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COMPLEX SUPERMANIFOLDS OF ODD DIMENSION BEYOND 5

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Abstract. Any non-split complex supermanifold is a deformation of a split supermanifold. These deformations are classified by group orbits in a non-abelian cohomology. For the case of a split supermanifold with no global nilpotent even vector fields, an injection of this non-abelian cohomology into an abelian cohomology is constructed. The cochains in the non-abelian complex appear as exponentials of cochains of nilpotent even derivations. Necessary conditions for a recursive construction of these cochains of derivations are analyzed up to terms of degree six. Results on classes of examples of supermanifolds of odd dimension beyond 5 are deduced.

Complex supermanifolds appear as deformations of split complex supermanifolds $(M, \mathcal{O}_{\Lambda E})$, where $E \to M$ is a complex vector bundle and $\mathcal{O}_{\Lambda E}$ denotes the sheaf of holomorphic sections of ΛE. These deformations of a split complex supermanifold can be parametrized by $H^0(M, Aut(E))$-orbits in a non-abelian first Čech cohomology $H^1(M, G_E)$ (see [1]). The cocycles of this cohomology appear as exponentials of nilpotent derivations u in $C^1(M, Der^{(2)}(\Lambda E))$ (see [4]). Here $Der^{(2)}(\Lambda E)$ denotes the even derivations of $\mathcal{O}_{\Lambda E}$ that increase the degree by at least two. In detail, the cochain u has to satisfy the non-abelian cocycle condition $\mathbf{d}\exp(u) := (\exp(u_{ij})\exp(u_{jk})\exp(u_{ki}))_{ijk} = Id$. Due to nilpotency, the appearing exponential series are finite and their length increases with the rank of E. So the non-abelian cocycle condition on u becomes more and more complicated with higher odd dimension.

A naturally arising computational question is how to find suitable u that yield supermanifold structures. We are aiming at the questions:

(A) Is it possible to express the non-abelian cocycle condition on u up to non-abelian coboundaries as conditions in the abelian cohomology given by $H^1(M, Der^{(2)}(\Lambda E))$?

The $\mathbb{Z}$-grading of $\mathcal{O}_{\Lambda E}$ induces a $\mathbb{Z}$-grading on $Der^{(2)}(\Lambda E)$. So u is a finite sum $u_2 + u_4 + u_6 + \dots$. Let $2 \leq 2q \leq rank(E)$.

(B) What are the necessary and sufficient conditions on a sum $u_2 + \dots + u_{2q-2}$ to be extendable to a $u \in C^1(M, Der^{(2)}(\Lambda E))$ that defines a supermanifold structure?

In the first section we answer question (A) for split complex supermanifolds with no global even vector fields that increase the degree by two or more. Speaking of

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automorphisms of the split supermanifold, this condition $H^0(M, Der^{(2)}(\Lambda E)) = 0$ can be reformulated as follows: there is no automorphism whose degree preserving part is the identity but the identity itself. Under this condition we construct a well-defined injection $\sigma_D : H^1(M, G_E) \to H^1(M, Der^{(2)}(\Lambda E))$. This generalizes a result on supermanifolds of odd dimension up to 5 in [2]. Note at this point that differing methods for determining the cohomology $H^1(M, G_E)$ can be found in [3].

For question B, assume that $u_{(2q-2)} = u_2 + \cdots + u_{2q-2}$ satisfies the non-abelian cocycle condition $\mathbf{d}\exp(u_{(2q-2)}) = Id$ up to terms of degree $2q$ and higher. The necessary and sufficient condition for the existence of a $u_{(2q)} := u_{(2q-2)} + u_{2q}$ satisfying $\mathbf{d}\exp(u_{(2q)}) = Id$ up to terms of degree $2q+2$ and higher is $pr_{2q}\mathbf{d}\exp(u_{(2q-2)}) \in B^2(M, Der_{2q}(\Lambda E))$. Here pr_{2q} denotes the projection onto the degree $2q$ component. In general it is not at all clear that $pr_{2q}\mathbf{d}\exp(u_{(2q-2)})$ lies in either $Z^2(M, End_{2q}(\Lambda E))$ or $C^2(M, Der_{2q}(\Lambda E))$. However, we show in the second section that for $q = 2, 3$, the condition $pr_{2q}\mathbf{d}\exp(u_{(2q-2)}) \in Z^2(M, End_{2q}(\Lambda E))$ is automatically satisfied. Even better, the condition $pr_6\mathbf{d}\exp(u_{(4)}) \in Z^2(M, Der_6(\Lambda E))$ only depends on u_2. This yields results for several classes of examples where the cohomology $H^2(M, Der_{2q}(\Lambda E))$ vanishes.

