambda}, \tag{30}$$

with two other integration constants a and b. In view of (28), the equation for $r(t)$ is $1 - r' = \lambda e^r$, and it has a solution

$$r(t) = \log\frac{Ae^t}{1+Be^t}. \tag{31}$$

Taking into consideration (30) and (31), we find the functions $f_2(t,p), f_1(t,p)$ and $f(t,p)$. For the latter, we get

$$f(t,p) = \exp(-h(p) - r(t)) = (Ae^{-t} + B)\log(ap+b). \tag{32}$$

Now, we are able to repeat the procedure in the end of Section 2, that is, to set $p = 0, 1$, in equation (32), in order to find the boundaries $L(t)$ and $U(t)$. Inverting $f(t,p)$ w.r.t. p yields

$$v(t,x) = a^{-1}\exp\left(\frac{x}{Ae^{-t}+B}\right) - \frac{b}{a}. \tag{33}$$

Of course, one should take care that the integration constants A, B, a and b are such that the function $f(t,p)$ is non-negative and increasing in p. This will be the case if $B > 0, A > -B, a > 0$ and $b > 1$. Since the process X_t is recurrent, the probability of hitting $L(t)$ before $U(t)$ is $1 - v(t,x)$.

Theorem 3. *Let the process X_t satisfy equation* (24) *and $L(t)$ and $U(t)$ be boundaries*

$$\begin{aligned} L(t) &= (Ae^{-t}+B)\log b, \\ U(t) &= (Ae^{-t}+B)\log(a+b), \end{aligned}$$

with constants A, B, a and b that meet the conditions mentioned above. If $\tau(t,x)$ and $v(t,x)$ are being given by formulas (2) *and* (33), *respectively, then*

$$P(X_{\tau(t,x)} = U(t)) = v(t,x), \tag{34}$$

$$P(X_{\tau(t,x)} = L(t)) = 1 - v(t,x), \tag{35}$$

provided $L(t) < x < U(t)$.

4. A pinned Brownian motion

A pinned Brownian motion is a conditional process satisfying the stochastic differential equation

$$\begin{aligned} dX_t &= \frac{b - X_t}{T-t}dt + dB_t, \\ X_0 &= x. \end{aligned}$$

For this process equation (3) reduces to

$$v_t + \frac{b-x}{T-t}v_x + \frac{1}{2}v_{xx} = 0. \tag{36}$$

The following three substitutions transform (36) into equation (13). First, we set $v(t,x) = v_1(T-t, b-x)$, and for $v_1(t,x)$ we get

$$-(v_1)_t - \frac{x}{t}(v_1)_x + \frac{1}{2}(v_1)_{xx} = 0. \tag{37}$$

The next substitution is $v_1(t,x) = v_2(t^{-1}, x)$, and equation (37) turns into

$$t^2(v_2)_t - tx(v_2)_x + \frac{1}{2}(v_2)_{xx} = 0.$$

Dividing it by t^2, we get that $v_2(t,x)$ satisfies

$$(v_2)_t - \frac{x}{t}(v_2)_x + \frac{1}{2t^2}(v_2)_{xx} = 0. \tag{38}$$

Finally, we set $v_2(t,x) = v_3(t,tx)$ in (38) to see that v_3 solves equation (13). Taking into consideration all these substitutions, we obtain

$$v(t,x) = v_3\left(\frac{1}{T-t}, \frac{b-x}{T-t}\right).$$

Making use of (7) for both v, f and v_3, f_3, we easily get that

$$f(t,p) = b - (T-t)f_3\left(\frac{1}{T-t}, p\right),$$

$f_3(t,p)$ being exit probability levels of a Brownian motion, corresponding to boundaries $L_3(t) = f_3(t,0)$ and $U_3(t) = f_3(t,1)$. The last two functions are decreasing in x and p, respectively, and in fact they define the exit probability levels for the lower boundary. The corresponding exit probabilities and probability levels for the upper boundary are $1 - v(t,x)$ and $f(t, 1-p)$, respectively. We have established the following result.