We fix the notation. Let $E \to M$ be a holomorphic vector bundle on a complex manifold M. Denote by $\mathcal{O}_E$ its sheaf of sections, by $\mathcal{O}_{\Lambda E}$ the associated exterior algebra, by $Aut(\Lambda E)$ the sheaf of automorphisms of the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf of algebras $\mathcal{O}_{\Lambda E}$ and by $Der(\Lambda E)$ and $End(\Lambda E)$ the sheaves of even $\mathbb{C}$-linear derivations, resp. endomorphisms of $\mathcal{O}_{\Lambda E}$. Note that the last two sheaves carry a natural $2\mathbb{Z}$-grading $Der_{2k}(\Lambda E)$, resp. $End_{2k}(\Lambda E)$, given by the condition $u(\mathcal{O}_{\Lambda^j E}) \subset \mathcal{O}_{\Lambda^{j+2k} E}$ for all $j \geq 0$. Furthermore set for $k \geq 0$,

$$\begin{aligned} &Der^{(2k)}(\Lambda E) := \bigoplus_{\ell=k}^{\infty} Der_{2\ell}(\Lambda E)\ , && End^{(2k)}(\Lambda E) := \bigoplus_{\ell=k}^{\infty} End_{2\ell}(\Lambda E)\ , \\ &Der_{(2k)}(\Lambda E) := \bigoplus_{\ell=1}^{k} Der_{2\ell}(\Lambda E) \qquad \text{and} && End_{(2k)}(\Lambda E) := \bigoplus_{\ell=1}^{k} End_{2\ell}(\Lambda E)\ . \end{aligned}$$

Denote for $k \geq 1$ the induced projections by $pr_{2k} : End^{(2)}(\Lambda E) \to End_{2k}(\Lambda E)$ and further $pr_{(2k)} : End^{(2)}(\Lambda E) \to End_{(2k)}(\Lambda E)$. Let $G_E \subset Aut(\Lambda E)$ denote the subsheaf of automorphisms satisfying $(\varphi - Id)(\mathcal{O}_{\Lambda^j E}) \subset \bigoplus_{k \geq 1} \mathcal{O}_{\Lambda^{j+2k} E} \quad \forall j \geq 0$. It was shown in [4] that the exponential $\exp : End(\Lambda E) \to Aut(\Lambda E)$ yields a bijection between $Der^{(2)}(\Lambda E)$ and G_E. We frequently use that $pr_{(2q)} \circ f \circ pr_{(2q)} = pr_{(2q)} \circ f$ for $f = \exp$ or $\log$. In the following $\mathbf{d} : C^1(M, Aut(\Lambda E)) \to C^2(M, Aut(\Lambda E))$ denotes the coboundary map with respect to composition. In contrast denote by $d : C^1(M, End^{(2)}(\Lambda E)) \to C^2(M, End^{(2)}(\Lambda E))$ the coboundary map with respect to addition.

Starting on the other hand with a complex supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, the nilpotent elements $\mathcal{O}_{\mathcal{M}}^{nil}$ define the locally free $\mathcal{O}_M$-module $\mathcal{O}_{\mathcal{M}}^{nil}/(\mathcal{O}_{\mathcal{M}}^{nil})^2$ yielding a holomorphic vector bundle E. Denote by $Aut(E)$ the sheaf of automorphisms of the vector bundle E over the identity. It is shown in [1] that the isomorphism classes of complex supermanifolds defining in this way the isomorphism class of a fixed vector bundle E over M are parametrized by the $H^0(M, Aut(E))$-orbits on the Čech cohomology $H^1(M, G_E)$. Note that this cohomology is defined with respect to the composition of maps and hence is non-abelian in general.

1. Embedding non-abelian in abelian cohomology

Aiming at an embedding of the cohomology $H^1(M, G_E)$ into the abelian cohomology of sheaves of $\mathcal{O}_M$-modules $H^1(M, Der^{(2)}(\Lambda E))$, define for $q \geq 2$ the maps

$$\begin{array}{cccc} R_{2q}: & C^1(M, Der^{(2)}(\Lambda E)) & \longrightarrow & C^2(M, End_{2q}(\Lambda E)), \\ & u & \longmapsto & pr_{2q}\big(\mathbf{d}\exp(pr_{(2q-2)}(u))\big). \end{array}$$

Denote

$$\tilde{C}^1(M, Der^{(2)}(\Lambda E)) := \{u \in C^1(M, Der^{(2)}(\Lambda E)) \mid \exp(u) \in Z^1(M, G_E)\}.$$

Note that $pr_{2q}\big(\mathbf{d}\exp(u)\big) = d(pr_{2q}u) + pr_{2q}(\mathbf{d}\exp(pr_{(2q-2)}(u)))$. It follows that $\exp(u) \in Z^1(M, G_E)$ is equivalent to $R_{2q}(u) = -d(pr_{2q}(u))$ for all $q \geq 2$ and $d(pr_2(u)) = 0$. Hence the images of $\tilde{C}^1(M, Der^{(2)}(\Lambda E))$ under the maps R_{2q} lie in $B^2(M, Der_{2q}(\Lambda E))$, respectively. Note that the maps R_{2q} only depend on the component in $Der_{(2q-2)}(\Lambda E)$ of the argument. Hence it is possible to choose maps

$$D'_{2q}: \tilde{C}^1(M, Der^{(2)}(\Lambda E)) \to C^1(M, Der_{2q}(\Lambda E)), \qquad \text{for } q \geq 2,$$

that factorize over $pr_{(2q-2)}: \tilde{C}^1(M, Der^{(2)}(\Lambda E)) \to C^1(M, Der^{(2)}(\Lambda E))$ and satisfy the equation $dD'_{2q}(u) = R_{2q}(u)$ for all $u \in \tilde{C}^1(M, Der^{(2)}(\Lambda E))$. Now set for $q \geq 2$:[1]

$$\begin{aligned} F_{2q}: &C^0(M, Der^{(2)}(\Lambda E)) \times C^1(M, Der^{(2)}(\Lambda E)) \longrightarrow C^1(M, End^{(2)}(\Lambda E)) \\ &(v, u) \longmapsto pr_{(2q)}(\exp(pr_{(2q-2)}(v)).\exp(pr_{(2q-2)}(u))). \end{aligned}$$

Set $\lambda(v,u) := \log(\exp(v).\exp(u))$ for (v,u) in the product $C^0(M, Der^{(2)}(\Lambda E)) \times C^1(M, Der^{(2)}(\Lambda E))$ and note that $u \in \tilde{C}^1(M, Der^{(2)}(\Lambda E))$ yields the property $\lambda(v,u) \in \tilde{C}^1(M, Der^{(2)}(\Lambda E))$. Continue the D'_{2q}, $q \geq 2$, to maps

$$\begin{aligned} D_{2q}: \ H^0(M, Aut(E)) \times C^0(M, Der^{(2)}(\Lambda E)) \times \tilde{C}^1(M, Der^{(2)}(\Lambda E)) \\ \longrightarrow C^1(M, Der_{2q}(\Lambda E)) \end{aligned}$$

via $D_{2q}(Id, 0, u) := D'_{2q}(u)$ and

$$\varphi.D_{2q}(\varphi, v, u) := D_{2q}(Id, 0, \varphi.\lambda(v,u)) + pr_{2q}(\log(F_{2q}(\varphi.v, \varphi.u))) \tag{1.1}$$

and set $D = \sum_{q=2}^{\infty} D_{2q}$.

Proposition 1.1. *If there is a choice of maps D'_{2q}, $q \geq 2$, satisfying*

$$D'_{2q}(\lambda(v,u)) = D'_{2q}(u) - pr_{2q}(\log(F_{2q}(v,u))) \tag{1.2}$$

for all $(v,u) \in C^0(M, Der^{(2)}(\Lambda E)) \times \tilde{C}^1(M, Der^{(2)}(\Lambda E))$, then the induced map

$$\sigma_D: \ H^1(M, G_E) \longmapsto H^1(M, Der^{(2)}(\Lambda E))$$

given by $\sigma_D([\exp(u)]) = [D(Id, 0, u) + u]$ is well-defined and injective. If additionally the D'_{2q} are $H^0(M, Aut(E))$-equivariant, then σ_D is $H^0(M, Aut(E))$-equivariant.