Theorem 4. *Let $v_3(t,x)$ and $f_3(t,p)$ be solutions to Problem 2 for a Brownian motion, corresponding to boundaries $L_3(t) = f_3(t,0)$ and $U_3(t) = f_3(t,1)$. Then, the functions $v(t,x)$ and $f(t,p)$ given by the formulas*

$$v(t,x) = 1 - v_3\left(\frac{1}{T-t}, \frac{b-x}{T-t}\right), \tag{39}$$

$$f(t,p) = b - (T-t) f_3\left(\frac{1}{T-t}, 1-p\right) \tag{40}$$

are solutions to the same problem for a pinned Brownian motion, corresponding to boundaries $L(t) = f(t,0)$ and $U(t) = f(t,1)$.

Now, we are in a position to combine this result with Theorem 2 and obtain a number of solutions to Problem 2 for a pinned Brownian motion. For example, let us take $v(t,x)$ and $f(t,p)$ from the point (i) of Theorem 2. Then, (39) and (40) yield

$$\begin{aligned} v(t,x) &= 1 - \left(\left(\frac{b-x}{T-t}\right)^2 - \frac{1}{T-t} - B\right)/A, \\ f(t,p) &= b - (T-t)\left(\sqrt{\frac{1}{T-t} + A(1-p) + B}\right), \\ L(t) &= f(t,0),\ U(t) = f(t,1),\ L(t) \le x \le U(t). \end{aligned}$$

As expected, $f(t,p) \to b,\ t \uparrow T$. Moreover, $f_t(t,p) \to \infty,\ t \uparrow T$.

We obtain another interesting example if we consider the trivial case of boundaries $L_3(t) = c,\ U_3(t) = d > c$. Then,

$$\begin{aligned} v_3(t,x) &= \frac{x-c}{d-c},\ c \le x \le d, \\ f_3(t,p) &= c + (d-c)p,\ 0 \le p \le 1, \end{aligned}$$

and Theorem 4 yields

$$\begin{aligned} v(t,x) &= 1 - \left(\frac{b-x}{T-t} - c\right)/(d-c), \\ f(t,p) &= b - (T-t)(c + (d-c)(1-p)), \\ L(t) &= f(t,0),\ U(t) = f(t,1),\ L(t) \le x \le U(t). \end{aligned}$$

In this case, the exit probability levels of the pinned Brownian motion are straight lines that intersect at a point (T,b).

5. Bessel processes

The Bessel processes are diffusion processes with an infinitesimal generator

$$L_d = \frac{1}{2}\frac{d^2}{dx^2} + \frac{d-1}{2x}\frac{d}{dx}, \tag{41}$$

where the number $d > 0$ is an index of the process. They satisfy the stochastic differential equation

$$dX_t = \frac{d-1}{2X_t}dt + dB_t.$$

We are going to apply Theorem 1 to these processes in order to find a large family of two-sided boundaries for which the problem (3)-(5) has an explicit solution.

In view of (41), equation (3) reduces to

$$v_t + \frac{d-1}{2x} v_x + \frac{1}{2} v_{xx} = 0. \tag{42}$$

On the other hand, making use of (9), we obtain the following equation for the function $f(t,p)$:

$$f_{pp} = (d-1)\frac{f_p^2}{f} - 2 f_t f_p^2. \tag{43}$$

Repeating the same arguments as in Section 2 and making use of the substitution (12), we get that functions $r(t)$ and $h(p)$ must satisfy equations (15) and (16) for some constants α and β. If relations (15) and (16) hold, then the function $y(s)$ satisfies the second-order ODE

$$y'' = (d-1)\frac{(y')^2}{y} - \frac{\alpha}{s} y' - \beta s (y')^3. \tag{44}$$

The solutions to equations (15) and (16) are given by formulas (18) and (19). Obviously, the constants A and B in equation (19) must satisfy $B > 0$ and $A+B > 0$ if $\alpha \neq 1$. As for equation (44), its solution depends essentially on the constants α and β. Firstly, we consider the case $\alpha \neq 1,\ \beta \neq 0$. By means of two substitutions equation (44) reduces to Ricatti's equation. We set $y_1(s) = y(s^{1-\alpha})$ and obtain the following equation for $y_1(s)$:

$$y_1'' = (d-1)\frac{(y_1')^2}{y_1} - cs(y_1')^3,\ c = \beta(1-\alpha). \tag{45}$$