Proof. For $\exp(u) \in Z^1(M, G_E)$ we have $dD_{2q}(Id, 0, u) = dD'_{2q}(u) = R_{2q}(u) = -d(pr_{2q}(u))$ for all $q \geq 2$ and $dpr_2(u) = 0$. So we find

$$D(Id, 0, u) + u \in Z^1(M, Der^{(2)}(\Lambda E)).$$

Further note that

$$pr_{(2q)}(\exp(v).\exp(u)) = dpr_{2q}(v) + pr_{2q}(u) + F_{2q}(v,u).$$

[1] We denote $(exp(v).exp(u))_{ij} = \exp(v_i)\exp(u_{ij})\exp(-v_j)$.

Using this, $H^0(M, Aut(E))$-equivariance of λ and F_{2q}, and reasons of degree:

$$\begin{aligned} pr_{2q}(\varphi.\lambda(v,u)) &= pr_{2q}\log(pr_{(2q)}(\exp(\varphi.v).\exp(\varphi.u))) \\ &= pr_{2q}\log(dpr_{2q}(\varphi.v) + pr_{2q}(\varphi.u) + F_{2q}(\varphi.v,\varphi.u)) \\ &= pr_{2q}(d\varphi.v + \varphi.u) + pr_{2q}(\log(F_{2q}(\varphi.v,\varphi.u))). \end{aligned}$$

So

$$\begin{aligned} &\sigma_D([\varphi.(\exp(v).\exp(u))]) = \sigma_D([\exp(\varphi.\lambda(v,u))]) \\ &= [D(Id,0,\varphi.\lambda(v,u)) + d\varphi.v + \varphi.u + \textstyle\sum_{q=2}^{\infty} pr_{2q}(\log(F_{2q}(\varphi.v,\varphi.u)))], \end{aligned}$$

differing from $\varphi.\sigma_D([\exp(u)])$ with (1.1) and (1.2) by

$$d\varphi.v + D(Id,0,\varphi.u) - \varphi.D(Id,0,u).$$

For $\varphi = Id$, the first part of the proposition follows. The second statement follows for $H^0(M, Aut(E))$-equivariant D'_{2q}. □

Corollary 1.2. *If $H^0(M, Der^{(2)}(\Lambda E)) = 0$, then there is a D such that σ_D is well-defined and injective.*

Proof. We show that there exists a choice of the D'_{2q} satisfying (1.2). First we have to check that D'_{2q} can be well-defined as a map satisfying (1.2), that is, that $\lambda(v,u) = u$ includes $pr_{2q}(\log(F_{2q}(v,u))) = 0$ for all $q \geq 2$. We follow by induction that $\lambda(v,u) = u$ includes $v = 0$: $pr_0(v) = 0$. Assume that $pr_{2s}(v) = 0$ for all $s < q$; then

$$0 = pr_{2q}(\lambda(v,u) - u) = pr_{2q}(\log(\exp(pr_{(2q-2)}(v) + pr_{2q}(v)).\exp(u)) - u) = pr_{2q}(dv),$$

and due to $H^0(M, Der^{(2)}(\Lambda E)) = 0$ we have $pr_{2q}(v) = 0$. Finally,

$$pr_{2q}(\log(F_{2q}(0,u))) = pr_{2q}(pr_{(2q-2)}(u)) = 0.$$

Second we check that (1.2) does not contradict the derivative conditions $dD'_{2q}(u) = R_{2q}(u)$ for $q \geq 2$ and $d(pr_2(u)) = 0$. Deriving (1.2), this is equivalent to checking whether

$$pr_{2q}(\mathbf{d}\exp(pr_{(2q-2)}(u)) - \mathbf{d}\exp(pr_{(2q-2)}(\lambda(v,u))) - d\log(F_{2q}(v,u))) = 0. \tag{1.3}$$

Now for reasons of degree:

$$\begin{aligned} pr_{(2q-2)}(\lambda(v,u)) &= pr_{(2q-2)}(\log(\exp(v).\exp(u))) = pr_{(2q-2)}(\log(F_{2q}(v,u))) \\ &= pr_{(2q)}(\log(F_{2q}(v,u))) - pr_{2q}(\log(F_{2q}(v,u))) \end{aligned}$$

and

$$pr_{(2q)}(\exp(pr_{(2q-2)}(\lambda(v,u)))) = pr_{(2q)}(F_{2q}(v,u)) - pr_{2q}(\log(F_{2q}(v,u))).$$

Using this, (1.3) is equivalent to

$$pr_{2q}(\mathbf{d}\exp(pr_{(2q-2)}(u)) - \mathbf{d}F_{2q}(v,u)) = 0.$$

This always holds since for reasons of degree:

$$\begin{aligned} \mathbf{d}F_{2q}(v,u) &= pr_{(2q)}\mathbf{d}(\exp(pr_{(2q-2)}(v)).\exp(pr_{(2q-2)}(u))) \\ &= pr_{(2q)}(\mathbf{d}\exp(pr_{(2q-2)}(u))). \end{aligned}$$

□

Here is a class of examples where the above results are of computational help.

Remark 1.3. Let M be a compact connected complex manifold, let L be a line bundle on M and set $E = m \cdot L$ for an $m \in \mathbb{N}$. Assuming the condition in Corollary 1.2, $H^1(M, G_E)$ can be injectively mapped to $V := H^1(M, Der^{(2)}(\Lambda E))$ by a σ_D. Further the elements in $H^0(M, Aut(E))$ are sections of a trivial bundle. Hence $H^0(M, Aut(E))$ can be identified with $GL_m(\mathbb{C})$. Set $G = U_m$ and note that the D'_{2q} defining D can be assumed to be G-equivariant by an averaging process. This choice does not interfere with (1.2) due to $H^0(M, Aut(E))$-equivariance of λ and $pr_{2q} \circ \log \circ F_{2q}$. Now the equivalence classes of supermanifold structures associated with $E \to M$ appear as collections of orbits under a linear representation of the compact group G on the finite dimensional vector space V. These orbits can be determined in contrast to those of the $GL_m(\mathbb{C})$-action on the pointed set $H^1(M, G_E)$. So this approach effectively reduces the problem of computability of equivalence classes of supermanifold structures in the mentioned setting.

2. Constructing non-split supermanifolds from cochains of nilpotent derivations

We have seen that a necessary condition on an element $u \in C^1(M, Der^{(2)}(\Lambda E))$ for $\exp(u) \in Z^1(M, G_E)$ (or equivalently $u \in \tilde{C}^1(M, Der^{(2)}(\Lambda E))$) is $R_{2q}(u) \in B^2(M, Der_{2q}(\Lambda E))$ for $q \geq 2$ and $d(pr_2(u)) = 0$. In particular, the weaker condition $R_{2q}(u) \in Z^2(M, End_{2q}(\Lambda E))$ for $q \geq 2$ has to be satisfied. For shortening the notation we denote $u = \sum_{k=1}^{\infty} u_{2k}$ with $u_{2k} \in C^1(M, Der_{2k}(\Lambda E))$. For $q = 2$ we see, using $(du_2)_{ijk} = u_{2,ij} + u_{2,jk} + u_{2,ki} = 0$, that

$$
\begin{aligned}
R_4(u)_{jkl} =& \frac{1}{2}(u_{2,jk}^2 + u_{2,kl}^2 + u_{2,lj}^2) + u_{2,jk}u_{2,kl} + u_{2,jk}u_{2,lj} + u_{2,kl}u_{2,lj} \\
=& \frac{1}{2}(du_2^2)_{jkl} + u_{2,lj}^2 + u_{2,jk}u_{2,kl} + u_{2,jk}u_{2,lj} + u_{2,kl}u_{2,lj} \\
=& \frac{1}{2}(du_2^2)_{jkl} + u_{2,jk}u_{2,kl}.
\end{aligned} \tag{2.1}
$$

Hence again by $du_2 = 0$,

$$
(dR_4(u))_{ijkl} = u_{2,jk}u_{2,kl} - u_{2,ik}u_{2,kl} + u_{2,ij}u_{2,jl} - u_{2,ij}u_{2,jk} = 0.
$$