The second substitution, $y_2(s) = y_1(\ln(s))$, reduces equation (45) to

$$y_2'' = (d-1)\frac{(y_2')^2}{y_2} + y_2' - c(y_2')^3.$$

The last equation admits reducing of the order by means of standard substitutions:

$$y_2' = z(y_2),\ y_2'' = zz'. \tag{46}$$

The function z satisfies Ricatti's equation

$$z' = 1 + \frac{d-1}{y_2} z - cz^2. \tag{47}$$

The corresponding second order linear ODE is

$$u'' - \frac{d-1}{y_2} u' - cu = 0. \tag{48}$$

The relation between the solutions of equation (47) and equation (48) is

$$z = \frac{u'}{cu}.$$

Equation (48) is well-known in the literature (see Kamke [6]), and its solution admits a representation in terms of Bessel functions. We have

$$u(y_2) = y_2^{\mu} Z_{\mu}(i\sqrt{c} y_2),\ \mu = d/2,$$

where

$$Z_{\mu}(x) = C_1 I_{\mu}(x) + C_2 K_{\mu}(x),$$

$I_\mu(x)$ and $K_\mu(x)$ being the modified Bessel functions. Consider the functions

$$Y(y_2) = \int \frac{dy_2}{z(y_2)} = \int \frac{u(y_2)dy_2}{cu'(y_2)},\ Y_1 = e^Y. \tag{49}$$

We assume that both constants C_1 and C_2 can be chosen in such a way that the functions Y and Y_1 are monotone. Then, in view of the first relation in (46) we obtain

$$\begin{aligned} y_2(s) &= Y^{-1}(s), \\ y_1(s) &= Y^{-1}(\ln(s)) = Y_1^{-1}(s), \end{aligned}$$

where Y^{-1} and Y_1^{-1} are functions inverse to Y and Y_1, respectively. Finally, for the function $f(t,p)$ we get

$$f(t,p) = y(h(p)r(t)) = y_1((h(p)r(t))^{1-\alpha}) = Y_1^{-1}\left((Ap+B)e^{ct/2}\right). \tag{50}$$

It follows from Theorem 1 that to find solutions to Problem 2, we need to know the function Y_1^{-1} in the right-hand side of equation (50). Thus, we are in a position to state our main result.

Theorem 5. *Let integration constants C_1 and C_2 be chosen in such a way that the function $Y(y_2)$ in equation* (49) *is increasing w.r.t. y_2. Then, a solution to Problem* 2 *is the triple*

$$\begin{aligned} L(t) &= Y_1^{-1}\left(Be^{ct/2}\right), \\ U(t) &= Y_1^{-1}\left((A+B)e^{ct/2}\right), \\ v(t,x) &= \frac{Y_1(x)e^{-ct/2} - B}{A}, \end{aligned}$$

where $A > 0,\ B > 0,\ L(t) \le x \le U(t)$.

If the function $Y(y_2)$ is decreasing, then $B > 0,\ -B < A < 0$, and the solution is

$$\begin{aligned} L(t) &= Y_1^{-1}\left((A+B)e^{ct/2}\right), \\ U(t) &= Y_1^{-1}\left(Be^{ct/2}\right), \\ v(t,x) &= \frac{Y_1(x)e^{-ct/2} - B}{A},\ L(t) \le x \le U(t). \end{aligned}$$

5.1. **Special cases $\beta = 0$ or $\alpha = 1$.** These two cases need special attention since the approach demonstrated above does not work. The case $\beta = 0$ is trivial. Indeed, in that case both boundaries are straight lines parallel to the time axis, and equation (3) is reduced to an easy first order ODE.