So $R_4(u) \in Z^2(M, End_4(\Lambda E))$ independently of the choice of the element $u_2 \in Z^1(M, Der_2(\Lambda E))$. We now analyze the condition $R_6(u) \in Z^2(M, End_4(\Lambda E))$. Therefore we study $R_6(u_2)$ with $u_2 \in Z^1(M, Der_2(\Lambda E))$ first. By direct calculation we have

$$
\begin{aligned}
R_6(u_2)_{ijk} = \; & u_{2,ij}u_{2,jk}u_{2,ki} \\
& + \frac{1}{2}(u_{2,ij}u_{2,jk}^2 + u_{2,ij}^2 u_{2,jk} + u_{2,ij}u_{2,ki}^2 + u_{2,ij}^2 u_{2,ki} \\
& \qquad + u_{2,jk}u_{2,ki}^2 + u_{2,jk}^2 u_{2,ki}) \\
& + \frac{1}{6}(u_{2,ij}^3 + u_{2,jk}^3 + u_{2,ki}^3) \\
= \; & u_{2,ij}u_{2,jk}u_{2,ki} + \frac{1}{2}[u_{2,ij}, u_{2,jk}^2] - \frac{1}{3}(u_{2,ij}^3 + u_{2,jk}^3 + u_{2,ki}^3).
\end{aligned} \tag{2.2}
$$

Further by direct calculation using $du_2 = 0$:

$$\begin{aligned}
R_6(u)_{ijk} =& R_6(u_2)_{ijk} + u_{2,ij}u_{4,jk} + u_{4,ij}u_{2,jk} + u_{2,ij}u_{4,ki} \\
&\qquad + u_{4,ij}u_{2,ki} + u_{2,jk}u_{4,ki} + u_{4,jk}u_{2,ki} \\
&+ \frac{1}{2}(u_{2,ij}u_{4,ij} + u_{4,ij}u_{2,ij} + u_{2,jk}u_{4,jk} \\
&\qquad + u_{4,jk}u_{2,jk} + u_{2,ki}u_{4,ki} + u_{4,ki}u_{2,ki}) \\
=& R_6(u_2)_{ijk} + u_{4,ij}(\frac{1}{2}u_{2,ij} + u_{2,jk} + u_{2,ki}) + \frac{1}{2}u_{2,ij}u_{4,ij} \\
&+ u_{4,jk}(\frac{1}{2}u_{2,jk} + u_{2,ki}) + (\frac{1}{2}u_{2,jk} + u_{2,ij})u_{4,jk} \\
&+ \frac{1}{2}u_{4,ki}u_{2,ki} + (\frac{1}{2}u_{2,ki} + u_{2,ij} + u_{2,jk})u_{4,ki} \\
=& R_6(u_2)_{ijk} + \frac{1}{2}([u_{2,ij}, u_{4,ij}] + [u_{2,jk}, u_{4,jk}] - [u_{2,ki}, u_{4,ki}]) \\
&\qquad + [u_{2,ij}, u_{4,jk}] \\
=& R_6(u_2)_{ijk} + \frac{1}{2}(d[u_2, u_4])_{ijk} + [u_{2,ij}, u_{4,jk}].
\end{aligned} \tag{2.3}$$

So it follows that $R_6(u) \in Z^2(M, End_6(\Lambda E))$ is equivalent to

$$dR_6(u_2) = -d([u_{2,ij}, u_{4,jk}])_{ijk}. \tag{2.4}$$

The left hand side is $(dR_6(u_2))_{ijkl} = R_6(u_2)_{jkl} - R_6(u_2)_{ikl} + R_6(u_2)_{ijl} - R_6(u_2)_{ijk}$. The summand $\frac{1}{3}(u_{2,ij}^3 + u_{2,jk}^3 + u_{2,ki}^3)_{ijk} = \frac{1}{3}du_2^3$ in (2.2) has no contribution, and we find with $du_2 = 0$ and $u_{2,jk}^2 + u_{2,kl}^2 - u_{2,jl}^2 = (du_2^2)_{jkl}$:

$$\begin{aligned}
&(dR_6(u_2))_{ijkl} \\
=& u_{2,jk}u_{2,kl}u_{2,lj} - u_{2,ik}u_{2,kl}u_{2,li} + u_{2,ij}u_{2,jl}u_{2,li} - u_{2,ij}u_{2,jk}u_{2,ki} \\
&+ \frac{1}{2}([u_{2,jk}, u_{2,kl}^2] - [u_{2,ik}, u_{2,kl}^2] + [u_{2,ij}, u_{2,jl}^2] - [u_{2,ij}, u_{2,jk}^2]) \\
=& u_{2,jk}u_{2,kl}u_{2,lj} - u_{2,ik}u_{2,kl}u_{2,li} + u_{2,ij}u_{2,jl}u_{2,li} - u_{2,ij}u_{2,jk}u_{2,ki} \\
&- \frac{1}{2}[u_{2,ij}, (du_2^2)_{jkl}].
\end{aligned} \tag{2.5}$$

For the right hand side of (2.4) we have with $du_2 = 0$:

$$\begin{aligned}
-(d([u_{2,ij}, u_{4,jk}])_{ijk})_{ijkl} =& -[u_{2,jk}, u_{4,kl}] + [u_{2,ik}, u_{4,kl}] - [u_{2,ij}, u_{4,jl}] + [u_{2,ij}, u_{4,jk}] \\
=& \quad [u_{2,ij}, u_{4,jk} + u_{4,kl} + u_{4,lj}] = [u_{2,ij}, (du_4)_{jkl}].
\end{aligned}$$

Under the stronger necessary condition and $du_4 = -R_4(u)$ we obtain

$$-(d([u_{2,ij}, u_{4,jk}])_{ijk})_{ijkl} = -[u_{2,ij}, R_4(u)_{jkl}]. \tag{2.6}$$

Inserting (2.5), (2.6), and (2.1) into (2.4), $R_6(u) \in Z^2(M, End_6(\Lambda E))$ is equivalent to vanishing of

$$u_{2,jk}u_{2,kl}u_{2,lj} - u_{2,ik}u_{2,kl}u_{2,li} + u_{2,ij}u_{2,jl}u_{2,li} - u_{2,ij}u_{2,jk}u_{2,ki} + [u_{2,ij}, u_{2,jk}u_{2,kl}].$$

By $du_2 = 0$ this is always satisfied. Hence we summarize:

Proposition 2.1. *Let $u \in C^1(M, Der^{(2)}(\Lambda E))$. The following implications of necessary conditions for* $\exp(u) \in Z^1(M, G_E)$ *exist:*

If $du_2 = 0$ is satisfied, then $R_4(u) \in Z^2(M, End_4(\Lambda E))$ is satisfied.

If $du_2 = 0$ and $du_4 = -R_4(u)$ are satisfied, then $R_6(u) \in Z^2(M, End_6(\Lambda E))$ is satisfied and

$$R_6(u) \in Z^2(M, Der_6(\Lambda E)) \qquad \Leftrightarrow \qquad R_6(u_2) \in C^2(M, Der_6(\Lambda E)).$$

Proof. The second part of the second statement follows from (2.3). □

In the following we denote:

$$\tilde{Z}^1(M, Der_2(\Lambda E)) := \big\{u_2 \in Z^1(M, Der_2(\Lambda E)) \;\big|\; \begin{array}{ll} & R_4(u_2) \in C^2(M, Der_4(\Lambda E)) \\ \text{and} & R_6(u_2) \in C^2(M, Der_6(\Lambda E))\big\}. \end{array}$$

It follows with Corollary 1.2:

Corollary 2.2. *Let $E \to M$ be a vector bundle of rank 6 or 7. Assume further that $H^2(M, Der_4(\Lambda E)) = H^2(M, Der_6(\Lambda E)) = 0$.*

The necessary and sufficient condition on a cochain $u_2 \in C^1(M, Der_2(\Lambda E))$ for the existence of a $u \in \tilde{C}^1(M, Der^{(2)}(\Lambda E))$ with $pr_2(u) = u_2$ is $u_2 \in \tilde{Z}^1(M, Der_2(\Lambda E))$. If in addition $H^0(M, Der^{(2)}(E)) = 0$, then there is a D such that

$$\sigma_D : H^1(M, G_E) \to \frac{\tilde{Z}^1(M, Der_2(\Lambda E))}{B^1(M, Der_2(\Lambda E))} \oplus H^1(M, Der_4(\Lambda E)) \oplus H^1(M, Der_6(\Lambda E))$$

is a well-defined bijection.