The case $\alpha = 1$ is far more interesting. Here, we find a solution to Problem 2 in this case when $d = 3$. Now, equation (44) becomes

$$y'' = 2\frac{(y')^2}{y} - \beta s(y')^3 - \frac{y'}{s}. \tag{51}$$

The substitution $y' = z(s,y)$ reduces it to the following quasi-linear first order PDE for the new unknown function $z(s,y)$:

$$z_s + zz_y = 2\frac{z^2}{y} - \beta s z^3 - \frac{z}{s}. \tag{52}$$

The equation (52) closely relates to the following first order linear PDE:

$$w_s + zw_y + \left(2\frac{z^2}{y} - \beta s z^3 - \frac{z}{s}\right) w_z = 0. \tag{53}$$

Namely, if $w(s, y, z)$ is a solution to equation (53), then the function $z(s, y)$ defined as an implicit function from the relation

$$w(s, y, z(s, y)) = 0$$

is a solution to equation (52). Thanks to a lucky guess, we have found the following partial solution to equation (52):

$$z = \frac{3}{\beta s y}.$$

Making use of it, we are looking for a first integral to equation (53) in the form

$$w = \varphi(s, y)(3z^{-1} - \beta s y),$$

where $\varphi(s, y)$ is a new unknown function. Substituting the right-hand side of the last formula in equation (53), we find that the function $\varphi(s, y)$ should satisfy the following conditions:

$$2\varphi - y\varphi_y = 0, \;\; \varphi + s\varphi_s = 0,$$
$$\beta s y \varphi_s - 3\varphi_y + \left(\beta y + \frac{6}{y}\right)\varphi = 0.$$

The third condition above follows from the first two. On the other hand, they yield

$$\varphi = \frac{y^2}{s}.$$

Thus, the first integral of equation (53) is

$$w_1 = \frac{3y^2}{sz} - \beta y^3.$$

Solving the equation $w_1 = C_1$ w.r.t. z, we find a solution to equation (52):

$$z = \frac{3y^2}{s(C_1 + \beta y^3)}.$$

Finally, solving the equation $y' = z$, we find the general solution to equation (51):

$$y^3 + \left(C_1 - \frac{6}{\beta}\ln s\right) y + C_2 = 0. \tag{54}$$

Now, we should distinguish between the following two cases:

5.1.1. *The case* $C_2 = 0$. In this case, besides the trivial solution $y = 0$, from equation (54) we get

$$y(s) = \sqrt{\frac{6}{\beta}\ln s - C_1}.$$

In the last equation we set $s = h(p)r(t) = B\exp(Ap + \beta t/2)$. Thus, we find the exit probability levels $f(t, p)$:

$$f(t, p) = \sqrt{Ap + 3t + B}, \;\; A > 0, \;\; B > 0. \tag{55}$$

5.1.2. *The case $C_2 < 0$.* This condition guarantees that the third order equation (54) has a positive solution. It is convenient to set $C_2 = -2C$, $C > 0$, $q = C_1 - 2\ln s/\beta$. Then, equation (54) reduces to

$$y^3 + 3qy - 2C = 0.$$

Cardano's formula yields

$$y(q) = \sqrt[3]{C + \sqrt{C^2 + q^3}} + \sqrt[3]{C - \sqrt{C^2 + q^3}}.$$

Setting again $s = h(p)r(t) = B\exp(Ap + \beta t/2)$, after reduction of constants, we find that

$$f(t,p) = \sqrt[3]{C + \sqrt{C^2 + (B - Ap - t)^3}} + \sqrt[3]{C - \sqrt{C^2 + (B - Ap - t)^3}}. \tag{56}$$

Making use of equations (55) and (56), we come to the following result.

Theorem 6. *Let $d = 3$. Then, besides the solutions given by Theorem 5, Problem 2 has the following solutions $(L(t), U(t), v(t,x))$:*

$$\begin{aligned}
L(t) &= \sqrt{B + 3t}, \\
U(t) &= \sqrt{A + B + 3t}, \\
v(t,x) &= \frac{x^2 - 3t - B}{A}, \\
L(t) &= \sqrt[3]{C + \sqrt{C^2 + (B - t)^3}} + \sqrt[3]{C - \sqrt{C^2 + (B - t)^3}}, \\
U(t) &= \sqrt[3]{C + \sqrt{C^2 + (B - A - t)^3}} + \sqrt[3]{C - \sqrt{C^2 + (B - A - t)^3}}, \\
v(t,x) &= \frac{B - t}{A} - \frac{2C - x^3}{3Ax},
\end{aligned}$$

where $A > 0$, $B > 0$, $C > 0$ and $L(t) \le x \le U(t)$.

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