Proof. By the proposition, $R_4(u_2) \in Z^1(M, Der_4(\Lambda E))$. Fix any cochain u_4 such that $R_4(u_2) = -du_4$. Now $R_6(u_2 + u_4) \in Z^1(M, Der_6(\Lambda E))$ is satisfied so there is a u_6 with $R_6(u_2 + u_4) = -du_6$. □

The condition $H^2(M, Der_4(\Lambda E)) = H^2(M, Der_6(\Lambda E)) = 0$ is satisfied e.g. on strongly 2-complete underlying manifolds. Moreover a Leray cover (for coherent sheaf cohomology) of M consisting only of 2 charts allows $R_{2q} \equiv 0$ and hence $\tilde{Z}^1(M, Der_2(\Lambda E)) = Z^1(M, Der_2(\Lambda E))$. Here are two classes of examples.

Example 2.3. Let $M = \mathbb{P}^n(\mathbb{C}) \backslash \mathbb{P}^{n-2}(\mathbb{C})$, $n \geq 2$, where $\mathbb{P}^{n-2}(\mathbb{C})$ is embedded in $\mathbb{P}^n(\mathbb{C})$ by $[w] \mapsto [0 : 0 : w]$. Further let E be any vector bundle on M of rank up to 7. There is a Leray cover with two components: two of the $n + 1$ standard coordinate charts of $\mathbb{P}^n(\mathbb{C})$. By Corollary 2.2 any $u_2 \in Z^1(M, Der_2(\Lambda E))$ can be continued to a $u \in C^1(M, Der^{(2)}(\Lambda E))$ with $pr_2(u) = u_2$ such that $\exp(u)$ defines a supermanifold structure on M.

For any vector bundle E and $q \geq 1$ we have the short exact sequence

$$0 \to Hom_{\mathcal{O}_M}(E, \Lambda^{2q+1}E) \to Der_{2q}(\Lambda E) \to Der(\mathcal{O}_M, \Lambda^{2q}E) \to 0,$$

the second arrow by continuation via graded Leibniz rule trivially on $\mathcal{O}_M$ and the third arrow by restriction. By the long exact sequence of cohomology we obtain exactness of

$$H^0(M, Hom_{\mathcal{O}_M}(E, \Lambda^{2q+1}E)) \to H^0(M, Der_{2q}(\Lambda E)) \to H^0(M, Der(\mathcal{O}_M, \Lambda^{2q}E))$$

where $Hom_{\mathcal{O}_M}(E, \Lambda^{2q+1}E) \cong \mathcal{O}_{\Lambda^{2q+1}E \otimes E^*}$ and $Der(\mathcal{O}_M, \Lambda^{2q}E) \cong \mathcal{O}_{\Lambda^{2q}E \otimes TM}$. Combining this we have

$$\begin{aligned} (2.7) \qquad & H^0(M, \mathcal{O}_{\Lambda^{2q+1}E \otimes E^*}) = H^0(M, \mathcal{O}_{\Lambda^{2q}E \otimes TM}) = 0 \quad \forall q \geq 1 \\ \Rightarrow \quad & H^0(M, Der^{(2)}(\Lambda E)) = 0. \end{aligned}$$

Example 2.4. Let $M = \mathbb{P}^1(\mathbb{C})$ and fix a sum of line bundles $E = \bigoplus_{i=1}^k \mathcal{O}(l_i)$ with $4 \le k \le 7$ and $l_1 \le \cdots \le l_k$ such that

$$l_{k-1} + l_k < -2 \qquad \text{and} \qquad l_{k-2} + l_{k-1} + l_k - l_1 < 0. \tag{2.8}$$

Now $Der(\mathcal{O}_M) = \mathcal{O}(2)$ and $H^0(M, \mathcal{O}(l)) = 0$ for $l < 0$ so $H^0(M, Der^{(2)}(\Lambda E)) = 0$ by (2.7). The standard coordinate charts of M yield a Leray cover with two components. Hence Corollary 2.2 produces a bijection:

$$\sigma_D : H^1(M, G_E) \to H^1(M, Der^{(2)}(\Lambda E)).$$

Further (2.8) ensures that $\dim(H^1(M, Der^{(2)}(\Lambda E))) \gg 0$, yielding many non-split supermanifold structures.

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